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ERRATA.

IN THE PREFACE.

In page xv, line 14 from the bottom, inſtead of *place*, rea

IN THE BOOK.

In page 5, line 20, inſtead of *indicunt*, read *inducunt*.

In page 7, line 13 from the bottom, inſtead of *Auctorum*,

In page 21, line 6, inſtead of 3)2(2 : 2, read 3)3(2 : 2.

And in the ſame page 21, line 11, inſtead of 3)2(4 : 3, r

In page 23, line 6 from the bottom, inſtead of *tabulà*, re

In page 25, line 16, inſtead of $\frac{\overline{n-3}.e}{r}$, read $\frac{\overline{n-3}.c}{r}$.

In page 27, line 5, after the word "primam" dele the fig

And in the ſame page 27, line the 3d from the botto
$\frac{n.n-1}{1.2}$, read $\frac{n.\overline{n-1}}{1.2}$.

In page 28, line 10, after the word "ſubquintuplum," in

And in the ſame page 28, line 15, inſtead of $\frac{n.\,n-1}{1.2}$, read

In page 30, the bottom line, inſtead of $-\frac{1}{2}nn$; read $+$

In page 32, in the laſt line but one of the lines that are
ſide of the page, inſtead of $-\frac{1}{12}nn$, read $-\frac{3}{20}nn$.

In page 50, line 2 from the bottom, inſert the mark "
"Mathematicks."

In page 69, line 21, inſtead of *of rows*, read *or rows*.

In page 72, in the note at bottom, inſtead of *Alteratio*
nations.

In page 73, line 6, inſtead of 252, 462, read 210, 330.

In page 87, dele the figure of 1 at the end of the firſt

In page 101, line 3, inſtead of 71, read 74.

In page 102, line 12, inſtead of 71, read 74.

In page 103, line 12, inſtead of $\frac{n \times \overline{f+e+d+e+b}}{r}$
$\frac{n \times \overline{f+e+d+c+b+a}}{r}$

ERRATA.

In page 112, line 16, inſtead of $\frac{n.\,n-1.\,\overline{n-2}.\,\overline{n-3}}{2\times 3\times 4}$, read $\frac{n.\,\overline{n-1}.\,\overline{n-2}.\,\overline{n-3}}{2\times 3\times 4}$.

And in the ſame page 112, line 17, inſtead of $\frac{n.\,\overline{n-1}.\,\overline{n-2}.\,\overline{n-3}.\,n-4}{2\times 3\times 4\times 5}$, read $\frac{n.\,\overline{n-1}.\,\overline{n-2}.\,\overline{n-3}.\,\overline{n-4}}{2\times 3\times 4\times 5}$.

In page 117, line 11 from the bottom, inſtead of $Ba^{1-2}b^{1}$, read $Ba^{1-2}b^{2}$.

In page 144, line 7 from the bottom, inſtead of $-56x$, read $-56x^{5}$.

In page 151, in the bottom line, inſtead of $-\frac{n}{\ }Ax$, read $-\frac{n}{1}Ax$.

In page 158, line 3, inſtead of $21x^{5}+36x^{6}$, read $+21x^{5}+28x^{6}+36x^{7}$.

In page 183, lines 6, 7, and 8, the figure of 2 is not clear in the powers of 12 in the numerators of the fractions $\frac{\overline{12}|^{7}}{2}$, $\frac{12\times\overline{12}|^{7}}{8}$, and $\frac{7\times\overline{12}|^{6}}{2}$.

In page 184, line 5 from the bottom, inſtead of 116,12, read 116,122.

In page 196, line 8 from the bottom, inſtead of $\frac{11}{11}$, read $\frac{n^{11}}{11}$.

In page 197, the top line, inſtead of $\frac{5n}{6}$, read $\frac{5n}{66}$.

In page 218, line 3, inſtead of

$$+nBx^{n-1}d^{2}+n\times\frac{n-1}{2}Bx^{n-2}d^{3}+n\times\frac{n-1}{2}\times\frac{n-2}{3}Bx^{n-3}d^{4},\ \&c$$

read $+nBx^{n-1}d+n\times\frac{n-1}{2}Bx^{n-2}d^{2}+n\times\frac{n-1}{2}\times\frac{n-2}{3}Bx^{n-3}d^{3}$, &c.

And in the ſame page 218, line 4, inſtead of

$+n-1\times Cx^{n-2}d^{3}+n-1\times\frac{n-2}{2}Cx^{n-3}d^{4}$, &c, read

$+\overline{n-1}\times Cx^{n-2}d+\overline{n-1}\times\frac{n-2}{2}\times Cx^{n-3}d^{2}$, &c.

And again in the ſame page 218, line 5, inſtead of $+n-2\,Dx^{n-3}d^{4}$, &c, read $+\overline{n-2}\times Dx^{n-3}d$, &c.

In page 221, line 5 from the bottom, inſtead of $\frac{n\times n-1\times n-2}{2.3.4.5.6}$, read $\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}$.

And in the ſame page 221, line 2 from the bottom, inſtead of $\frac{n \times n-1 \times n-2\, x^{n-3}}{2.3.4.5.6}$, read $\frac{n \times \overline{n-1} \times \overline{n-2} \times x^{n-3}}{2.3.4.5.6}$.

And again in the ſame page 221, the bottom line, inſtead of $\frac{n \times n-1 \times n-2}{2.3.4.5.6}$, read $\frac{n \times \overline{n-1} \times \overline{n-2}}{2.3.4.5.6}$.

In page 239, line 11, inſtead of a_m, read a^m.

In page 264, line 6 from the bottom, inſtead of $\overline{1+1}|$, read $\overline{1+1}|^{n}$.

In page 298, line 6, inſtead of *bc*, read *bbc*.

In page 302, line 10 from the bottom, the firſt figure after 10, which ſhould be a 3, is not clear.

In page 340, line 16 from the bottom, inſtead of 391, read 191.

In page 341, line 15 from the bottom, inſtead of 3, 67, read 31, 67.

In page 350, line 11, inſtead of 92 × 76, read 29 × 67.

In page 369, line 9 from the bottom, column 17, inſtead of p, read 7.

In page 371, line 6 from the bottom, column 14, inſtead of 19, read 37.

In page 374, line 11 from the bottom, column 10, inſtead of 73, read 107.

In page 435, line 9 from the bottom, inſtead of 3, read 2.

In page 446, line 13 from the bottom, inſtead of $\frac{4m^2n^3}{\overline{mm+nn}|^2}$, read $\frac{4m^2n^2}{\overline{mm+nn}|^2}$.

In pages 470, 471, 472, and 473, the title at the top of the pages is wrong. It ſhould be, *Of the Extraction of the Cube-root by Mr. de* Lagny*'s Method of Approximation.*

In page 472, lines 8 and 9 from the bottom, inſtead of 3.264, read 0.3264.

In page 488, line 4, after the letter *a*, inſert the firſt mark of a parentheſis, to wit (.

In page 495, line 7, inſtead of $\frac{a^3 - c \times a}{c + 2a^3}$, read $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$.

In page 511, line 4 from the bottom, inſtead of $- Ca^{m-2}z^2$, read $+ Ca^{m-2}z^2$.

And in the ſame page 511, line 2 from the bottom, inſtead of $+ Ca^{m-2}$, read $+ Ca^{m-2}z^2$.

In page 540, lines 10 and 15, inſtead of $\sqrt{}^{3}$, read $\sqrt{}^{m}$.

In

ERRATA.

In page 542, line 9 from the bottom, instead of $\frac{2a \times a^{m} - N}{\overline{m-1} \times N + \overline{m+1} \times a^{m}}$, read $\frac{2a \times \overline{a^{m} - N}}{\overline{m-1} \times N + \overline{m+1} \times a^{m}}$.

In page 561, in the Note, line 19 from the bottom, instead of $- b^2x^3$, read $- b^2x^3$.

THE

PREFACE.

IT is well known to perſons acquainted with the hiſtory of the Mathematicks, that Sir Iſaac Newton's celebrated Theorem concerning the powers of a binomial quantity, ſuch as $a + b$, was communicated by him to the world, in the latter part of the laſt century, without a demonſtration. And many other writers on Algebra ſince Sir Iſaac's death, and, amongſt the reſt, the famous Profeſſor *Saunderſon*, of the Univerſity of Cambridge, have followed his example in taking this important Theorem for granted, and delivering it to their readers without attempting to demonſtrate it. For the chapter on this ſubject in the ſecond volume of the Profeſſor's *Elements of Algebra*, in two volumes, quarto, contains nothing more than a full and clear deſcription of the Theorem, with an application of it to a good number of well-choſen examples, by way of illuſtration. This neglect of demonſtrating ſo important a propoſition has always appeared to me very ſtrange; as the great merit and glory of the mathematical Sciences conſiſts in the certainty of the principles on which they are founded, and the clearneſs and regularity with which all the ſubſequent concluſions obtained in them are deduced from thoſe fundamental principles. There have been, however, other eminent Mathematicians who have ſupplied this great omiſſion, and given us juſt and accurate demonſtrations of this Theorem in ſome of the more obvious and important caſes of it, though not, perhaps, in all its caſes. And of theſe I conſider Mr. *James Bernoulli* (who was Profeſſor of Mathematicks at Baſil, or Baſle, in Switzerland, in the latter part of the laſt century,) as one of the moſt ſucceſsfull. For, in the 3d chapter of the ſecond part of his excellent Treatiſe on the Doctrine of Chances, intitled, *Ars Conjectandi*, (which was publiſhed at Baſil in the year 1713, in a ſmall quarto volume, ſome years after his death,) there is a demonſtration of this celebrated Theorem in the firſt, or ſimpleſt, caſe of it, (or when m, or the index of the power of the binomial quantity $a + b$, is an affirmative whole number,) that is deduced from the very nature of Multiplication and the properties of the Figurate numbers, in the cleareſt and moſt accurate manner poſſible. So that no demonſtration of it ought to be expected, or need be deſired, that ſhall exceed, or even equal, this in point of accuracy and perſpicuity, though ſome others may, perhaps, be ſomewhat ſhorter. This demonſtration I was therefore deſirous of making more generally known to the Students of the Mathematicks; and with that view I reſolved to re-publiſh it, together with ſo much of the concomitant text of Mr. Bernoulli's ſaid valuable Treatiſe, as was neceſſary to the thorough underſtanding of it, in a volume of a moderate ſize and price. This was the inducement that gave riſe to the preſent publication.

To anſwer this purpoſe in the moſt effectual manner, I thought it would be beſt to re-publiſh the whole of the three firſt chapters of the ſecond Part of the ſaid Treatiſe of Mr. James Bernoulli, together with the Preface to the ſaid ſecond Part; but without the firſt Part of the ſame work, becauſe the ſaid firſt part, (though in itſelf important and curious, and eſſential, I doubt not, to the full underſtanding of *the Doctrine of Chances*,) is not at all neceſſary to the underſtanding of the ſecond Part, which treats of the Doctrine of Permutations and Combinations, and begins, in the moſt diſtinct and elementary manner, with the firſt foundations of that doctrine. And further, as there are many perſons in England that are fond of the Mathematical Sciences without having much acquaintance with the Latin language, I have, in order to render the contents of theſe three valuable chapters acceſſible to ſuch perſons, tranſlated theſe chapters into Engliſh, and ſubjoined the tranſlation to the original text in Latin; ſo that the reader may chuſe in which of the two languages he will peruſe them. And in this tranſlation I have expreſſed myſelf in a fuller manner than Mr. Bernoulli had adopted in the original, becauſe I had obſerved that the great degree of brevity with which Mr. Bernoulli had expreſſed himſelf had rendered ſome parts of the original rather obſcure. And I have likewiſe added a few notes both to the original and the tranſlation, where the text ſeemed to me to require them.

And further, in the latter part of the tranſlation of theſe chapters, I have alſo done ſomething more than merely tranſlate them. For, as I obſerved that Mr. Bernoulli's concluſions concerning the properties of the Figurate numbers, (which he had applied to the demonſtration of the binomial theorem in the firſt, or ſimpleſt, caſe of it, or when m, or the index of the power of the binomial quantity $a + b$, was *an affirmative whole number*,) might eaſily be applied to the demonſtration of the binomial theorem in another caſe of it, to wit, in that caſe of it in which m, or the index of the binomial quantity $a + b$, is *a negative whole number*, I drew up ſome additional articles, that are not contained in Mr. Bernoulli's text, for this purpoſe. Theſe additional articles, (which contain a demonſtration of the binomial theorem in *the caſe of integral and negative powers*, or in the caſe of the quantity $\overline{a+b}\rvert^{-m}$,) extend from page 123 to page 166; after which the tranſlation of Mr. Bernoulli's text is reſumed, and continues to page 213.

Theſe three chapters contain a moſt accurate and diſtinct explanation of the fundamental parts of the Doctrine of Permutations and Combinations, and of the moſt remarkable properties of the Figurate numbers, which, it is well known, are of the moſt extenſive uſe in various branches of the Mathematicks. And they likewiſe contain an application of the properties of theſe important numbers to the ſummation of the ſquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to any propoſed number n, and to the ſummation of the cubes, and of the fourth powers, and of the fifth powers, and of all the following powers, of the ſame numbers, (which is a matter of much nicety and difficulty, and was formerly a great object of inquiry to Mathematicians,) as well as to the demonſtration of the binomial theorem in the caſe of integral and affirmative powers, and (with the articles I have added to it in pages 123, &c, to 166,) in the caſe of integral and negative powers. All which, together, makes a conſiderable body of very uſeful mathematical learning.

Immediately after theſe three chapters of Mr. James Bernoulli's *Ars Conjectandi*, I have re-publiſhed the tenth Mathematical Eſſay of the late very learned and ingenious Mathematician, Mr. Thomas Simpſon, of Woolwich Academy, which is a ſolution of the following Problem, to wit, "*To find the ſum of any ſeries of powers whoſe roots are in* "*arithmetical progreſſion, as* $\overline{m+d}|^{n}$, $\overline{m+2d}|^{n}$, $\overline{m+3d}|^{n}$, " $\overline{m+4d}|^{n}$, $\overline{m+5d}|^{n}$, - - - - - - x^{n}, *the letters* m, d, *and* " n, *denoting any numbers whatſoever.*" This Eſſay of Mr. Simpſon had been alluded to in a note to the tranſlation of the foregoing extract from Mr. Bernoulli's book, at the bottom of page 213; and it is ſo nearly connected with the ſubject of the latter part of that extract relating to the ſums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, that I thought it would be agreeable to the reader to have it laid before him immediately after the ſaid extract; and therefore I cauſed it to be reprinted in that place. It extends from page 214 to page 224.

The next Tract is one of my own compoſition, and contains *An Inveſtigation and Demonſtration of Sir Iſaac Newton's*

Binomial Theorem in the caſe of integral and affirmative powers; in which the law of the generation of the numeral co-efficients of the terms of the ſeries which is equal to the quantity $\overline{a + b}|^{m}$, *is diſcovered by a conjecture grounded on the obſervation of the law of the ſaid co-efficients in ſome particular examples; but, when ſo diſcovered, is ſhewn to be true univerſally in all other integral and affirmative powers whatſoever of the ſaid binomial quantity, by a ſtrict and accurate demonſtration.* This Tract begins in page 227, and ends in page 268, and contains, as I believe, the beſt and moſt ſatisfactory demonſtration of the Binomial Theorem in the caſe of integral and affirmative powers that has yet been given of it, next to that contained in the foregoing Extract from Mr. James Bernoulli's Treatiſe, intitled *Ars Conjectandi.* The conjectural inveſtigation of the law of the numeral co-efficients of the terms of the ſeries that is equal to $\overline{a + b}|^{m}$, given in this Tract, is ſuggeſted by Profeſſor Saunderſon, in the ſecond volume of his Algebra, in the chapter on the Binomial Theorem; where (as I before obſerved,) the reader will find a good explanation and illuſtration of the ſaid celebrated Theorem by a variety of examples, both in the caſe of integral powers and in the caſe of roots and other fractional powers, and even in the caſe of negative powers and of powers that are both fractional and negative; but no demonſtration of it in any caſe, not even in that of integral and affirmative powers. And the following ſtrict demonſtration of this Theorem in the caſe of integral and affirmative powers, (which begins in page 252, and ends with page 264,) is nearly the ſame with that which is given by Mr. John Stewart, of Aberdeen, in the 6th ſection of his Commentary on Sir Iſaac Newton's curious little Tract, intitled, *Analyſis per Æquationes numero terminorum infinitas,* or *Analyſis by Equations of an infinite number of Terms.* See his edition of Newton's Treatiſe *on the Quadrature of Curves,* and of the ſaid Tract, intitled, *Analyſis, &c,* with his learned comments on both, in one volume, quarto, publiſhed at London in the year 1745, page 471, art. 155.

This Tract, concerning the ſaid conjectural inveſtigation and ſubſequent general demonſtration of the Binomial Theorem in the caſe of integral and affirmative powers, contains the ſubſtance of two Tracts publiſhed in the year 1792, in the ſecond volume of the Collection of Mathematical Tracts, in quarto, called, *Scriptores Logarithmici*, to wit, the 15th Tract, which extends from page 153 to page 169, of the ſaid ſecond volume, and the 23d, or laſt, Tract in the ſaid volume, which extends from page 587 to page 591.

Next to this Tract on the Inveſtigation and Demonſtration of Sir Iſaac Newton's Binomial Theorem, I have republiſhed a Tract of the learned Dr. John Wallis, of Oxford, on the ſame Doctrine of Permutations and Combinations, which is the ſubject of the foregoing Extract from Mr. James Bernoulli's work above-mentioned. This Tract was publiſhed with Dr. Wallis's Algebra in the year 1685, under the title of *A Diſcourſe of Combinations, Alternations, and Aliquot Parts*, and is mentioned by Mr. James Bernoulli in the foregoing Extract of his *Ars Conjectandi*, in the Scholium in pages 29 and 166, as a well-known and valuable Treatiſe on the properties of the Figurate numbers. And it does indeed contain a great deal of excellent and curious matter concerning thoſe numbers, and the other ſubjects of which it treats, but without that accuracy and regularity in the manner of deducing the concluſions of it one from another, which diſtinguiſh the foregoing chapters of Mr. Bernoulli's work. However, on account of its intrinſick merit, and its relating to the ſame ſubjects, in a great meaſure, as the ſaid Extract from Mr. Bernoulli's book, I thought it would be agreeable to my readers to ſee a re-publication of it in the ſame volume with the ſaid Extract, and therefore I have given it a place in this Collection. It begins in page 271, and extends to page 351.

Much of this Diſcourſe of Dr. Wallis relates to *Prime*, or *Incompoſit*, numbers, and to curious arithmetical queſtions depending on them, And in one part of it, to wit, in page 318, the Doctor ſpeaks of the great convenience of having at hand a Table of Prime Numbers ſet down in regular order, to be referred to when we want to know into what prime numbers a given odd number may be reſolved. And he mentions a very uſeful Table of this kind that had been drawn up by a Mr. *Thomas Brancker*, M. A. and publiſhed by him in the year 1668, in an Appendix to an Engliſh tranſlation, made by him, of *Rhonius*'s *Algebra*, which had been publiſhed in the German language at Zurich in Switzerland, in the year 1659, under the title of *Algebra Rhonii, Germanicè.* This Engliſh tranſlation of Rhonius's Algebra was publiſhed by Mr. Brancker under the inſpection, and with the aſſiſtance, of Dr. John Pell, an eminent Mathematician in the reign of King Charles the Second, and ſome conſiderable additions were made to the tranſlation by Dr. Pell himſelf; which has given occaſion to the book's being ſometimes ſpoken of by ſubſequent writers of Mathematicks, and amongſt others by Dr. Wallis himſelf in this Diſcourſe, page 319, by the name of *Dr. Pell's Algebra.*

This Table of Prime Numbers Dr. Wallis ſet a high value on, inſomuch that he took the pains to examine it carefully throughout, and to correct the few errors that he found in it; ſo that now, with his corrections, it may be conſidered as very accurate. This Table therefore, together with the Appendix in which it is contained, I have here cauſed to be re-printed immediately after the foregoing Diſcourſe of Dr. Wallis. It contains not only all the Prime numbers that are leſs than 100,000, but all the odd numbers whatſoever that are leſs than that number, (except ſuch odd numbers as end with the figure of 5, and are therefore evidently diviſible by the number 5,) and it diſtinguiſhes the Prime numbers from the other odd numbers, by annexing to them the letter p; and it annexes likewiſe to every other odd number (that is not a Prime, or Incompoſit, number, but is the product of the multiplication of two, or more, leſſer numbers,) the leaſt of the prime numbers into which it may be reſolved. This Appendix, with the ſaid Table of odd numbers contained in it, extends from page 353 to page 416.

The next Tract in this Collection relates to the Rational Numbers that will expreſs the Sides of Right-angled Triangles, and contains two methods of finding as many ſets of numbers as we pleaſe that ſhall have this property. The firſt of theſe methods begins in page 417, and ends in page 431, and the ſecond reaches from page 431 to page 448; after which I have inſerted a Table of the Squares of the ſeveral natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, as far as 100, together with two additional columns adjoining to the column of the ſaid ſquares, in the former of which I have ſet down the differences of the ſaid ſquares, and in the latter the differences of thoſe differences, or the ſecond differences of the ſquares themſelves; which ſecond differences are all equal to each other, and to the number 2. This Table begins in page 449, and is accompanied with ſome remarks which extend to page 457. This Tract has a conſiderable reſemblance to ſome parts of the foregoing Diſcourſe of Dr. Wallis, and may afford ſome amuſement to ſuch readers as are fond of contemplating the properties of numbers.

The next Tract relates to the Cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, and to the differences of the ſaid cubes, and the differences of the ſaid differences, or the ſecond differences of the ſaid cubes themſelves, and to the differences of the ſaid ſecond differences, or the third differences of the cubes; which third differences are all equal to each other, and to the number 6. And in pages 460, 461, and 462, I have exhibited a Table of the Cubes of all the ſaid natural numbers as far as 100, together with the 1ſt, 2d, and 3d differences of the ſaid cubes in adjoining columns; after which follows an extract from a learned letter of the celebrated Mr. Leibnitz to Mr. Oldenburgh, the Secretary of the Royal Society of London, dated from London on the 3d day of February, 1672-3, relating to the ſubject of the differences of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, and to the ſeveral ſucceſſive orders of ſuch differences, and to the ultimate equality of the ſeveral ſucceſſive differences in the ſecond, or third, or fourth, or fifth, or other ſubſequent, order of the ſaid differences, according to the height of the power to which the ſaid numbers are raiſed, and relating to other curious properties of numbers. This letter is in Latin, and extends from page 463 to page 469, and is re-printed from the celebrated *Commercium Epiſtolicum* of Mr. John Collins, and other eminent Mathematicians of the latter part of the laſt century, that was firſt printed by the order of the Royal Society in the year 1712, and was afterwards re-printed in the year 1722. The remaining part of this Tract, from page 469 to page 504, relates to Monſieur de Lagny's Method of Extracting the Cube-roots of Numbers by Approximation, and begins with ſhewing the uſefulneſs of the foregoing Table of Cubes, in finding the firſt near value of the cube-root ſought, to one, or two, places of figures, which is to be the baſis of a further approximation to it by Mr. de Lagny's method. The four expreſſions given by Mr. de Lagny for the ſecond near value of the cube-root that is ſought, are ſtated in pages 470 and 471; and in the following pages to page 483, examples are given of the extraction of the cube-roots of the three numbers 2, 231, and 37,945, and of the long number 696,536,483,318,640,035,073,641,037, to a great degree of exactneſs, by means of ſome of the ſaid expreſſions, after the firſt near values of the ſaid cube-roots to one or two places of figures, have been obtained by the help of the foregoing table of the cubes of the firſt hundred numbers. Theſe four examples (in which the ſeveral proceſſes are ſtated very much at length,) will, I apprehend, be ſufficient to make the reader familiarly acquainted with the method of uſing the ſaid expreſſions of Mr. de Lagny in the extraction of the cube-roots of numbers, and at the ſame time to convince him of the great uſefulneſs of theſe expreſſions for effecting that purpoſe; and they will likewiſe ſhew the uſefulneſs of the foregoing Table of Cubes, in obtaining the firſt near values of the cube-roots ſought, to one, or two, places of figures, from which the more accurate values of them are afterwards derived by means of Mr. de Lagny's expreſſions. After theſe four examples of Mr. de

Lagny's method of Approximation, follows a Scholium (in pages 484, 485, and 486,) concerning the invention of theſe expreſſions, and of Mr. Raphſon's and Sir Iſaac Newton's methods of extracting the Cube-roots, and other higher roots, of given numbers, and even the roots of affected equations of any order, by ſimilar approximations; which methods were invented by thoſe eminent Mathematicians before the publication of theſe expreſſions of Mr. de Lagny. And, then, (in pages 486, 487, 488, &c, to page 500,) I have given very full and accurate inveſtigations of the foregoing expreſſions of Mr. de Lagny, which had been only ſtated in pages 470 and 471, and illuſtrated by examples in the following pages, from page 471 to page 483. And, laſtly, in pages 501, 502, 503, 504, I have given a further illuſtration of the ſaid expreſſions of Mr. de Lagny, by applying ſome of them to the extraction of ſome of the cube-roots which had been obtained in the foregoing examples by means of others of them; with a view to make a compariſon between the different expreſſions given for the ſame purpoſe by Mr. de Lagny, and to diſcover which of them are the moſt exact, or the moſt eaſy to practice, and in which caſes it will be moſt adviſeable to reſort to ſome of them in preference to the others. This Tract (which begins in page 459, and ends in page 504,) I conſider as a very uſeful one to young ſtudents of Arithmetick and Algebra.

Having in the foregoing Tract very fully explained, and illuſtrated by examples, Mr. de Lagny's method of Extracting the Cube-roots of given numbers by Approximation, I proceed in the next Tract to ſtate his general method of Extracting any higher Roots whatſoever of given numbers by ſimilar Approximations. All theſe approximations are grounded on the ſame principle, and conſiſt in putting ſome letter of the alphabet, as a, for the known part of the root ſought, (which known part is found by conjecture, or otherwiſe, as the caſe may admit,) and putting ſome other letter, as z, for the unknown difference by which x, or the true root of the given number (which may be called N,) exceeds, or falls ſhort of, the firſt value a, (which is ſuppoſed to be known,) and then ſubſtituting $a + z$, or $a - z$, inſtead of x, in the original equation $x^5 = N$, or $x^7 = N$, or, in general, $x^m = N$, (whereby the ſaid equation will be transformed into another equation in which z will be the only unknown quantity,) and, laſtly, in reſolving this transformed equation (of which z is the root,) as if it was only a quadratick equation, or omitting, or expunging from it, all the terms that involve any higher power of z than the ſquare. By ſuch a reſolution of this transformed equation Mr. de Lagny obtains a value of z that approaches nearly to its true value: and conſequently, by ſubſtituting this near value of z, inſtead of z, in the binomial quantity $a + z$, or $a - z$, (which is equal to x, or the root ſought, or the mth root of the given number N,) he obtains a near value of $a + z$, or $a - z$, or a ſecond near value of x, or $\sqrt[m]{N}$, which is much nearer to its true value than a, or its firſt near value, was. To explain in a full and diſtinct manner this method of extracting the mth root of any given number N, and to illuſtrate it by a few examples of the extraction of ſome high roots of given numbers, by means of the general expreſſions of the values of $a + z$ and $a - z$ derived from it, is the object of the preſent Tract. And, as the inveſtigations neceſſary for this purpoſe are very general, and, from that circumſtance, are rather more ſubtle and difficult than the inveſtigations in the preceeding Tract, (which related only to the extraction of the cube-roots of given numbers,) I have taken great pains to ſet down all the ſteps in them in regular order, as clearly and plainly as I could; which may make them appear longer than might, perhaps, have been expected, but will, in fact, enable the reader to make himſelf perfect maſter of them in leſs time than if they had been compreſſed within a narrower compaſs. The general expreſſions that are thus inveſtigated, are no leſs than four; to wit, two near values of $a + z$, obtained by conſidering the aforeſaid transformed equation (ariſing from the ſubſtitution of $a + z$, inſtead of x, in the original equation $x \quad = N$,) as a quadratick equation, and reſolving it, as ſuch, in two different manners, to wit, firſt, imperfectly, and ſecondly, in an accurate manner; and two near values of $a - z$ obtained in like manner, by conſidering the other transformed equation, (ariſing from the ſubſtitution of $a - z$, inſtead of x, in the original equation $x^m = \mathrm{N}$,) as a quadratick equation, and reſolving it, as ſuch, in two different manners, to wit, firſt, imperfectly, and ſecondly, in an accurate manner. In order to perform theſe inveſtigations the more eaſily and diſtinctly, I have divided the ſubject into two caſes, with Problems correſponding to them, according as x, or $\sqrt[m]{\mathrm{N}}$, is greater, or leſs, than its firſt value a, or is equal to $a + z$, or to $a - z$. The firſt caſe, or that in which x, or $\sqrt[m]{\mathrm{N}}$, is equal to $a + z$, is conſidered in the firſt Problem; and the ſecond caſe, or that in which x, or $\sqrt[m]{\mathrm{N}}$, is equal to $a - z$, is conſidered in the ſecond Problem: and from the Solution of the firſt Problem we obtain the two following expreſſions, to wit, $a + \dfrac{2a \times \overline{\mathrm{N} - a^m}}{\overline{m-1} \times \mathrm{N} + \overline{m+1} \times a^m}$ and

$$a + \sqrt{\frac{aa}{\overline{m-1}^2} + \frac{2 \times \overline{\mathrm{N} - a^m}}{m \times \overline{m-1} \times a^{m-2}}} - \frac{a}{m-1},$$

for near values of the binomial quantity $a + z$, or for ſecond near values of x, or $\sqrt[m]{\mathrm{N}}$; and from the Solution of the ſecond Problem we obtain the two following expreſſions, to wit, $a - \dfrac{2a \times \overline{a^m - \mathrm{N}}}{\overline{m-1} \times \mathrm{N} + \overline{m+1} \times a^m}$ and

$$a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} - \frac{2 \times \overline{a^m - \mathrm{N}}}{m \times \overline{m-1} \times a^{m-2}}},$$

for near values of the reſidual quantity $a - z$, or for ſecond near values of x, or $\sqrt[m]{\mathrm{N}}$. And the Solutions of both theſe Problems are illuſtrated by a few ſuitable examples, placed at the end of each ſolution reſpectively, of the extraction of different roots of given numbers by means of the ſaid general expreſſions obtained in the preceeding ſolutions. The Solution of the firſt of theſe Problems begins in page 508, and ends in page 516; and is followed by three examples, which begin in page 516 and end in page 525: after which I have inſerted a Scholium containing a compariſon between Mr. de Lagny's aforeſaid method of extracting the roots of given numbers, and Mr. Raphſon's method of performing the ſame thing; which is ſomewhat ſimpler and eaſier than Mr. de Lagny's method, though not quite ſo exact. For the difference between the two methods conſiſts only in this, that, whereas Mr. de Lagny reſolves the transformed equation ariſing from the ſubſtitution of $a + z$ inſtead of x in the original equation $x^m = N$, as if it was a quadratick equation, omitting all the terms of it that involve any higher power of z than its ſquare, Mr. Raphſon reſolves the ſame equation as if it was a mere ſimple equation, or omits all the terms of it that involve any higher power of z than its ſimple power, or z itſelf; which makes his expreſſion of the near value of $a + z$, or of the ſecond near value of x, or $\sqrt[m]{}N$, derived from the ſaid transformed equation, a good deal ſimpler and eaſier to manage than thoſe of Mr. de Lagny. This Scholium extends from page 525 to page 529, and is followed by a fourth example of the extraction of the root of a very long number by Mr. de Lagny's method, which extends to page 534. The Solution of the ſecond of the ſaid Problems begins in page 536, and extends to page 546, and is followed by two examples of the extraction of the roots of given numbers by means of the general expreſſions obtained in it, that extend from page 547 to page 554. And then the Tract concludes with ſome Obſervations, in pages 555 and 556, on the ſeveral different methods that may be taken for the extraction of the roots of numbers.

This Tract, as well as the laſt before it, concerning the Extraction of the Cube-roots of given Numbers, will, I hope, be found to be of great uſe to the Students of Arithmetick and Algebra.

The laſt Tract in this Collection is intitled, *Obſervations on Mr. Raphſon's Method of reſolving Affected Equations of all degrees by Approximation.* It begins in page 559, and ends in page 590; and its contents may be deſcribed as follows. The firſt part of it, as far as page 571, is intended, partly, to remove ſome difficulties that occur in reading Mr. Raphſon's excellent Treatiſe on the Reſolution of all Equations, (whether pure or affected,) by Approximation, intitled, *Analyſis Æquationum Univerſalis*, which difficulties are not inherent in the ſubject itſelf, or neceſſarily belonging to his method of reſolving equations, but have ariſen merely from his having unfortunately adopted the doctrine and language of *negative roots* of equations, by which the Science of Algebra, or Univerſal Arithmetick, has been diſgraced and rendered obſcure and difficult, and diſguſting to men of a juſt taſte for accurate reaſoning, ever ſince its introduction by Harriot and Des Cartes. The firſt part of this Tract is, I ſay, intended, partly, to remove ſome difficulties of this kind, in the ſaid Treatiſe of Mr. Raphſon, and, partly, to illuſtrate his method of reſolving high equations in other caſes, or where no negative roots are mentioned, by performing the reſolution of one of the equations given by him in his examples, to wit, of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, in a very full and diſtinct manner, with every ſtep of the reſolution, and the reaſonings upon which it is grounded, ſet forth at length, agreeably to the principles laid down by him in the beginning of the ſaid Treatiſe, inſtead of reſorting (as he has done in his reſolution of the ſame example, and in thoſe of all his other examples,) to the repeated application of a general theorem, or canon, that he has deduced from the ſaid principles: becauſe that way of performing the ſaid reſolution, by means of a theorem, or canon, affords much leſs ſatisfaction to the mind of the reader, or operator, in the uſe of it, than he would receive by performing the reſolution of the equation by the immediate application of the principles themſelves, as I have done, in the reſolution here given of the ſaid equation. And the following part of this Tract contains a compariſon between Mr. Raphſon's method of Reſolving

Equations by Approximation, and Sir Iſaac Newton's method of Reſolving them alſo by Approximation, (which, after the firſt proceſs of the approximation, or the diſcovery of the ſecond near value of the root of the equation, differs a little from Mr. Raphſon's method,) in order to diſcover which of the two methods deſerves to be reckoned the moſt convenient. This compariſon between theſe two methods of reſolving equations by approximation, (the reſult of which is, that Mr. Raphſon's method appears to me, upon the whole, more convenient than Sir Iſaac Newton's,) reaches from page 571 to page 586: and the few remaining pages of this Tract, from page 586 to page 590, relate partly to the method of trying the exactneſs of the near value of *x*, or the root of the propoſed equation, which has been obtained by either of the ſaid two methods of Approximation, and, partly, to the method of finding *a*, or the firſt near value of *x*, or of the root of the propoſed equation, to a moderate degree of exactneſs, in certain difficult caſes, to wit, in thoſe caſes in which the propoſed equation either has, or (from the changes of the ſigns of its terms from + to —, and from — to +,) ſeems to have, more than one real and affirmative root.

In the next place I have re-publiſhed a uſeful Table of Numbers, from a book intitled *The Calculator*, publiſhed in octavo in the year 1747, by the late learned Mr. James Dodſon, being a Table of the Square-roots and Cube-roots of all the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, to 180, carried to ſeven places of figures; which may often be the means of ſaving a Student of theſe Sciences ſome time and pains in performing the calculations that may occur in them. This Table is contained in pages 591 and 592.

And in the laſt place I have re-publiſhed a Table of the Square-roots of all the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, as far as 1000, and likewiſe of the Reciprocals of all the ſaid numbers, or of the values of the fractions $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, &c, as far as $\frac{1}{1000}$, expreſſed in decimal fractions, from the fourth volume of Dr. Charles Hutton's *Miſcellanea Mathematica*, publiſhed in the year 1775, in four little volumes, duodecimo. This Table begins in page 595, and ends in page 604, and, with Dr. Hutton's explanatory account of it in pages 605 and 606, concludes the preſent volume.

ARTIS CONJECTANDI

PARS SECUNDA,

CONTINENS

DOCTRINAM DE PERMUTATIONIBUS ET COMBINATIONIBUS.

PROOEMIUM.

INFINITAM varietatem, quæ cùm in naturæ operibus, tùm in actionibus mortalium elucet, quáeque præcipuam hujus univerſi pulchritudinem conſtituit, non aliunde quàm ex diverſimodâ compoſitione, mixturâ & tranſpoſitione partium ejus inter ſe originem ducere palàm eſt. Sed, quia multitudo rerum ad effectum aliquem producendum concurrentium ſæpenumerò tanta eſt támque varia, ut difficillimum ſit recenſere vias omnes, quibus earundem compoſitio, vel mixtura, fieri, vel non fieri, poteſt, hinc fit ut nullum ſit vitium, in quod homines etiàm maximè prudentes & circumſpecti frequentiùs incidant illo, quod Logici communitèr appellant *inſufficientem enumerationem partium*; adeò quidem ut non verear dicere, hanc unicam ferè ſcaturiginem eſſe infinitorum, eorúmque graviſſimorum, errorum, quos in ratiociniis noſtris circà res tùm cognoſcendas tùm agendas quotidiè committimus. Quarè merito ſuo utiliſſima cenſenda eſt ars, *combinatoria* dicta, quæ huic mentis noſtræ defectui medetur, docétque ſic enumerare modos omnes poſſibiles, ſecundùm quos res plures permiſceri, tranſponi, vel conjungi, invicèm poſſunt, ut certi ſimus, nos nullum eorum prætermiſiſſe, qui inſtituto noſtro conducere valent. Quanquam enim hoc negotii eatenùs ſit conſiderationis Mathematicæ, quatenùs in ſubducendo calculo terminatur; ſi tamen uſum & neceſſitatem ſpectes, univerſale prorſùs eſt & ita comparatum, ut ſine illo nec ſapientia Philoſophi, nec Hiſtorici exactitudo, nec Medici dexteritas, aut Politici prudentia, conſiſtere queat. Argumento ſit hoc unicum, quòd omnis horum labor in *conjectando*, & omnis conjectura in trutinandis cauſarum complexionibus aut combinationibus verſatur. Unde quoque nonnulli eximii viri, ac nominatìm Schootenius, Leibnitius, Walliſius, Preſletus, materiam hanc ſibi tractandam ſumpſêre, ne quis exiſtimet nova eſſe hìc omnia quæ prolaturi ſumus; tametſi quædam non contemnenda de noſtro adjecimus, imprimis demonſtrationem generalem & facilem proprietatis numerorum figuratorum, cui cætera pleraque innituntur, & quam nemo, quod ſciam, ante nos dedit eruítve. Cùm itaque nondum plenum Artis ſyſtema habeamus, tùm verò, ne illa quæ habemus aliunde petere ſit opus, viſum eſt totam Doctrinam ab ovo ordiri, ac, ne quid indemonſtratum relinquatur, ex primis fundamentis eruere; quod tamèn breviter fiet & ſuccinctè, nec niſi in quantum inſtituti noſtri ratio exigere videtur. Totam Tractationem ad duo ſumma capita referimus, quorum unum Permutationum, alterum Combinationum doctrinam perſequitur; cui accedit tertium, quod utraſque mixtìm contemplatur.

CAPUT I.

DE PERMUTATIONIBUS.

PERMUTATIONES rerum voco variationes, juxtà quas, ſervatâ eâdem rerum multitudine, ordo ſitúſque inter ipſas diverſimodè permutatur.

Itaque ſi quæratur, quoties nonnullæ res tranſponi vel permiſceri invicèm poſſint, ſic ut ſemper accipiantur omnes ſolo ordine ſitûve mutato, dicentur quæri omnes permutationes rerum illarum.

Res autem permutandæ vel omnes poſſunt eſſe diverſæ, vel aliquot earum eædem; quæ quidem per totidem Alphabeti literas, ſive diverſas ſive eaſdem, commodè deſignabuntur.

1. *Si res omnes permutandæ ſunt diverſæ:*

CUM numerus permutationum in rebus pluribus iniri nequeat, niſi idem priùs in omnibus aliis numero paucioribus compertus habeatur, liquet in hâc inquiſitione utendum viâ ſyntheticâ, hoc eſt, ordiendum nobis eſſe ab hypotheſibus omnium primis & ſimpliciſſimis:

Unius rei, vel literæ, *a*, una tantùm ſumptio vel poſitio eſt.

Duarum rerum, aut literarum, *a* & *b*, vel *a* præcedit & *b* ſequitur, vel præcedente *b* ſequitur *a*; unde duo ipſarum fiunt ordines *a b* & *b a*.

Tres, porrò, literæ *a*, *b*, *c*, ita collocari poſſunt, ut primus locus vel ipſi *a* vel *b* vel *c* concedatur: ſi *a* primum tenet locum, reliquæ duæ duobus, ut diximus, modis diſponi queunt: ſi *b* in primum locum transferatur, reliquarum duarum duplex itidem poterit eſſe poſitio; quod & intelligendum, ubi tertia *c* primam ſedem occupaverit. Unde trium literarum in univerſum ter duæ, ſeu 6, exiſtunt permutationes *abc*, *acb*: *bac*, *bca*: *cab*, *cba*.

Similitèr, ſi 4 extent literæ *a*, *b*, *c*, *d*, earum unaquæque primum obtinere locum poteſt, intereà dum tres reliquæ, ut nunc oſtenſum, ter bis, ſeu ſexiès, ordinem variabunt: quare cùm earum, quæ primo loco poni poſſunt, ſint quatuor, ſequitur omnes quatuor quater ter bis, ſeu quater ſexies, hoc eſt, vicies quater ſitum inter ſe permutare poſſe.

Ob eandem rationem accedente 5tâ literâ *e* inſtitui poſſunt quinquies tot variationes, quot in caſu præcedenti, hoc eſt, quinquies 24, ſeu 120. Et generalitèr, datis quotcunque literis, numerus permutationum, quas ſubire poſſunt omnes, totiès excedit numerum permutationum, quas recipiunt literæ unâ pauciores, quot ſunt unitates in dato literarum numero. Unde ſponte manat ſequens

Regula pro inveniendis omnibus permutationibus rerum quotcunque datarum.

OMNES numeri ab unitate ſe conſequentes naturali ordine, ad datum uſque rerum numerum incluſivè, ducantur in ſe invicèm; productum manifeſtabit quæſitum.

Putà, ſi datus rerum numerus ſit n, numerus permutationum erit $1.2.3.4.5.$ &c. uſque ad n; vel etiam (quia unitas non multiplicat) $2.3.4.5. \ldots\ldots n$. Nota, punctula numeris interjecta hìc et ubique in ſimili materiâ continuum numerorum in ſe ductum ſignificant. Exempli gratiâ, ſeptem rerum permutationes ſunt $2.3.4.5.6.7. = 5040$. Ratio patet ex dictis, operatio ex adjunctâ Tabellâ;

Numerus Rerum.	*Numerus Permutationum.*
1	1
	2
2	2
	3
3	6
	4
4	24
	5
5	120
	6
6	720
	7
7	5040
	8
8	40320
	9
9	362,880
	10
10	3,628,800
	3 6 28 800
11	39,916,800
	7 9 8 3 3 600
12	479,001,600

2. *Si rerum permutandarum nonnullæ ſunt eædem:*

Quòd ſi literæ una pluréſve recurrant ſæpiùs, hoc eſt, ſi in dato rerum numero aliquæ res ſimiles ſint ſive eædem; ut, ſi datæ ſint *a a a b c d*, ubi litera *a* ter repetitur; numerus permutationum multo minor evadit: ad quem inveniendum cogitandum eſt, quòd, ſi omnes eſſent diverſæ, putâ, ſi loco *a a a* ſcriberetur *a α* a, poſſent hæ tres literæ etiam nullâ cæterarum loco motâ inter ſe ſexiès tranſponi, per præcedentem Regulam; unde totidem diverſæ naſcerentur permutationes; at nunc cùm ſunt eædem, ſex iſtæ permutationes literarum *a α* a nullam univerſarum diſpoſitioni variationem indicunt, ac proinde pro unâ eâdemque habendæ ſunt: quod cùm de quâcunque diſpoſitione literarum paritèr ſit intelligendum, indicium præbet, numerum permutationum rerum datarum ſexiès, hoc eſt, totiès minorem eſſe numero permutationum, quas ſubire poſſent ſi omnes eſſent diverſæ, quotiès inter ſe permutâri queunt res ſimiles: ſed ſi omnes 6 literæ diverſæ exiſterent, permutari poſſent, juxtà præcedentem tabellam, 720. vicibus. Ergò nunc ubi tres ipſarum conveniunt, permutari duntaxat poterunt vicibus 120.

Iterùm ſi datæ ſint 6 literæ *a a a b b c*, ubi præter literam *a* quæ ter recurrit, etiam litera *b* bis repetitur; manifeſtum eſt, numerum permutationum adhuc bis minorem evadere, quàm in præcedenti caſu fuerat, adeóque ſolùm ad 60 ſe extendere: quandoquidem binæ quælibet permutationes, quæ ex ſolâ tranſpoſitione duplici literarum *b b*, ſi diverſæ eſſent, naſcerentur, nunc coïncidunt. Eodem pacto colligendum, ſi plures literæ repeterentur ſæpiùs, pro ſingulis earum numerum permutationum minui toties, quoties ſeorsìm inter ſe permutari poſſunt eædem literæ. Unde ratio habetur ſequentis Regulæ.

Regula pro inveniendis rerum permutationibus, cùm earum nonnullæ ſunt eædem.

NUMERUS permutationum, quas admitterent datæ res ſi omnes differentes eſſent, dividatur per numerum permutationum, quas ſubire poteſt res ſimilis ſecundùm multitudinem ſuam, ſi una ſit quæ ſæpiùs repetatur: aut per productum ex numeris permutationum, quas ſeorsìm recipere poſſunt ſingulæ res ſimiles ſecundùm multitudinem ſuam, ſi plures ſint quæ ſæpiùs recurrant; & quotiens exhibebit quæſitum.

Uſus doctrinæ Permutationum inſignis eſt in definiendo numero Anagrammatûm alicujus vocis. Exempli gratiâ; Tranſpoſitiones omnes poſſibiles literarum in voce *Roma* ſunt 1.2.3.4.=24, ob 4 differentes literas, per 1 Regulam: et in voce *Leopoldus* $\frac{362880}{2.2=4} = 90720$: et in voce *Studioſus* $\frac{362880}{2.6=12} = 30240$: ob 9 utrobique literas, intérque illas ibi geminum *l* et geminum *o*, hìc geminum *u* & triplex *ſ*; per 2 Regulam.

Hùc pertinent verſus nonnulli ob variationum multitudinem *Protei* dicti, quos inter celebrantur Lanſii, Scaligeri, Bauhuſii. Thomæ Lanſio hoc diſtichon debemus:

Lex, Rex, Grex, Res, Spes, Jus, Thus, Sal, Sol, (bona) *Lux, Laus:*
Mars, Mens, Sors, Lis, Vis, Styx, Pus, Nox, Fax, (mala) *Crux, Fraus.*

cujus ſinguli verſus per Regulam primam, ob 11 monoſyllaba (diſſyllabis vocibus *bona* & *mala* 5tæ ſemper regioni affixis)

fixis) ſalvâ metri lege variari poſſunt 39,916,800 vicibus. Et quanquam aliàs contingat, ut pleræque variationes in metri leges arietent, nec non ut plerique Anagrammatiſmi ſint non-ſignificantes & barbari; levi tamen plerumque induſtriâ opus eſt ad ſecernendum utiles ab inutilibus, illorúmque numerum ſeorſim ineundum, ſi aliquem in iis inquirendis ordinem obſerves. Quemadmodùm cernere eſt in hexametro à Bernhardo Bauhuſio, Jeſuitâ Lovanienſi, in laudem Virginis Deiparæ conſtructo:

Tot tibi ſunt dotes, Virgo, quot ſidera cœlo;

quem dignum peculiari operâ duxerunt plures viri celebres. Erycius Puteanus in libello, quem Thaumata Pietatis inſcripſit, variationes ejus utiles integris 48 paginis enumerat, eâſque numero ſtellarum, quarum vulgò 1022 recenſentur, accommodat, omiſſis ſcrupuloſiùs illis, quæ dicere videntur, tot ſidera cœlo eſſe, quot Mariæ dotes; nam Mariæ dotes eſſe multo plures. Eundem numerum 1022 ex Puteano repetit Gerhardus Voſſius cap. 7. de Scientiis Mathematicis; Preſtetus, Gallus, in primâ editione Elementorum Mathematicorum, paginâ 348, Proteo huic 2196 variationes attribuit; ſed, factâ reviſione in alterâ editione, tomo primo, paginâ 133, numerum earum dimidio ferè auctum ad 3276 extendit. Induſtrii Auctorum Lipſienſium Collectores menſe Junii 1686, in recenſione Tractatûs Walliſiani de Algebrâ, numerum in quæſtione (quem Auctor ipſe definire non fuit auſus) ad 2580 determinant. Et ipſe poſtmodùm Walliſius in editione Latinâ operis ſui, Oxoniæ anno 1693 impreſſâ, paginâ 494, eundem ad 3096 profert. Sed omnes adhuc à vero ſunt deficientes, adeò ut deluſam tot virorum, poſt adhibitas quoque ſecundas curas, in re levi perſpicaciam meritò mireris. Facto enim examine deprehendo, fœtum hunc Bauhuſianum, excluſis etiàm ſpondaïcis, admiſſis verò iis qui cæſurâ deſtituti ſunt, ſalvâ metri lege omninò tèr milliès, tercentiès, ac duodeciès variabilem eſſe. At prolixiùs de his agere tanti non intereſt, nec inſtitutum noſtrum patitur.

TYPUS VARIATIONUM VERSÛS BAUHUSIANI [1].

Tot tibi ſunt dotes, Virgo, quot ſidera cælo.

Quintam Regionem Hexametri occupat vel

- *Sidera*, quam vocem excipit aut vox
  - *Diſſyllaba una*, nempe vel
    - *Cælo*, ac tum vox *Tibi* inter ſex reliquas occupat locum vel
      - *Secundum*, præcedente voce nunc
        - *Monoſyllabâ*, eâque vel
          - *Tot*, cui caſui reſpondent - Variationes 24
          - *Sunt*, - - - 24
          - *Quot*, - - 24
        - *Diſſyllabâ Virgo*, - - 24
      - *Tertium*, præeuntibus
        - *Unâ monoſyllabâ & unâ diſſyllabâ*, primas tenente vel
          - *Monoſyllabâ*, *Tot*, quam excipit alterutra
            - *Dotes*: 6 } *Virgo*: 6 } 12
            - *Sunt*, - 12
            - *Quot*, - - 12
          - *Diſſyllabâ*, *Dotes*, quam ſequitur
            - *Tot*: 6 } *Sunt*: 6 } *Quot*: 6 } - - 18
            - *Virgo*, - - - 18
        - *Duabus diſſyllabis*, nempe, *Dotes Virgo*, 6
      - *Quartum*, præcedentibus
        - *Tribus monoſyllabis*, - - 12

Duabus monoſyllabis cum diſſyllabâ Virgo, 12
Una monoſyllabâ & duabus diſſyllabis, 36
Quintum, præmiſſis
Tribus monoſyllabis cum unâ diſſyllabâ, 48
Duabus monoſyllabis cum totidem diſſyllabis,
quarum poſterior *Virgo,* - 18
Sextum, - - - 120
a b c d
420..420

Dotes, unde totidem variationes, quot in *Cælo,*
nempe - - - 420
Virgo, unde rursùs totidem, quot in *Cælo,* exceptis ſolùm illis 60 variationibus, ubi poſtrema ſyllaba in *Virgo* correpta eſt; quibus proin demptis ex 420, remanent - - 360
Monoſyllabæ duæ, eæeque
Quot ſunt, vel *Sunt quot;* voce *Tibi* occupante locum vel
Secundum, primo relicto voci
Monoſyllabæ, Tot: - - 12
Diſſyllabæ, Virgo: - - 12
Tertium, præcedentibus
Monoſyllabâ cum diſſyllabâ, - 24
Duabus diſſyllabis, quarum poſt *Virgo,* 8
Quartum, præeuntibus
Monoſyllabâ cum duabus diſſyllabis, 36
Tribus diſſyllabis, quarum ultima *Virgo,* 4
Quintum, - - - 48
144..144

Tot ſunt, vel, *Sunt tot,* totidem - 144
Tot quot, aut, *Quot tot,* totidem - - 144
1632

Tibi, quam vocem ſequitur vox
Diſſyllaba una, eâque vel
Cælo, voce *Sidera* occupante locum aut

Primum,	120	
Secundum,	48	
Tertium, præmiſſis vel		
Duabus monoſyllabis,	36	
Duabus diſſyllabis,	12	
Quartum, præeuntibus duabus monoſyllabis & unâ diſſyllabâ,	72	
Quintum, præcedentibus duabus monoſyllabis, totidémque diſſyllabis,	72	
	360..	360
Dotes, totidem quot in *Cælo*		360
Virgo, totidem		360
Monoſyllabæ duæ, eæque		
Quot ſunt, vel *Sunt quot:* voce *Sidera* tenente locum		
Primum:	48	
Secundum, poſt diſſyllabam vocem,	36	
Tertium, poſt duas diſſyllabas,	24	
Quartum, poſt tres diſſyllabas,	12	
	120	.120
Tot ſunt, vel, *Sunt tot,* totidem		120
Tot quot, vel, *Quot tot,* totidem		120
Monoſyllabâ unâ (quo caſu ante *Tibi* ſemper habetur *Virgô*), nempe vel		
Sunt, voce *Sidera* locum poſſidente aut		
Primum:	24	
Secundum:	12	
Tertium, præcedentibus *duabus monoſyllabis,*	4	
duabus diſſyllabis,	4	
Quartum,	12	
Quintum,	24	
	80...	80
Tot, totidem quot in *Sunt,*		80
Quot, totidem		80

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CAP.

CAP. II.

DE COMBINATIONIBUS, IÍSQUE PRIMÒ CONSIDERATIS SIMPLICITÈR.

COMBINATIONES rerum ſunt conjunctiones, juxtà quas ex datâ rerum multitudine nonnullæ eximuntur, intérque ſe conjunguntur nullo ordinis ſitûs-ve ipſarum reſpectu habito.

Idcircò cùm quæritur, quoties ex dato rerum numero vel binæ, vel ternæ, vel quaternæ, &c. accipi poſſint, ſic ut nunquam omnes eædem res ſumantur ſæpiùs quàm ſemèl, dicentur quæri omnes combinationes diverſæ rerum datarum.

Numerus, ſecundùm quem res datæ conjunguntur, dicitur *Exponens* Combinationis: Ita, ſi res binæ ſumuntur, Exponens erit 2; ſi ternæ, 3; ſi quaternæ, 4. Res verò ſecundùm hos exponentes junctæ dicuntur *Binarii*, *Ternarii*, *Quaternarii*, &c. vel *Biniones*, *Terniones*, *Quaterniones*, &c. & conſonantèr etiàm *Uniones*, vel *Unitates*, quando res ſumuntur ſingulæ, & *Nulliones* cùm nulla planè ſumitur.

Conjunctiones ipſas nonnulli vocant *Combinationes*, *Conternationes*, *Conquaternationes*, &c. quas omnes vulgò unâ voce *Combinationum* complecti ſolent, tametſi hæc vox ſtrictiori ſignificatu propriè non niſi illas conjunctiones indigitare videatur, quibus res binæ invicèm junguntur. Quamobrem alii generaliori voce *Complicationum* vel *Complexionum* uti malunt: alii magìs appoſitè *Electiones* vocant, ut & illæ ſubintelligi poſſint rerum acceptiones, quibus res ſingulæ ſeorsìm ſumuntur, aut quibus etiàm nulla planè ſumitur.

Res autèm quæ inter ſe combinandæ ſunt, vel omnes poſſunt eſſe diverſæ, vel aliquot ipſarum eædem; eæque vel ita combinari debent, ut in nullâ combinatione res eadem ſæpiùs contineatur, quàm ipſa reperitur in toto rerum numero: vel ſic, ut in eâdem combinatione res eadem etiàm ſæpiùs recurrere, hoc eſt, ut ſecum ipſâ quoque combinari poſſit. Iterúmque quæri poteſt numerus combinationum vel ſecundùm omnes exponentes conjunctìm, vel ſecundùm ſingulos ſeorsìm. Atque inſupèr circà unumquemque horum combinandi modorum plures formari poſſunt quæſtiones & problemata, è quibus illa tantùm delibabimus, quæ in ſequentibus alicui uſui fore judicamus.

1. *Si res omnes combinandæ ſunt diverſæ, ìnque nullâ combinatione eadem res bìs occurrere debet, invenire omnes Combinationes ſimplicitèr ſive ſecundùm omnes exponentes conjunctìm.*

SUNTO combinandæ modis omnibus literæ *a*, *b*, *c*, *d*, *e*, &c. Fiant tot ſeries quot literæ, hoc modo: In primâ ſerie ponatur ſola litera *a*.

In ſecundâ ponatur *b*, nunc ſeorsìm, nunc junctìm cum *a*, ut habeatur *ab* vel *ba*. Eadem enim conjunctio eſt, quæ *b* cum *a*, & *a* cum *b* jungit, cùm nullus ordinis, ſitûs-ve ipſarum inter ſe, reſpectus haberi ſupponatur.

In tertiâ collocetur *c*, eâque primò ſola, dein juncta, partìm cum *a* & *b*, ut fiant biniones *ac*, *bc*; partìm cum ipſo binione *ab*, ut fiat ternio *abc*.

a.

b. ab.

c. ac. bc. abc.

d. ad. bd. cd. abd. acd. bcd. abcd.

e. ae. be. ce. de. abe. ace. bce. ade. bde. cde. abce. abde. acde. bcde. abcde.

In quartâ ponatur *d*, primò ſola, deinde juncta cum ſingulis præcedentium literarum *a*, *b*, *c*, ſingulíſque earum tum binariis *ab*, *ac*, *bc*, tum ternario *abc*; ut fiant novi biniones *ad*, *bd*, *cd*, terniones *abd*, *acd*, *bcd*, & quaternio *abcd*.

Similitèr quintæ ſeriei agmen ducat litera *e*, quam primò ingrediatur ſola, dein juncta cum omnibus præcedentium ſerierum electionibus. Eâdémque methodo procedendum eſſet, ſi plures eſſent datæ literæ. Quâ ratione ſatis manifeſtum eſt, datas literas in iſtis ſeriebus omnifariàm inter ſe junctas eſſe, nullámque earum fieri poſſe electionem, quæ non in unâ harum ſerierum reperiatur, ſed & nullam eſſe quæ alicubi bìs occurrat; adeóque omnes unà ſeries ſuppeditaturas omnes electiones poſſibiles, quæ circà datas literas inſtitui queunt.

Harum igitur numerus initur facilè, ſi conſideretur quòd in quâlibet ſemper ſerie una ampliùs inveniri debeat electio, quàm in antecedentibus omnibus ſeriebus ſimul: quoniam litera, quæ illius ſeriei caput eſt, ibidem ſemel ponitur ſola, & præteréà unà aſſumit ſecum omnes electiones præcedentium ſerierum. Hinc enim ſequitur, quia in primâ ſerie eſt electio unica, fore in ſecundâ electiones duas, in tertiâ 4, in quartâ 8, & ſic deinceps in progreſſione geometricâ duplâ: quandoquidem progreſſionis duplæ ab unitate hanc quoque naturam eſſe conſtat *; ut ſumma terminorum quotlibet unitate aucta ſequentem terminum exhibeat. Quócircà ſumma electionum in ſeriebus omnibus æqualis eſt ſummæ terminorum totidem progreſſionis duplæ ab unitate, hoc eſt, per modò memoratam proprietatem, ipſi termino ſubſequenti

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* Hoc autèm ita demonſtrari poteſt.

PROPOSITIO.

Sit ſeries terminorum in geometricâ ratione unitatis ad numerum binarium continuò creſcentium, ſcilicet, 1, 2, 4, 16, 32, 64, 128, 256, &c, uſque ad n terminos. Horum terminorum ſumma vocetur S. Manifeſtum eſt ultimum, ſive maximum, hujus ſeriei terminum fore æqualem $\overline{2|}^{\,n-1}$. Augeatur jam hæc ſeries uno adjecto termino, ſcilicet, $2 \times \overline{2|}^{\,n-1}$, ſeu $\overline{2|}^{\,n}$. Dico, quòd novus terminus $\overline{2|}^{\,n}$ erit æqualis $S + 1$, ſive ſummæ S omnium priorum terminorum unà cum unitate.

DEMONSTRATIO.

Duplicando terminos ſeriei S, ſive $1 + 2 + 4 + 8 + 16 + 32 + 64 + \&+ 2^{n-1}$, orietur ſeries $2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c. + \overline{2|}^{\,n}$, cujus termini omnes, excepto ultimo $\overline{2|}^{\,n}$, ſunt reſpectivè æquales terminis omnibus prioris ſeriei, excepto primo 1; hoc eſt, $2S$ erit $= S - 1 + \overline{2|}^{\,n}$. Ergò $2S + 1$ erit $= S + \overline{2|}^{\,n}$, et $\overline{2|}^{\,n}$ erit $= 2S + 1 - S$, ſeu $S + 1$.

Q. E. D.

ejuſdem progreſſionis unitate multato; qui quidem terminus ſubſequens idem eſt cum producto binarii toties, ſive tot vicibus, poſiti & in ſe ducti, quot ipſum in progreſſione termini præcedunt, hoc eſt, quot ſunt ſeries, quarum electiones quæruntur. Unde talis exurgit.

Regula pro inveniendis omnibus electionibus rerum datarum ſecundùm omnes exponentes :

A Producto binarii toties, ſive tot vicibus, poſiti & multiplicati in ſe, quot ſunt datæ res, auferatur unitas : reliquum indicabit quæſitum.

Hoc eſt, poſito rerum datarum numero n, numerus omnium electionum ſimplicitèr, putà, omnium unionum, binionum, ternionum, &c. erit $2^n - 1$. Hinc ſi nullionem ſeu electionem, quâ ex rebus datis nulla ſumitur, quæque in quâvis rerum multitudine una ſemper eſt & unica, ſimùl comprehendas, fiet numerus ille 2^n: ſin cum nullione ipſos quoque uniones reſeces, quorum numerus ipſi rerum numero perpetuò æquatur, erit numerus binionum, ternionum, cæterarúmque complexionum $2^n - n - 1$. Exempli gratiâ. Septèm planetarum conjunctiones, vel complicationes, omnes diverſæ ſunt $2^7 - 1 = 2.2.2.2.2.2.2 - 1 = 128 - 1 = 127$; unde ſi demas electiones 7, quibus ſinguli planetæ ſeorsìm accipiuntur, quáeque propriè non conjunctiones, ſed disjunctiones planetarum ſunt, relinquetur numerus omnium conjunctionum ſtrictè dictarum, quibus planetæ vel bini, vel terni, vel quaterni, vel quini, vel ſeni, vel denique ſepteni junguntur, $2^7 - 7 - 1 = 120$. Sic etiàm duodecìm, uti vocant, Regiſtra, ſeu fiſtularum ordines, in organo pneumatico, quibus ſonus, mox ſibilans, mox tremebundus, efficitur, aut alitèr modificatur, variari poſſunt $2^{12} - 1 = 4095$ vicibus.

Nota: Si quis examinet ſeries combinationum ſuprà in typo expoſitas, obſervabit in qualibet ſerie (ſolâ primâ exceptâ,

ceptâ, quæ unicum unionem *a* complectitur) numerum electionum ſecundùm exponentes pares æquari numero electionum ſecundùm impares: ſaltèm, cùm id in aliquot ab initio ſeriebus verum deprehenderit, idem quoque in ſerie proximè ſequente locum habere concludet. Nam litera, quæ illius ſeriei caput eſt, juncta præcedentium ſerierum electionibus iis, quæ impares exponentes habent, parium; & iis viciſsìm quæ pares habent juncta, imparium; exponentium complexiones efficit: adſciſcens verò primæ ſeriei unionem *a*, paris; & ipſa per ſe ſola accepta, imparis; exponentis electionem conſtituit: unde & in hâc ſerie numerum harum numero illarum æquari conſtat. In omnibus igitur ſeriebus ſimùl ſumtis numerus electionum ſecundùm impares exponentes numerum electionum ſecundùm pares unitate ſuperabit; aut, ſi his inſupèr nullionem accenſeas, æquabit. Quocircà, cùm numerus omnium electionum ſimplicitèr, incluſo nullione, oſtenſus ſit 2^n, erit ejus ſemiſſis, ſive poteſtas binarii proximè minor, 2^{n-1}, numerus electionum ſecundùm ſolos impares; &, dempto rursùm nullione, $2^{n-1}-1$ numerus electionum ſecùndùm ſolos pares exponentes. Idem quoque demonſtrabitur infrà in coroll. 6. cap. 4.

CAP.

CAP. III.

DE COMBINATIONIBUS SECUNDÙM SINGULOS EXPONENTES SEORSIM; UBI DE NUMERIS FIGURATIS, EORUMQUE PROPRIETATIBUS AGITUR.

EX typo combinationum præcedentis capitis manifeſtum ſit, literam quæ cujuſlibet ſeriei caput eſt, adjunctam unionibus ſerierum præcedentium efficere ſuæ ſeriei biniones, adjunctam binionibus efficere terniones, ternionibus quaterniones, & ſic porrò : adeóque numerum binionum in quâvis ſerie æquari ſummæ unionum in omnibus ſeriebus antecedentibus, numerum ternionum ſummæ binionum, numerum quaternionum ſummæ ternionum, & generalitèr numerum combinationum ſecundùm datum quemcunque exponentem in ſerie quâcunque æquari ſummæ combinationum omnium præcedentium ſerierum ſecundùm exponentem unitate minorem dato. Sequitur hinc, quòd

Uniones, quia in ſingulis ſeriebus reperiuntur ſinguli, omnes inter ſe conſtituunt ſeriem 1.1.1.1.1. &c. ſeu ſeriem unitatum.

Biniones in primâ ſerie nulli ſunt, in ſecundâ 1, in tertiâ $1+1=2$, in 4tâ $1+1+1=3$, in 5tâ $1+1+1+1=4$, &c. proinde omnes biniones inter ſe conſtituunt ſeriem 0.1.2.3.4.5. &c, hoc eſt, ſeriem *numerorum* arithmeticè progreſſionalium, ſive *Lateralium*.

Terniones in primâ & ſecundâ ſerie nulli ſunt, in 3tiâ 1, in 4tâ $1+2=3$, in 5tâ $1+2+3=6$, in 6tâ $1+2+3+4=10$. &c. Omnes itaque ordine accepti ſeriem conficiunt 0.0.1.3.6.10.15. &c. hoc eſt, ſeriem *numerorum*, ut vocant, *Trigonalium*, ſeu *Triangularium*.

Quaterniones in tribus primis ſeriebus nulli ſunt, in 4tâ 1, in 5tâ $1+3=4$, in 6tâ $1+3+6=10$, in 7mâ $1+3+6+10=20$. &c. qui omnes ordine aſſumti ſeriem efficiunt 0.0 0.1.4.10.20. &c. ſeriem, videlicet, *Pyramidalium*.

Pari ratione Quiniones omnes ſeriem conſtituunt *Trianguli-pyramidalium* 0. 0. 0. 0. 1. 5. 15. 35. &c. Seniones ſeriem *Pyramido-pyramidalium* 0. 0. 0. 0. 0. 1. 6. 21. &c. aliáeque combinationes ſecundùm altiores exponentes efficiunt alias atque alias ſeries numerorum figuratorum altioris generis in infinitum.*

Et ſic occaſione doctrinæ Combinationum in ſpeculationem inſperatam *numerorum figuratorum* incidimus; quâ appellatione vulgò inſigniuntur numeri, qui ex continuâ arithmeticè proportionalium, indéque ortorum numerorum, additione, vel collectione, generantur.

Ut verò hæ figuratorum numerorum ſeries ſub unum aſpectum caderent, eóque faciliùs comprehenderentur quæ de illis dicenda ſuperſunt, ſequentem appoſui tabellam, quam quis nullo negotio quouſque voluerit tum deorſùm tum dextrorſùm continuabit. Numeri barbari, ſeu Arabici, in ſiniſtro tabulæ margine adſcripti numerant columnas tranſverſas, & ſimùl rerum combinandarum multitudinem : numeri verò Romani in ſupremo margine conſpicui numerant columnas verticales & unà exponentes combinationum innuunt. Columnarum verticalium prima eſt ſeries monadum ſeu unitatum ; ſecunda ſeries numerorum naturalium, ſeu lateralium, ab unâ cyphrâ incipiens ; tertia ſeries trigonalium incipiens à cyphris duabus, quarta pyramidalium incipiens à tribus cyphris, quinta trianguli-pyramidalium incipiens à quatuor cyphris, & ſic deinceps.

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* De horum numerorum nominibus eſt inter auctores arithmeticos quædam variatio. Nam numeri 0.0.0.0.1.5.15.35, &c, qui hìc vocantur *Trianguli-pyramidales*, vocantur à quibuſdam ſcriptoribus, et, inter alios, à Nicolao Mercatore, in celeberrimâ ſuâ Logarithmotechniâ, *Trigono-trigonales*; et numeri 0.0.0.0.0.1.6.21.56, &c. qui hìc vocantur *Pyramido-pyramidales*, ab illo vocantur *Trigono-pyramidales*. Nomina quibus diverſi ordines numerorum figuratorum deſignantur apud *Mercatorem* ſunt quæ ſequuntur; ſcilicet, 1. unitates, 2. radices, 3. numeri trigonales, 4. pyramidales, 5. trigono-trigonales, 6. trigono-pyramidales, 7. pyramidi-pyramidales, 8. trigono-trigonò-pyramidales, 9. trigono-pyramidi-pyramidales, 10. pyramidi-pyramidi-pyramidales. Vide Scriptores Logarithmicos, tom. 1^{mum}, pag. 178.—— Ad evitandam hanc confuſionem nominum ſatius eſſe videtur diverſos horum numerorum ordines poſt quartum ordinem, ſive numerorum pyramidalium, diſtinguere ſolùm per numeros exponentes ordinum deſignandorum, appellando eos ſive numeros figuratos ordinis quinti, ſive ordinis ſexti, ſive ordinis ſeptimi, ſive octavi, ſive noni, ſive decimi, aut alius cujuſcunque ordinis.

TABULA COMBINATIONUM, SEU NUMERORUM FIGURATORUM.

EXPONENTES COMBINATIONUM.

Numeri Rerum Combinandarum.

	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
1.	1	0	0	0	0	0	0	0	0	0	0	0
2.	1	1	0	0	0	0	0	0	0	0	0	0
3.	1	2	1	0	0	0	0	0	0	0	0	0
4.	1	3	3	1	0	0	0	0	0	0	0	0
5.	1	4	6	4	1	0	0	0	0	0	0	0
6.	1	5	10	10	5	1	0	0	0	0	0	0
7.	1	6	15	20	15	6	1	0	0	0	0	0
8.	1	7	21	35	35	21	7	1	0	0	0	0
9.	1	8	28	56	70	56	28	8	1	0	0	0
10.	1	9	36	84	126	126	84	36	9	1	0	0
11.	1	10	45	120	210	252	210	120	45	10	1	0
12.	1	11	55	165	330	462	462	330	165	55	11	1

Habet hæc tabula proprietates planè eximias & admirandas; præterquàm enim quòd Combinationum myſterium in illâ latere jam oſtendimus, notum eſt interioris geometriæ peritis, præcipua etiàm totius reliquæ matheſeos arcana ibi deliteſcere. Nos proprietatum aliquas hìc delibabimus, & quidem delibabimus tantùm, nullius niſi primariæ illius, quæ propoſito noſtro inſervit, demonſtrationem accuratiorem allaturi, cùm cæteræ vel ex hâc oſtendi poſſint, vel ex ipſâ tabellæ conſtructione & numerorum figuratorum geneſi ſatis pateſcant.

Mirificæ Proprietates Tabulæ Combinationum:

1. Columnarum verticalium ſecunda incipit ab unâ cyphrâ, tertia à cyphris duabus, quarta à tribus: & generalitèr columna c à cyphris $c - 1$.

2. Columnarum verticalium termini primi ſignificativi à ſiniſtrâ dextrorſùm obliquè deſcendendo ordine ſumpti reddunt ipſos terminos primæ columnæ verticalis, ſecundi ſecundæ, tertii tertiæ, & ita deinceps: putà, primi conſtituunt ſeriem monadum, ſecundi lateralium, tertii trigonalium, &c.

3. Secundus ab unitate terminus columnæ verticalis cujuſlibet æquatur ipſius columnæ numero.

4. Terminus quivis tabellæ æquatur ſummæ omnium ſuperiorum præcedentis columnæ verticalis.

5. Quilibet terminus æquatur duobus aliis immediatè ſuprà ſe poſitis, quorum unus eſt in eâdem verticali columnâ, alter in præcedente.

6. Columnæ cujuſvis tranſverſæ termini ab unitate aliquouſque creſcunt, deinde per eoſdem gradus rursùm decreſcunt. Idem intellige de ſummis columnarum verticalium æque-altarum, ceu terminis ſequentis columnæ tranſverſæ per quartam proprietatem.

7. Columnarum verticalium æque-altarum baſes, ſive termini columnæ tranſverſæ cujuſlibet, primus quidem & ultimus ſignificativus perpetuò inter ſe æquantur, ut et ſecundus & penultimus, tertius & antepenultimus, atque ita porrò, ſi columna pluribus terminis ſignificativis conſtet.

8. Quin & ſumptis ab initio columnis verticalibus quotcunque cum totidem tranſverſis, collectiſquè in unam ſummam qui in eâdem verticali ſibi reſpondent terminis, erit ſumma prima æqualis penultimæ, ſecunda antepenultimæ, tertia proantepenultimæ, & ſic deinceps. Exhibent enim hæ ſummæ ipſos columnæ tranſverſæ ſequentis terminos, primo excepto. Confer proprietates 4 & 7. Exempli gratiâ:

Quinque primæ columnæ tum verticales tum tranſverſæ ſunt :

1. 0. 0. 0. 0.
1. 1. 0. 0. 0.
1. 2. 1. 0. 0.
1. 3. 3. 1. 0.
1. 4. 6. 4. 1.

---

5. 10. 10. 5. 1. Termini ſextæ columnæ tranſverſæ, primo excepto.

9. Columnæ tranſverſæ ordine exhibent coëfficientes omnium poteſtatum à radice aliquâ binomiâ genitarum; nempe ſecunda coëfficientes radicis 1. 1. tertia quadrati 1. 2. 1. quarta cubi 1. 3. 3. 1. quinta biquadrati 1. 4. 6. 4. 1. & ſic porrò.

10. Summæ ſerierum tranſverſarum progrediuntur in continuâ ratione duplâ : ſummarum verò ſummæ ab initio collectæ terminos conſtituunt progreſſionis duplæ unitate multatos; putà

$$\begin{array}{rcl} 1 & = & 1 \\ 1+1 & = & 2 \\ 1+2+1 & = & 4 \\ 1+3+3+1 & = & 8 \\ 1+4+6+4+1 & = & 16 \end{array}$$

$$\begin{array}{rcrcl} 1 & = & 1 & = & 2-1 \\ 1+2 & = & 3 & = & 4-1 \\ 1+2+4 & = & 7 & = & 8-1 \\ 1+2+4+8 & = & 15 & = & 16-1 \\ 1+2+4+8+16 & = & 31 & = & 32-1 \end{array}$$

fluit ex iis quæ in præcedente capite de Combinationibus ſimpliciter ſpectatis dicta ſunt.

11. Termini ſeriei verticalis cujuſlibet ordine diviſi per terminos collaterales ſeriei præcedentis (initio vel ab unitate vel à ſuis reſpectivè cyphris facto) exhibent quotos arithmeticè proportionales, quorum communis differentia eſt fractio, cujus numerator eſt unitas, & denominator ipſe numerus, ſive

ſecundus ab unitate terminus ſeriei dividentis. Exempli gratiâ :

Diviſ.)	*divid.*	(*quot.*
1)	1	(2 : 2
2)	3	(3 : 2
3)	6	(4 : 2
4)	10	(5 : 2
5)	15	(6 : 2

Diviſ.)	*divid.*	(*quot.*
1)	0	(0 : 2
2)	1	(1 : 2
3)	2	(2 : 2
4)	6	(3 : 2
5)	10	(4 : 2

Diviſ.)	*divid.*	(*quot.*
1)	1	(3 : 3
3)	2	(4 : 3
6)	10	(5 : 3
10)	20	(6 : 3
15)	35	(7 : 3

Diviſ.)	*divid.*	(*quot.*
1)	0	(0 : 3
3)	1	(1 : 3
6)	4	(2 : 3
10)	10	(3 : 3
15)	20	(4 : 3

Non difficultèr hæc proprietas, ſi opus foret, deduci poſſet ex ſequente.

12. Summa terminorum quotcunque ſeriei verticalis cujuſlibet à ſuis reſpectivè cyphris incipientis ad ſummam terminorum totidem ultimo æqualium eam habet rationem, quam habet unitas ad illius ſeriei numerum; hoc eſt, aggregatum numerorum quotcunque lateralium ab unâ cyphrâ ſeriem auſpicantium eſt ad aggregatum numerorum totidem maximo eorum, ſeu ultimo, æqualium, ut 1 ad 2; trigonalium à cyphris duabus, ut 1 ad 3; pyramidalium à tribus, ut 1 ad 4; &c. Idem quoque valet de ratione, quam habet ſumma terminorum ſeriei cujuſlibet ab unitate incipientis ad ſummam totidem maximum ſequenti termino æqualium. Exempli gratiâ:

				0	6		
0	3	1	5	0	6	1	15
1	3	2	5	1	6	3	15
2	3	3	5	3	6	6	15
3	3	4	5	6	6	10	15
6. 12 :: 1. 2		10. 20 :: 1. 2		10. 30 :: 1. 3		20. 60 :: 1. 3	

0	10			&c.
0	10	1	56	
0	10	4	56	
1	10	10	56	
4	10	20	56	
10	10	35	56	
15.60 :: 1.4		70.280 :: 1.4		

Cùm inter affectiones numerorum figuratorum hæc præcipua ſit, eadémque ſcopo noſtro primario inſerviat, viſum hìc eſt exponere methodum, quâ talem proprietatis ἀπόδειξιν exhibeo, quæ ſimùl & ſcientifica ſit, & propoſitum univerſalitèr concludat. Quem in finem ſequentia præſtruo lemmata:

LEMMA PRIMUM.

Summa terminorum quotlibet primæ ſeriei ad ſummam totidem terminorum ultimo æqualium rationem habet æqualitatis, ſive ut 1 ad 1.

DEMONSTRATIO.

Cùm enim ſeries meris conſtet unitatibus, erit ſumma terminorum quotlibet, ſumma tot unitatum, hoc eſt, tot terminorum ultimo æqualium, quot ſunt termini.

Q. E. D.

LEMMA SECUNDUM.

In quâlibet ſerie à ſuis reſpectivè cyphris incipiente, ſi quota eſt ipſa inter ſeries, tot ab initio ſumantur termini, erit ſumma terminorum omnium ad ſummam totidem ultimo æqualium, ut 1 ad ſeriei numerum.

DEMONSTRATIO.

Numerus enim cyphrarum quamcunque ſeriem auſpicantium unitate minor eſt ſeriei numero, per proprietatem primam. His igitùr ſi accedat ſequens terminus, numerus terminorum ſeriei numero æquabitur. Sed terminus, qui proximè cyphras ſequitur, eſt unitas, per proprietatem ſecundam. Unde terminorum aggregatum æquatur unitati, & aggregatum totidem ultimo æqualium æquatur ipſi ſeriei numero. Quarè conſtat Propoſitio.

LEMMA TERTIUM.

In quâcunque numerorum ſerie, ſi ſumma terminorum ab initio ſumptorum ad ſummam totidem ultimo æqualium perpetuò eandem habeat rationem, quotcunque accipiantur termini, putà ut 1 ad R, ita ut ſumma terminorum æquetur ſummæ totidem ultimo æqualium diviſæ per R; erit numerus terminorum aſſumptorum ablato R ad eundem numerum unitate mulctatum, ut ſumptorum penultimus ad ultimum.

DEMONSTRATIO.

Sumpti ſint ab initio termini quotlibet A.B.C.D. quorum numerus ſit N, penultimus C, & ultimus D. Eſt utique $A + B + C = A + B + C + D - D$, hoc eſt, (per hypotheſin) $\frac{C \text{ in } \overline{N - 1}}{R}$ eſt $= \frac{D \text{ in } N}{R} - D$, & proinde, æque-multiplicando, C in $\overline{N - 1}$ erit $= D$ in $N - D$ in $R = D$ in $\overline{N - R}$, adeóque $N - R : N - 1 :: C : D$. Q. E. D.

LEMMA QUARTUM.

In tabulà numerorum figuratorum ſi duæ ſint columnæ verticales contiguæ, in quarum priore quotlibet ab initio termini ad totidem ultimo eorum æquales habeant conſtantem rationem, ut 1 ad r; habeant verò in poſteriore termini aliquot ab initio ſumpti ad totidem ſumptorum ultimo æquales rationem ut 1 ad $r + 1$: habebit quoque, addito ſequenti termino, ſumma omnium terminorum unà cum adjecto ad tot terminos adjecto æquales, quot ſunt cum adjecto termini, rationem ut 1 ad $r + 1$.

DEMONSTRATIO.

Sumpti ſint in poſteriore columnâ termini E.F.G.H, quos proximè ſequatur I; atque ſumantur in columnâ immediatè præcedente termini totidem A.B.C.D; ſumptorum verò utrinque numerus ſit n. Erit $r\,H =$ (ex numerorum figuratorum geneſi per proprietatem quartam) r in $A + B + C =$ (per hypotheſin) $n - 1$ in $C =$ (per lemma tertium) $n - r$ in D; quare $n - r : H :: r : D ::$ (per hypotheſin) $n : A + B + C + D ::$ (ex numeratorum figuratorum geneſi per proprietatem quartam) $n.$ I. Unde $\overline{n - r}$ in $I = n\,H =$ (per hypotheſin) $\overline{r + 1}$ in $E + F + G + H$; adeoque $n - r : r + 1 :: E + F + G + H : I$, &, componendo, $n + 1 : r + 1 :: E + F + G + H + I : I$, hoc eſt, $E + F + G + H + I : n + 1$ in $I :: 1 : r + 1$ *. Q. E. D.

Cum

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* Hæc demonſtratio præ nimiâ brevitate mihi videtur eſſe obſcura. Poteſt verò explicari et, ut opinor, ſatìs perſpicua reddi, modo ſequente.

Sumpti ſint in poſteriore columnâ termini E.F.G.H.; quos proximè ſequatur I; atque ſumantur in columnâ immediatè præcedente termini totidem A.B.C.D: ſumptorum verò utrinque numerus ſit n. Et ſit ſumma quotlibet terminorum A.B.C.D. ad totidem ultimo eorum æquales in ratione 1 ad r; et ſit ſumma terminorum E.F.G.H. ad n terminos ipſi H, eorum ultimo, æquales, hoc eſt, ad quantitatem $n \times H$, in ratione 1 ad $r + 1$. Dico, quòd ſumma omnium terminorum E. F. G. H. I erit ad $n + 1$ terminos ipſi I æquales, hoc eſt, ad $\overline{n + 1} \times I$, ut 1 ad $r + 1$.

DEMONSTRATIO.

Ex numerorum figuratorum geneſi, per proprietatem quartam ſuprà memoratam, erit $r \times H$ æqualis $r \times \overline{A + B + C}$, ideóque (per hypotheſin) æqualis $\overline{n - 1} \times C$, atque idcircò (per lemma tertium) æqualis $\overline{n - r} \times D$. Erit igitùr $n - r$ ad H ut r ad D. Sed (per hypotheſin) $A + B + C + D$ eſt ad $n \times D$ ut 1 ad r; et proinde (permutando) $A + B + C + D$ erit ad 1 ut $n \times D$ ad r, et (invertendo) 1 erit ad $A + B + C + D$ ut r ad n D.

Eſt

Cum olìm horum Fratri * copiam feciſſem, animadvertit ille poſſe demonſtrationem elegantèr abbreviari, poſtremis tribus lemmatibus in unum conflatis, hoc modo:

LEMMA.

In tabulâ numerorum figuratorum ſi ſumma terminorum ab initio ſeriei verticalis cujuſvis ad ſummam totidem maximo æqualium ubique rationem habeat ut 1 ad r, habebit ſumma terminorum ſeriei proximè ſequentis ad ſummam totidem maximo æqualium rationem ut 1 ad $r + 1$.

DEMONSTRATIO.

Sint ſeries ſequentes $a.\ b.\ c.\ d.$ &c. & $o.\ g.\ h.\ i.$ &c. numerus terminorum prioris ſit n, poſterioris $n + 1$. Eſt primò $q + p + l + i + h + g + o =$ (ex hypotheſi & geneſi numerorum figuratorum per proprietatem quartam) $\frac{nf}{r} + \frac{\overline{n-1}.e}{r} + \frac{\overline{n-2}.d}{r} + \frac{\overline{n-3}.e}{r} + \frac{\overline{n-4}.b}{r} + \frac{\overline{n-5}.a}{r} = \frac{n.\overline{f+e+d+c+b+a}}{r}$

$$n\left\{\begin{array}{lcr} a & - - & o \\ b & - - & g \\ c & - - & h \\ d & - - & i \\ e & - - & l \\ f & - - & p \\ & & q \end{array}\right\}n+1.$$

Eſt autem $n \times 1$, ſeu n, ad 1, ut $n \times r$ eſt ad r. Ergò, ex æquo, $n \times 1$, ſeu n, erit ad A+B+C+D ut $n \times r$ ad n D, hoc eſt, ut r ad D. Erit igitùr $n - r$ ad H ut n ad A+B+C+D.

Sed (ex numerorum figuratorum geneſi, per proprietatem quartam ſuprà memoratam) terminus I eſt æqualis A + B + C + D.

Erit igitùr $n - r$ ad H ut n ad I, et proinde $\overline{n-r} \times \mathrm{I}$ erit $= n \times \mathrm{H}$.

Sed, per hypotheſin, E + F + G + H eſt ad $n \times$ H ut 1 ad $r + 1$; atque ideò $\overline{\mathrm{E+F+G+H}} \times \overline{r+1}$ eſt æqualis $n \times \mathrm{H} \times 1$, ſeu $n \times$ H.

Erit igitùr $\overline{n-r} \times \mathrm{I} = \overline{\mathrm{E+F+G+H}} \times \overline{r+1}$; atque ideò erit $n - r$ ad $r + 1$ ut E+F+G+H ad I, et proinde (componendo) erit $n - r + r + 1$, ſeu $n + 1$, ad $r + 1$ ut E + F + G + H + I ad I, et (permutando) $n + 1$ ad E+F+G+H+I ut $r + 1$ ad I, et (invertendo) E+F+G+H+I ad $n + 1$ ut I ad $r + 1$, et (multiplicando conſequentes per I) E+F+G+H+I ad $\overline{n+1} \times \mathrm{I}$ ut I ad $\overline{r+1} \times \mathrm{I}$, hoc eſt, ut 1 ad $r+1$.

Q. E. D.

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* Johanni Bernoullio.

$\frac{-e-2d-3c-4b-5a}{r}$ = (ex geneſi numerorum figuratorum) $\frac{nq-p-l-i-b-g}{r}$. Ergò $rq+r.\overline{p+l+i+b+g} = nq-p-l-i-b-g$; factâque tranſlatione convenienti, $\overline{r+1} \times \overline{p+l+i+b+g} = nq-rq$. Dividatur utrinque per $r+1$, erit $p+l+i+b+g = \frac{nq-rq}{r+1}$; additóque q habebitur $q+p+l+i+b+g = \frac{nq-rq}{r+1} + q = \frac{\overline{n+1}\times q}{r+1}$, hoc eſt, $g+b+i+l+p+q$ erit ad $\overline{n+1} \times q$ ut 1 ad $r+1$. Q. E. D.

Sequitur nunc *Propoſitio principalis*, quæ talis eſt.

PROPOSITIO PRINCIPALIS.

In tabulâ numerorum figuratorum ſumma terminorum quotlibet à ſuis reſpectivè cyphris incipientium ad ſummam totidem ultimo æqualium: Item ſumma terminorum quotvis incipientium ab unitate ad ſummam totidem ultimum ſequenti æqualium: in ſerie primâ, ſeu monadum, eſt ut 1 ad 1; in ſerie ſecundâ, ſeu lateralium, ut 1 ad 2; in tertiâ, ſeu trigonalium, ut 1 ad 3; in quartâ, ſeu pyramidalium, ut 1 ad 4, & generalitèr in ſerie quâcunque ut 1 ad illius ſeriei numerum.

DEMONSTRATIO PRIMÆ PARTIS HUJÚSCE PROPOSITIONIS.

De primâ ſerie conſtat ex primo lemmate: de ſecundâ, tertiâ, quartâ, &c. è reliquis. Nam, quia ſumma terminorum quotlibet ad ſummam totidem ultimo æqualium in primâ ſerie eſt ut 1 ad 1, erit, vi horum lemmatum, in ſecundâ ut 1 ad 1+1 = 2; &, quia in ſecundâ eſt ut 1 ad 2, erit in tertiâ ut 1 ad 2+1 = 3; & propterea etiàm in quartâ ut 1 ad 3+1 = 4; in quintâ ut 1 ad 4+1 = 5; & generalitèr in ſerie c ut 1 ad c. Q. E. D.

DEMON-

DEMONSTRATIO SECUNDÆ PARTIS HUJUSCE PROPOSITIONIS.

Quia rationem 1 ad $r+1$ memoratam in ultimo lemmate hìc interpretamur per rationem 1 ad c, erit $r = c - 1 =$ (per proprietatem primam 1) numero cyphrarum, à quibus columna c incipit. Quarè, cum in dicto lemmate repertum ſit $g+h+i+l+p = \frac{\overline{n-r}\times q}{r+1} = \frac{\overline{n-r}\times q}{c}$, ſequitur quòd $g+h+i+l+p$ (ſumma terminorum quorum numerus eſt n) ſe habet ad q in $n-r$ (numerum terminorum minùs numero cyphrarum) ſicut 1 ad c; hoc eſt, ſumma terminorum quotlibet ab unitate incipientium ad totidem terminos ſequenti ultimum æquales, ut 1 ad c*. Q. E. D.

CONSECTARIUM.

Ex hâc oſtenſâ proprietate facile nunc eſt invenire tùm terminum optatum, tùm ſummam terminorum ſeriei cujuſlibet. Sumpti intelligantur termini æque-multi ex pluribus continuè columnis, & ſit numerus ſumptorum ab initio cujuſque columnæ n, adeóque numerus terminorum ab unitate (excluſis cyphris initialibus) in ſecundâ columnâ $n-1$, in tertiâ $n-2$, in quartâ $n-3$, atque ita deinceps, per primam proprietatem: quo poſito, quæſitum ita colligo. Summa terminorum n primæ columnæ, nempe, n unitates, ſeu $\frac{n}{1}$, æquatur termino $\overline{n+1}$.no, hoc eſt, termino ſequenti ultimum, ſecundæ columnæ, per quartam proprietatem, ex tabulæ geneſi. Quarè termini hujus in $n-1$ (numerum terminorum ab unitate ſecundæ columnæ) ducti ſubduplum, ſeu $\frac{n.\overline{n-1}}{1.2}$, per duodecimam proprietatem æquale eſt aggregato terminorum ſecundæ columnæ, & ſimùl (per quartam proprietatem) ipſi termino ſequenti ultimum tertiæ columnæ.

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* Vide ſuper hâc materiâ opera ipſius Johannis Bernoullii, edita Lauſannæ anno Domini 1742, Tomum tertium, paginam 521, in 47mâ Lectione de Calculo Integralium.

Unde ſimilitèr hujus termini in $n-2$ (numerum terminorum ab unitate tertiæ columnæ) ducti ſubtriplum, nempe $\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}$, æquatur (per duodecimam proprietatem) aggregato terminorum tertiæ columnæ, inſimûlque (per quartam proprietatem) ipſi termino ſequenti ultimum quartæ. Quocircà & hujus termini in $n-3$ (numerum terminorum ab unitate quartæ columnæ) ducti ſubquadruplum, putà $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}$, exhibet ſummam terminorum quartæ columnæ, unàque terminum qui ſequitur ultimum quintæ; & rursùs iſtius termini in $n-4$ ducti ſubquintuplum nempe, $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4.5}$, producit ſummam terminorum columnæ quintæ, & ſimùl terminum qui excipit ultimum ſextæ; atque ita conſequentèr. E quibus igitùr infertur, quòd ſumma terminorum n primæ columnæ ſit $\frac{n}{1}$, ſecundæ $\frac{n.\overline{n-1}}{1.2}$, tertiæ $\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}$, quartæ $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}$, quintæ $\frac{n.\overline{n-1}\,\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4.5}$, & generalitèr columnæ c, $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}\ldots.\overline{n-c+1}}{1.2.3.4.5\ldots.c}$. Et, quia quælibet harum quantitatum etiàm exprimit terminum $n+1$ ſequentis columnæ, ſequitur quòd ipſe illius terminus optatus, ſeu ultimus, n habeatur mutato ſolummodò ubique n in $n-1$; adeóque quòd terminus optatus, ſecundæ columnæ ſit $\frac{n-1}{1}$, tertiæ $\frac{\overline{n-1}.\overline{n-2}}{1.2}$, quartæ $\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3}$, quintæ $\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4}$, &, generalitèr, columnæ c, $\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}\ldots.\overline{n-c+1}}{1.2.3.4\ldots.c-1}$.

SCHOLIUM.

Multi, ut hoc in tranſitù notemus, numerorum figuratorum contemplationibus vacârunt (quos inter Faulhaberus & Remmelini Ulmenſes, Walliſius, Mercator in Logarithmotechniâ, Preſtetus, aliíque); ſed qui proprietatis hujus demonſtrationem univerſalem dederit & ſcientificam, novi neminem. Walliſius in Arithmeticâ Infinitorum fundamentum ſuæ methodi jacturus, rationes quas habent ſeries quadratorum, cuborum, aliarúmque poteſtatum, numerorum naturalium ad ſeriem totidem maximo æqualium, inductione inveſtigat; indéque in propoſitione 176, ad contemplationem numerorum trigonalium, pyramidalium, reliquorúmque figuratorum tranſit. Sed ſatius fuiſſet fortéque naturæ rei convenientius, ſi vice verſâ tractationem numerorum figuratorum, eámque univerſali & accuratâ demonſtratione munitam, præmiſiſſet, ac tum demùm ad poteſtatum ſummas inveſtigandas perrexiſſet. Præterquam enim quòd modus demonſtrandi per inductionem parùm ſcientificus eſt, inſupérque pro quâlibet ſerie peculiarem operam depoſcit; illa utique omnium judicio præcedere debent, quæ cæteris naturâ ſunt priora & ſimpliciora, quales videntur eſſe numeri figurati præ poteſtatibus, tùm quòd illi additione, hæ multiplicatione generantur, tùm, & præcipuè, quòd ſeries figuratorum à ſuis reſpectivè cyphris incipientes ad ſeries æqualium rationem habent exactè ſubmultiplicem, qualem non habere poſſunt ſeries poteſtatum (ſaltèm in terminis numero finitis) abſque aliquo exceſſu vel defectu, quicunque cyphrarum numerus ipſis præfigatur. De cætero namque ex cognitis figuratorum ſummis nihilo difficiliùs inveſtigari poterunt poteſtatum ſummæ, atque ex his priores collegit auctor; quod quomodo fiat, paucis oſtendam.

Inveſtigatio

Inveſtigatio Summarum quæ proveniunt ex additione quadratorum, cuborum, quadrato-quadratorum, et ſequentium poteſtatum numerorum naturalium 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. *ex numerorum figuratorum ſummis derivata.*

Proponatur ſeries numerorum naturalium ab unitate 1. 2. 3. 4. 5. &c. uſque ad n, & quærantur omnium ipſorum, item omnium quadratorum, cuborum, et ſequentium poteſtatum ex ipſis, ſummæ. Quoniam in tabulâ combinationum terminus ſecundæ columnæ indefinitè eſt $n - 1$, & ſumma omnium terminorum, hoc eſt, ſumma omnium $n - 1$, ſeu $\int . \overline{n - 1}$, per conſectarium præcedens inventa, eſt $\frac{n . \overline{n-1}}{1 . 2} = \frac{nn - n}{2}$, erit $\int . \overline{n - 1}$, ſive $\int n - \int 1$, $= \frac{nn - n}{2}$, & proinde $\int n = \frac{nn - n}{2} + \int 1$; ſed $\int 1$ (ſumma omnium unitatum) eſt n; quarè ſumma omnium n, ſeu $\int n$, erit $= \frac{nn - n}{2} + n = \frac{1}{2} nn + \frac{1}{2} n$.

Porrò cum terminus, tertiæ columnæ indefinitè acceptus per idem conſectarium ſit $\frac{\overline{n - 1} . \overline{n - 2}}{1 . 2} = \frac{nn - 3n + 2}{2}$, & ſumma omnium terminorum (hoc eſt), omnium $\frac{nn - 3n + 2}{2}$) $\frac{n . \overline{n - 1} . \overline{n - 2}}{1 . 2 . 3} = \frac{n^3 - 3nn + 2n}{6}$; erit $\int \overline{\frac{nn - 3n + 2}{2}}$ ſive $\int \frac{1}{2} nn - \int \frac{3}{2} n + \int 1 = \frac{n^3 - 3nn + 2n}{6}$, & $\int \frac{1}{2} nn = \frac{n^3 - 3nn + 2n}{6} + \int \frac{3}{2} n - \int 1$; ſed $\int \frac{3}{2} n = \frac{3}{2} \int n =$ (per modò oſtenſa) $\frac{3}{4} nn + \frac{3}{4} n$, & $\int 1 = n$: unde his ſubſtitutis ſit $\int \frac{1}{2} nn = \frac{n^3 - 3nn + 2n}{6} + \frac{3nn + 3n}{4} - n = \frac{1}{6} n^3 + \frac{1}{4} nn + \frac{1}{12} n$, ejúſque duplum $\int nn$ (ſumma quadratorum ex omnibus n) $= \frac{1}{3} n^3 - \frac{1}{2} nn + \frac{1}{6} n$.

Rurſùs, quia terminus n quartæ columnæ eſt $\frac{n-1.n-2.n-3}{1\,.\,2\,.\,3} = \frac{n^3-6nn+11n-6}{6}$, & ſumma omnium terminorum $\frac{n.n-1.n-2.n-3}{1\,.\,2\,.\,3\,.\,4} = \frac{n^4-6n^3+11nn-6n}{24}$, erit utique $\int \frac{n^3-6nn+11n-6}{6}$, hoc eſt, $\int \frac{1}{6}\,n^3 - \int nn + \int \frac{11}{6}\,n - \int 1 = \frac{n^4-6n^3+11nn-6n}{24}$; indéque $\int \frac{1}{6}\,n^3 = \frac{n^4-6n^3+11nn-6n}{24} + \int nn - \int \frac{11}{6}\,n + \int 1$. Et quoniam per modò inventa $\int nn = \frac{1}{3}\,n^3 + \frac{1}{2}\,nn + \frac{1}{6}\,n$; nec non $\int \frac{11}{6}\,n$ ſive $\frac{11}{6}\int n = \frac{11}{12}\,nn + \frac{11}{12}\,n$, & $\int 1 = n$; hinc, factâ horum ſubſtitutione, emerget $\int \frac{1}{6}\,n^3 = \frac{n^4-6n^3+11nn-6n}{24} + \frac{1}{3}\,n^3 + \frac{1}{2}\,nn + \frac{1}{6}\,n - \frac{11}{12}\,nn - \frac{11}{12}\,n + n = \frac{1}{24}\,n^4 + \frac{1}{12}\,n^3 + \frac{1}{24}\,nn$, ejúſque proin ſextuplum $\int n^3$ (ſumma cuborum) $= \frac{1}{4}\,n^4 + \frac{1}{2}\,n^3 + \frac{1}{4}\,nn$. Atque ſic porrò ad altiores gradatìm poteſtates pergere, levíque negotio ſequentem adornare laterculum licet:

Summa Poteſtatum.

$$\int n = \tfrac{1}{2}nn + \tfrac{1}{2}n.$$
$$\int nn = \tfrac{1}{3}n^3 + \tfrac{1}{2}nn + \tfrac{1}{6}n.$$
$$\int n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}nn.$$
$$\int n^4 = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 * - \tfrac{1}{30}n.$$
$$\int n^5 = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 * - \tfrac{1}{12}nn.$$
$$\int n^6 = \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 * - \tfrac{1}{6}n^3 * + \tfrac{1}{42}n.$$
$$\int n^7 = \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 * - \tfrac{7}{24}n^4 * + \tfrac{1}{12}nn.$$
$$\int n^8 = \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 * - \tfrac{7}{15}n^5 * + \tfrac{2}{9}n^3 * - \tfrac{1}{30}n.$$
$$\int n^9 = \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 * - \tfrac{7}{10}n^6 * + \tfrac{1}{2}n^4 * - \tfrac{1}{12}nn.$$
$$\int n^{10} = \tfrac{1}{11}n^{11} + \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 * - 1n^7 * + 1n^5 * - \tfrac{1}{2}n^3 * + \tfrac{5}{66}n.$$

Quìn imò qui legem progreſſionis terminorum in hoc laterculo deſcriptorum attentiùs inſpexerit, eundem etiàm continuare poterit abſque his ratiociniorum ambagibus, Sumptâ enim c pro poteſtatis cujuſlibet exponente, fit ſumma ma omnium n^c, ſeu $\int n^c$, $= \frac{1}{c+1} n^{c+1} + \frac{1}{2} n^c + \frac{c}{2} A n^{c-1} + \frac{c.\overline{c-1}.\overline{c-2}}{2.3.4} B n^{c-3} + \frac{c.\overline{c-1}.\overline{c-2}.\overline{c-3}.\overline{c-4}}{2.3.4.5.6} C n^{c-5} + \frac{c.\overline{c-1}.\overline{c-2}.\overline{c-3}.\overline{c-4}.\overline{c-5}.\overline{c-6}}{2.3.4.5.6.7.8} D n^{c-7}$ . . . . . & ita deinceps, exponentem poteſtatis ipſius n continuè minuendo binario, quouſque perveniatur ad n vel nn. Literæ capitales A, B, C, D, &c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, &c. nempe $A = \frac{1}{6}$, $B = -\frac{1}{30}$, $C = \frac{1}{42}$, $D = -\frac{1}{30}$. Sunt autem hi coëfficientes ita comparati, ut ſinguli cum cæteris ſui ordinis coëfficientibus complere debeant unitatem; ſic D valere diximus $-\frac{1}{30}$, quia $\frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9} (+ D) - \frac{1}{30} = 1$. Hujus laterculi beneficio intrà ſemi-quadrantem horæ reperi, quòd poteſtates decimæ, ſive quadrato-ſurſolidæ, mille primorum numerorum ab unitate in ſummam collecta efficiunt

91,409,924,241,424,243,424,241,924,242,500.

E quibus apparet, quàm inutilis cenſenda ſit opera Iſmaelis Bullialdi, quam conſcribendo tam ſpiſſo volumini Arithmeticæ ſuæ Infinitorum impendit, ubi nihil præſtitit aliud, quàm ut primarum tantum ſex poteſtatum ſummas (partem ejus quod unicâ nos conſecuti ſumus paginâ) immenſo labore demonſtratas exhiberet.

De ſeriebus ſerierum figuratarum analogis.

Antequam caput hoc finiamus, paucis adhuc indicare lubet quomodo, ſuppoſitis iis quæ de ſeriebus figuratis oſtenſæ ſunt, poſſint quævis etiàm aliæ ſeries figuratarum analogæ (quæ, ſcilicet, differentias ſuas primas, ſecundas, tertias, &c. æquales habent, adeóque ex continuâ additione terminorum alicujus ſeriei æqualium generantur) ad homologas figuratas reduci, ac proinde ſummari, vel poſtremi ipſarum termini inveniri. Sit ſeries quævis æqualium D, ex cujus additione naſcatur ſeries C, & ex hujus additione ſeries B, & ex hujus tandèm collectione ſeries A, ſumptis ad arbitrium primis ſerierum terminis d, c, b, a. Vocabitur ſeries A figuratarum analoga, cujus differentiæ primæ conſtituunt ſeriem B, ſecundæ ſeriem C, tertiæ ſeriem D, &c. Et quoniàm apparet, ſeriem A componi ex ſeriebus unitatum 1, 1, 1, 1, &c. lateralium 1, 2, 3, 4, &c. trigonalium 1, 3, 6, 10, &c. pyramidalium 1, 4, 10, 20, &c. in primos differentiarum terminos a, b, c, d, ſeorsìm ductis, quarúmque omnium poſtremi termini & ſummæ per ante dicta habentur, ipſius quoque hinc ſeriei A poſtremum terminum & ſummam terminorum obtineri poſſe conſtat; nimirùm, ſi numerus terminorum vocetur n, erit ultimus terminus ſeriei A $= a + \overline{n-1}.\,b + \frac{\overline{n-1}.\overline{n-2}}{2}\,c + \frac{\overline{n-1}.\overline{n-2}.\overline{n-3}}{2\,.\,3}\,d$; & ſumma omnium terminorum $= na + \frac{n.\overline{n-1}}{2}\,b + \frac{n.\overline{n-1}.\overline{n-2}}{2\,.\,3}\,c + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2\,.\,3\,.\,4}\,d$.

D	C	B	A.
d	c	b	a
d	$c+d$	$b+c$	$a+b$
d	$c+2d$	$b+2c+d$	$a+2b+c$
d	$c+3d$	$b+3c+3d$	$a+3b+3c+d$
d	$c+4d$	$b+4c+6d$	$a+4b+6c+4d$
d	$c+5d$	$b+5c+10d$	$a+5b+10c+10d$

A

TRANSLATION

OF THE

THREE FIRST CHAPTERS

OF THE

SECOND PART, OR BOOK,

OF

MR. JAMES BERNOULLI's EXCELLENT TREATISE,

INTITLED

ARS CONJECTANDI;

OR

"THE ART OF FORMING PROBABLE CONJECTURES CONCERNING EVENTS THAT DEPEND ON CHANCE."

Publiſhed in a ſmall Quarto Volume at Basil, or Basle, in
Switzerland, in the Year 1713.

THE PROŒMIUM, OR PREFACE, TO THE SECOND PART OF THE SAID TREATISE

DE ARTE CONJECTANDI.

IT is eaſy to perceive that the prodigious variety which appears both in the works of nature and in the actions of men, and which conſtitutes the greateſt part of the beauty of the univerſe, is owing to the multitude of different ways in which its ſeveral parts are mixed with, or placed near, each other. But, becauſe the number of cauſes that concur in producing a given event, or effect, is oftentimes ſo immenſely great, and the cauſes themſelves are ſo different one from another, that it is extremely difficult to reckon up all the different ways in which they may be arranged, or combined together, it often happens that men, even of the beſt underſtandings and greateſt circumſpection, are guilty of that fault in reaſoning which the writers on logick call *the inſufficient, or imperfect enumeration of parts, or caſes*: inſomuch that I will venture to aſſert, that this is the chief, and almoſt the only, ſource of the vaſt number of erroneous opinions, and thoſe too very often in matters of great importance, which we are apt to form on all the ſubjects we reflect upon, whether they relate to the knowledge of nature, or the merits and motives of human actions. It muſt therefore be acknowledged, that that art which affords a cure to this weakneſs, or defect, of our underſtandings, and teaches us ſo to enumerate all the poſſible ways in which a given number of things may be mixed and combined together, that we may be certain that we have not omitted any one arrangement of them that can lead to the object of our inquiry, deſerves to be conſidered as moſt eminently uſeful and worthy of our higheſt eſteem and attention. And this is the buſineſs of *the art, or doctrine of combinations.*

Nor is this art or doctrine to be conſidered merely as a branch of the mathematical ſciences. For it has a relation to almoſt every ſpecies of uſeful knowledge that the mind of man can be employed upon. It proceeds indeed upon mathematical principles in calculating the number of the combinations of the things propoſed: but by the concluſions that are obtained by it, the ſagacity of the natural philoſopher, the exactneſs of the hiſtorian, the ſkill and judgment of the phyſician, and the prudence and foreſight of the politician, may be aſſiſted; becauſe the buſineſs of all theſe important profeſſions is but *to form reaſonable conjectures* concerning the ſeveral objects which engage their attention, and all wiſe conjectures are the reſults of a juſt and careful examination of the ſeveral different effects that may poſſibly ariſe from the cauſes that are capable of producing them. And, I preſume, it was from a ſenſe of the great and general utility of this doctrine that ſeveral very eminent mathematicians have undertaken to treat of it in their public writings; and particularly Mr. Van Schooten (the learned commentator on Des Cartes's geometry), Mr. Leibnitz, Dr. Wallis, and

Monſieur Preſtet: ſo that the reader is not to conſider every thing he will meet with in this treatiſe as entirely new and of my invention. I have, however, made ſome improvements on the ſubject, and thoſe too of conſiderable importance, which I may juſtly call my own: and particularly I have diſcovered a general and eaſy demonſtration of the principal and moſt remarkable property of the figurate numbers, to wit, "that of the proportion between the ſum of any number of terms of a ſeries of figurate numbers of any order whatſoever to the ſum of the ſame number of terms all equal to the laſt term of the ſeries;" upon which property many of the following propoſitions in this book are founded: for of this property I believe no other writer has ever before given a demonſtration.

Indeed, none of the tracts hitherto publiſhed on this ſubject, can be ſaid to contain a full and ſatisfactory account of it. And therefore I have thought it would be agreeable to my readers to ſee it here treated in a regular manner, from the firſt and moſt ſimple principles on which it is founded, to the higher and more extenſive propoſitions which have been built upon them, without being under the neceſſity of referring to other books upon the ſubject. But, though, for theſe reaſons, I have laid down the very firſt elements of the doctrine, and have endeavoured to demonſtrate every thing as I went on, to the end that the chain of reaſoning might be uniform and compleat, I have done it in as conciſe a manner as I could, and only as far as was neceſſary to prepare the way to the ſubſequent and more important parts of the book. The greater part of the treatiſe conſiſts of two principal heads, of which the firſt contains the doctrine of *permutations*, and the ſecond contains the doctrine of *combinations*; which is followed by a third branch, which ſprings out of the two former, and treats of *permutations* and *combinations* joined together.

CHAPTER I.

CONCERNING PERMUTATIONS.

ARTICLE 1. BY the *permutations* of a number of things, I mean the ſeveral variations that may be made in their relative ſituations, or poſitions, or in the order in which they may be made to follow each other, while their number continues the ſame. So that, when it is propoſed to find in how many different ways a given number of things may be ranged, or diſpoſed, without omitting any of them, this is ſaid to be requiring the *number of their permutations*.

2. The things of which we are required to diſcover the number of permutations, may be either all diſtinguiſhed from each other by ſome plain mark, ſuch as a difference of ſhape or colour, as cubes from ſpheres, or black balls from white balls; or they may be exactly like each other, ſo as to be liable to be miſtaken one for another, as two ſpherical black balls of exactly the ſame ſize and weight. In the former caſe it will be proper to denote the ſeveral things by as many different letters of the alphabet; and in the latter caſe it will be convenient to denote ſo many of the things as are exactly like each other, by the ſame letter of the alphabet, repeated as often as any of the ſaid things which are like each other ſhall occur, as will be ſeen in the courſe of the following pages. We will firſt conſider the former of theſe caſes, or that in which all the things are diſtinguiſhed from each other.

The firſt Caſe of Permutations, in which all the things whoſe permutations are required to be aſſigned, are diſtinguiſhed from each other.

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3. As it is obvious that the number of changes of poſition that may happen in a great number of things cannot be determined without firſt knowing the number of the like changes of poſition that may happen in all leſſer numbers of them, it is manifeſtly neceſſary, in treating of this ſubject, to proceed in the ſynthetick method, and begin our reaſonings from the firſt and moſt ſimple caſes: which may be done as follows.

4. If there is only one thing to be arranged, which is denoted by the letter *a*, it can be taken, or ranged, only in one manner.

5. If there are two things clearly diſtinguiſhed from each other, which are denoted by the letters *a* and *b*, it is evident that we may either place *a* before *b*, or *b* before *a*; ſo that there will be two different ways of arranging them, to wit, *a b* and *b a*; or, in other words, there will be two *permutations* of them. Q. E. I.

6. If there be three things diſtinguiſhed from each other, and denoted by the three different letters *a*, *b* and *c*, it is evident that either of the three letters may be placed before the other two. Now, if *a* is placed firſt, the other two letters *b* and *c* may undergo two permutations, by what has been ſeen in the laſt article, and the three letters may be placed in theſe two poſitions, *a b c*, and *a c b*; and in like manner, if *b* is placed firſt, the other two letters *a* and *c* may undergo two permutations, and the three letters may be placed in the two following poſitions, to wit, *b a c* and *b c a*; and,

laſtly, if *c* is placed firſt, the other two letters *a* and *b* may undergo two permutations, and the three letters may be placed in the two following poſitions, to wit, *c a b* and *c b a*. Therefore the whole number of permutations which the order, or poſition, of the three letters, *a*, *b*, and *c* may undergo, is three times 2, or 6, to wit, *a b c*, *a c b*, *b a c*, *b c a*, *c a b*, and *c b a*. Q. E. I.

7. In like manner, if there are four different things clearly diſtinguiſhed from each other, and denoted by the four different letters *a*, *b*, *c*, and *d*, it is evident that either of the four may be placed before the other three, and that, while each of them is placed firſt, the other three may undergo 6 permutations, by what has been juſt now ſhewn in art. 6. Therefore the whole number of permutations which theſe four things, or letters, may undergo, will be four times 6, or 24. Q. E. I.

8. And, for the ſame reaſon, if there were five things denoted by the five different letters *a*, *b*, *c*, *d* and *e*, the number of their permutations would be five times as great as in the laſt caſe; or would be 5 times 24, or 120. And in general, whatever be the number of things or letters, the number of permutations, or changes of poſition, which they may be made to undergo, will be equal to the product that ariſes by multiplying the number of permutations of the next ſmaller number of things by the given number of them. So that, if the whole number of things, or letters, be n, and the number of permutations in $n - 1$ things, or letters, be N, the number of permutations in all the n letters, will be equal $n \times N$. And hence ariſes the following

Rule for diſcovering the whole number of permutations, or relative changes of poſition, which any given number n, *of things, may be made to undergo.*

9. Let all the numbers 1, 2, 3, 4, 5, 6, 7, &c, in their natural order, beginning from unity, up to the given number n, of things, or letters, whoſe permutations are to be inveſtigated, be multiplied one into the other; and the product $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times$ &c . . . $\times\, n$ will be the number of permutations that is required. Q. E. I.

10. It will be convenient ſometimes to uſe a full point [.] inſtead of the common mark of multiplication $\times$; and then $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times$ &c. $\times\, n$ will be $=$ $1 . 2 . 3 . 4 . 5 . 6 . 7 .$ &c $. n$, or (becauſe 1 has no effect in multiplication) $= 2 . 3 . 4 . 5 . 6 . 7 .$ &c $. n$; which will therefore be equal to the whole number of permutations, or changes of poſition, which n things may be made to undergo.

11. According to this rule, the number of permutations, or changes of poſition, which 7 different things may be made to undergo, is $2 . 3 . 4 . 5 . 6 . 7$, or 5040. Thus, for example, the different changes that may be rung upon ſeven different bells is 5040. The multiplications of theſe numbers into each other will appear in the following table;

The Number of Things.	*The Number of Permutations, or Changes of Poſition.*
1	1
	2
2	2
	3
3	6
	4
4	24
	5
5	120
	6
6	720
	7
7	5040
	8
8	40,320
	9
9	362,880
	10
10	3,628,800
	11
	3 628 800
	36 288 00
11	39,916,800
	12
	79 833 600
	399 168 00
12	479,001,600

12. We may ſee by this table how very faſt the number of permutations increaſes, as the number of things to be arranged becomes greater and greater. The four letters that compoſe the word *Roma* may be arranged in 24 different ways; but the ſix letters that compoſe the word *Romani* may be arranged in 720 different ways; and the ſeven letters that compoſe the word *Romanis* may be arranged in no leſs than 5040 different ways. We are now to conſider the ſecond caſe of permutations, in which ſome of the things to be arranged are exactly like others of them, ſo as not to be diſtinguiſhed from them.

The Second Caſe of Permutations, in which ſome of the things, the permutations of which are required to be aſſigned, are like others of them, ſo as not to be diſtinguiſhed from them.

13. If ſome of the things of which we are required to find the permutations, are exactly like others of them, ſo as not to be diſtinguiſhed from them, the number of permutations, or changes of poſition, which they may be made to undergo, will be much ſmaller than in the former caſe. Thus, for example, if there are ſix different things, whereof we are required to aſſign the permutations, but three of them are exactly like each other, ſo that it is impoſſible to diſtinguiſh either of them from the other two; as is the caſe with the ſix letters *a a a b c d*, in which the letter *a* occurs three times; the number of permutations which theſe ſix things, or letters, can undergo, will be much leſs than the number of permutations they could undergo, if they were all diſtinguiſhable from each other, as they were ſuppoſed to be in the former caſe. And the way of finding out how much leſs the number of permutations will be in this caſe than in the former caſe, will be to conſider how many per-

mutations; or changes of poſition, the three things which are exactly alike, and are denoted by the ſame letter *a*, might undergo, if they were unlike each other, and diſtinguiſhable one from the other, and then to ſubſtitute an unit, or one ſingle poſition, in lieu of all thoſe ſeveral permutations. Thus; for example, if, inſtead of the three things exactly alike which are denoted by the ſame letter *a*, we were to take three things that were unlike each other, and denote them by the three letters *a*, *α*, and a, that is, by an Italick *a*, a Greek *α*, and a Roman a, it is evident from what has been ſhewn in art. 6, that, without making any change in the poſition of the other letters, *b*, *c*, *d*, theſe three letters *a*, *α*, and a, might be placed in ſix different poſitions, inſtead of the one poſition *a a a* in which alone the three things perfectly alike, that were denoted by the ſame letter *a*, could be arranged. The number of permutations therefore in the ſix things denoted by the letters *a*, *α*, a, *b*, *c*, *d*, will be ſix times as great as that of the ſix things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, in which three of the things are alike, and denoted by the ſame letter *a*. And therefore, to find the number of permutations of the ſix things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, we muſt firſt find the whole number of permutations which they might undergo if they were all unlike each other, and denoted by the letters *a*, *α*, a, *b*, *c*, *d*, and then we muſt divide the ſaid number by 6, or the number of permutations which the three things denoted by the ſame letter *a* might undergo if they were unlike each other, and denoted by the three different letters *a*, *α*, and a. Now the whole number of permutations of ſix different things unlike each other, that are denoted by the letters *a*, *α*, a, *b*, *c*, *d*, has been ſhewn to be 720. Therefore the number of permutations of ſix different things, whereof three are perfectly like each other, and denoted by the ſame letter *a*, or of ſix different things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, will be $\frac{720}{6}$, or 120 *.

14. Again, if the ſix letters whereof we were required to find the permutations, were *a a a b b c*, in which, beſides the letter *a*, which is repeated three times, the letter *b* is alſo

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* The truth of this article may be made viſible to the eye in the following manner:

Let us (to avoid a great number of permutations, which would take up a great deal of room, and tend to confound the ſubject) ſuppoſe the three different letters a, *a*, and α, to be connected only with one more letter, to wit, *b*. Then, by art. 7, the whole number of permutations of theſe letters will be 24, to wit,

a, *a*, α, *b*,	a, *a*, *b*, α,	a, *b*, *a*, α,	*b*, a, *a*, α,
a, α, *a*, *b*,	a, α, *b*, *a*,	a, *b*, α, *a*,	*b*, a, α, *a*,
a, a, α, *b*,	*a*, a, *b*, α,	*a*, *b*, a, α,	*b*, *a*, a, α,
a, α, a, *b*,	*a*, α, *b*, a,	*a*, *b*, α, a,	*b*, *a*, α, a,
α, a, *a*, *b*,	α, a, *b*, *a*,	α, *b*, a, *a*,	*b*, α, a, *a*,
α, *a*, a, *b*,	α, *a*, *b*, a,	α, *b*, *a*, a,	*b*, α, *a*, a.

Now, let the Italick letter *a*, and the Greek letter α, be converted into the Roman letter a. And the foregoing permutations will thereby be converted into the following ones, to wit,

a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a;

of which the firſt ſix are all exactly alike, to wit, a a a *b*, and therefore muſt be reckoned as only one poſition, or permutation; and, in like manner, the next ſix are alſo all alike, to wit, a a *b* a, and therefore muſt be reckoned as only one poſition, or permutation; and the third ſix are alſo all alike, to wit, a *b* a a, and therefore muſt be reckoned as only one poſition, or permutation; and laſtly, the fourth ſix are alſo all alike, to wit, *b* a a a, and therefore muſt be reckoned as only one poſition, or permutation. So that, by the coincidence of ſix permutations into one in each of the four ſets of ſix permutations, the ſaid twenty-four different permutations will be reduced to only four, or $\frac{24}{6}$, different permutations, to wit, a a a *b*, a a *b* a, a *b* a a, *b* a a a.

And it is eaſy to ſee that the like reduction muſt take in the whole number of permutations that may happen amongſt any other given number of things that are all different and diſtinguiſhable from each other, when any other and leſſer number of the ſaid things are rendered like to, and undiſtinguiſhable from, each other.

repeated twice, it is evident that the number of the permutations which the ſaid letters could undergo, would be but half the number of the permutations of the ſix letters *a a a b c d*; becauſe every two permutations of theſe letters which would be diſtinguiſhable from each other when the two different letters *b* and *d* are made uſe of, will coincide, or become undiſtinguiſhable from each other, when *b* is inſerted inſtead of *d*. And therefore the number of the permutations of the ſix letters *a a a b b c* will be only $\frac{120}{2}$, or 60.

15. And in the ſame manner it may be ſhewn that, when ſeveral of the letters, of which we are required to aſſign the number of permutations, are repeated, or taken more than once, we muſt, for every ſuch repetition of the ſame letter, divide the number of permutations of the whole number of letters by the number of the permutations of ſo many different letters as there are repetitions of the ſame letter. And hence ariſes the following

Rule for diſcovering the number of permutations, or relative changes of poſition diſtinguiſhable from each other, which any given number n, *of things, whereof ſome are exactly like others, and cannot be diſtinguiſhed from them, may be made to undergo.*

16. Let the whole number of permutations, or changes of poſition, which the ſaid things might be made to undergo, if they were all unlike each other, and could be clearly diſtinguiſhed one from the other, be divided by the number of permutations, or changes of poſition, which the two, or more, things which are like each other, and are denoted by the ſame letter, might be made to undergo, if they were

unlike to each other, and clearly diſtinguiſhed from each other. And the quotient will be the number of permutations that is required. This is upon a ſuppoſition that, amongſt the things that are given, and of which we are required to find the number of permutations, there is only one ſet of things that are exactly like each other, and therefore denoted by the ſame letter.

But, if, amongſt the things of which we are required to find the number of permutations, there ſhould be two, or more, ſets of things that are exactly like each other, and therefore denoted by the repetition of the ſame letters, we muſt multiply the number of all the permutations which the firſt ſet of like things, denoted by the firſt letter that occurs more than once in the notation, might be made to undergo if they were all unlike each other, into the number of all the permutations which the ſecond ſet of like things, denoted by the ſecond letter that occurs more than once in the notation, might be made to undergo if they were all unlike each other, and further into the number of all the permutations which the third ſet of like things, denoted by the third letter that occurs more than once in the notation, might be made to undergo, if they were all unlike each other, and into the numbers of all the permutations which the fourth ſet, and the fifth ſet, and all the following ſets, of like things, denoted by the repetition of the ſame letters, might be made to undergo, if the things in each ſet were unlike each other: and the whole number of permutations, which all the n things that are given (and whereof we are required to find the number of permutations diſtinguiſhable from each other) might be made to undergo, if they were all unlike each other, muſt be divided by the product of the ſaid multiplication. The quotient will be the number of permutations diſtinguiſhable from each other, of the given number n of things, which was required to be found.

17. This doctrine of permutations is of great uſe in determining the number of anagrams that may be made of any propoſed word, or the number of different ways in which the letters that compoſe it may be arranged. Thus, for example,

ample, the letters that form the word *Roma* may be arranged in 2 . 3 . 4, or 24, different ways; and thoſe of the word *Romani* (which are ſix in number) may be arranged in 2 . 3 . 4 . 5 . 6, or 720, different ways; and thoſe of the word *Romanis* (which are ſeven in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7, or 5040, different ways; as we have ſeen in art. 12. In like manner the letters of the word *Trojanum* (which are eight in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7 . 8, or 40,320, different ways; and thoſe of the word *Doctrinam* (which are nine in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9, or 362,880, different ways. But the letters of the word *Leopoldus*, though they are alſo nine in number, cannot be arranged in ſo many different ways, becauſe of the repetition of the letters *l* and *o*, each of which occurs twice. The number of different ways in which the letters of this word can be arranged is $= \frac{362,880}{2 . 2}$, or $\frac{362,880}{4}$, or 90,720; becauſe the two *l*s, if they were different letters, would admit of two permutations, and the two *o*s, if they were different letters, would likewiſe admit of two permutations, and conſequently theſe numbers of permutations, to wit, 2 and 2, muſt (according to the foregoing rule) be multiplied into each other, ſo as to make the product 4, and then the number 362,880 (which is the whole number of permutations which nine different letters may be made to undergo) muſt be divided by it, which gives the quotient 90,720. And the letters of the word *Studioſus*, though likewiſe nine in number, will admit of only 30,240 permutations, becauſe of the repetition of the letter *u* twice, and the letter *ſ* three times. For the permutations which the two *u*s might be made to undergo, if they were different letters, are 2, and the permutations which the three *ſ*es might be made to undergo, if they were different letters, is 6; and the product that ariſes by multiplying 2 into 6 is 12. We muſt therefore divide 362,880 (which is the whole number of permutations of nine different letters) by 12; and the quotient 30,240 will be the number of all the permutations of the nine letters of the word *Studioſus* that will be different, or diſtinguiſhable from each other.

18. It is only by the aſſiſtance of this doctrine of permutations that all thoſe queſtions can be determined, which ſome learned and ingenious men have propoſed concerning the number of the variations, or tranſpoſitions of the words contained in certain verſes, which, on account of the great number of ſuch tranſpoſitions which may be made in them, have been called *Proteus verſes*, in alluſion to the Egyptian ſea-god of that name mentioned in Homer's Odyſſey, who was ſo famous for aſſuming many different ſhapes. The moſt celebrated of theſe verſes are thoſe which have been given us by Thomas Lanſius, and the learned Joſeph Scaliger, and Bernard Bauhuſius, a Jeſuit of the college at Louvain, in the Auſtrian Netherlands. The following two verſes we have from Thomas Lanſius:

Lex, Rex, Grex, Res, Spes, Jus, Thus, Sal, Sol, (bona) *Lux, Laus:*
Mars, Mors, Sors, Lis, Vis, Styx, Pus, Nox, Fæx, (mala) *Crux, Fraus.*

In each of theſe verſes there are eleven words of one ſyllable, and one word of two ſyllables, to wit, *bona* in the firſt, and *mala* in the ſecond. Theſe two words of two ſyllables muſt always remain in the ſame place, or within two words of the end of the lines, in order to preſerve the meaſure of the verſes, which requires that the fifth foot in each verſe ſhould be a dactyl. But the other eleven words in each verſe may be placed in any order, with reſpect to each other, that we pleaſe, without altering the meaſure of the verſes. Now the number of permutations, or changes of poſition, that eleven different things can undergo is 39,916,800, as appears from the table in art. 11. It follows therefore that the words of each of the two foregoing verſes may be tranſpoſed in 39,916,800 different ways, without ſpoiling the meaſure of them.

19. In ſome other inſtances of theſe *Proteus* verſes that have been given by ingenious writers on this ſubject, it happens that many of the tranſpoſitions of the words contained

in them are incompatible with the meaſure of the verſes, and ſome of them, from the irregular and ungrammatical order in which the words follow each other, ſeem to convey no ſenſe or meaning whatſoever, or, perhaps, in ſome caſes, a different ſenſe from that which the author intended. But in all theſe caſes a little attention and care will enable us to diſtinguiſh the uſeful tranſpoſitions from the abſurd ones, and to determine the numbers of tranſpoſitions of each ſort ſeparately, if we proceed by regular ſteps according to ſome order, or plan of admiſſion or excluſion, in making the enquiry. An inſtance of this kind occurs in the following Hexameter Latin verſe, which was made by the abovementioned Bernard Bauhuſius, the Jeſuit of Louvain, in honour of the bleſſed Virgin Mary, the mother of our Saviour Jeſus Chriſt; to wit,

Tot tibi ſunt dotes, Virgo, quot ſidera cælo.

On this celebrated verſe ſeveral men of great learning and reputation have beſtowed a great deal of attention. For, in the firſt place, *Ericius Puteanus*, in a little book which he publiſhed under the title of *Thaumata Pietatis*, has employed no leſs than 48 pages in reckoning up the ſeveral uſeful, or rational, tranſpoſitions that may be made of the words contained in it, and makes them amount to as many at leaſt as there are ſtars in the heavens, the number of which is uſually ſaid to be 1022; leaving out (through a religious reverence for the character of the Virgin Mary) all thoſe tranſpoſitions which ſeem to affirm that there are as many ſtars in the heavens as there are virtues in the Virgin's character, becauſe he thinks the number of the latter to be much greater than that of the former. And, 2dly, Gerard Voſſius, in the 7th chapter of his treatiſe intitled, *De Scientiis Mathematicis*, has affirmed the number of the tranſpoſitions which may be made in the words of this verſe without ſpoiling the ſenſe or the meaſure, to be 1022, as *Puteanus* had made it before him. And, 3dly, Monſieur *Preſtet*, a French mathematician, in the firſt edition of a book called " *The Elements of the Mathematicks*, page 348, has examined this *Proteus* verſe, and declared it to admit of 2196 tranſpoſitions

poſitions of its words without ſpoiling the ſenſe or the meaſure: and afterwards, in the ſecond edition of his ſaid work, vol. i. page 133, having re-conſidered the ſubject, has increaſed the number of theſe tranſpoſitions to almoſt half as many more, or 3276. And, 4thly, the induſtrious compilers of the Leipſic *Acta Eruditorum*, in the month of June, 1686, in giving an account of Dr. Wallis's Treatiſe of Algebra, have fixed the number of theſe tranſpoſitions (which Dr. Wallis himſelf had not in that treatiſe ventured to aſſign) at 2580. And, laſtly, Dr. Wallis himſelf, in a Latin edition of his works which he publiſhed afterwards in the year 1693, page 494, has carried the number of theſe tranſpoſitions to 3096. But all theſe writers have been miſtaken in their calculations, and have aſſigned wrong numbers for the ſolution of this queſtion; which cannot but ſeem rather ſurpriſing, as ſome of them had examined the ſubject twice over, and corrected their firſt concluſions. The true number of tranſpoſitions of its words which this famous hexameter verſe will admit of without ſpoiling either the ſenſe or the meaſure of it, that is, without admitting a ſpondee in the fifth place, but admitting ſuch tranſpoſitions as only deſtroy the *cæſura* of the verſe, I have found, upon a careful examination, to be 3312.

20. I here conclude the chapter on the doctrine of permutations, of which I hope the fundamental principles have been ſufficiently explained; and I proceed to conſider the doctrine of combinations, which is of no leſs uſe and importance than the former.

CHAPTER II.

CONCERNING COMBINATIONS.

DEFINITION 1.

21. BY the *combinations* of things, I mean the ſeveral different ways in which any given number of things may be joined, or connected with each other, without any regard to their relative poſitions, or the order in which they follow one another. So that, when a certain number *n* of things is given, and we are required to find in how many different ways theſe *n* things may be taken, by taking, firſt, two of them at a time, then three of them at a time, then four of them at a time, and ſo on in all other poſſible conjunctions, ſo that no one heap, or parcel, of them ſhall be taken more than once, we are ſaid to be required to find all the poſſible *combinations* of the ſaid given number of things.

DEFINITION 2.

22. The number of the things given which is directed to be joined together in one heap, or parcel, is called *the exponent* of the combination: ſo that if we are directed to combine them by pairs, or in parcels containing two a-piece, the exponent of the combination will be 2; if we are directed to combine them by triplets, or in parcels containing three a-piece, the exponent of the combination will be 3; and if we are directed to combine them by quadruplets, or in parcels containing four a-piece, the exponent of the combination will be 4; and, in general, if we are directed to combine them in parcels containing *m* a-piece, the exponent of the combination will be the number *m*.

DEFINITION 3.

23. And the ſeveral things combined in theſe different manners are called *pairs*, or *couplets*, and *triplets*, and *quadruplets*, &c, or *binaries*, *ternaries*, and *quaternaries*, &c, or *binions*, *ternions*, and *quaternions*, &c; that is, all the different conjunctions, or combinations, of any given number of things in parcels conſiſting of two things each, are called all the *pairs*, or *couplets*, or *binaries*, or *binions*, in the ſaid number of things; and all the different conjunctions or combinations of them in parcels conſiſting of three things each, are called all the *triplets*, or *ternaries*, or *ternions*, in them; and all the different conjunctions or combinations of them in parcels conſiſting of four things each, are called all the *quadruplets*, or *quaternaries*, or *quaternions*, in them. And ſimilar names may be found for theſe combinations, when the number of things contained in a ſingle parcel is greater than 4.

24. And when the things are taken ſingly, or ſeparately, or one by one, it will be convenient to denominate them *unaries*, or *unities*, or to give them ſome name that bears a reſemblance to the names by which we diſtinguiſh the ſeveral combinations of them with each other in parcels of two, or of three, or of four, or more together: becauſe, though, when they are taken ſingly, they cannot, in a ſtrict ſenſe, be ſaid to be *combined*, or the taking them ſingly cannot, in ſtrictneſs, be called a *combination* of them, yet in this doctrine of combinations it is often neceſſary to take into conſideration the number of them when taken ſingly, in order to determine all the variations that can be made upon them; and therefore, in a looſer and more extenſive ſenſe of the word *combinations*, the things, when taken ſeparately, are conſidered as undergoing one ſpecies of combination, or forming one claſs of the ſeveral claſſes of combinations which they may be made to undergo. This is a ſmall inaccuracy of language, ſimilar to that by which a unit is often called a *number*, though in ſtrictneſs a *number* means *two* or *more* *units*,

units, or ſingle quantities, joined together. But when due notice is given of what is meant by ſuch inaccurate expreſſions (which are often convenient for the purpoſe of avoiding a multiplicity of words) no miſtakes can ariſe from the uſe of them.

25. And for a like reaſon it will be convenient to give a name to the act of omitting to take them at all, either ſingly, or combined with each other, or to conſider ſuch omiſſion as one ſpecies of their combinations. Such an omiſſion of them may be called a *nullienation* (from the word *nullies*, which means *no-times*); and the *nothings*, or cyphers, ſet down, inſtead of the things themſelves, on theſe occaſions may be called *nullenaries* (like *binaries*, *ternaries*, and *quaternaries*) or *nullions*. The uſe of this ſort of odd language will appear in the courſe of the following pages.

26. Some writers have confined the word *combination* to the ſtrict original ſenſe of "taking things by *binaries*, or *pairs*, or *couplets*, only, or parcels conſiſting of two things;" and have called the taking them by *ternaries*, or parcels conſiſting of three things, *conternation*; and the taking them by *quaternaries*, or parcels conſiſting of four things, *conquaternation*; and have denominated the parcels conſiſting of two things each, that may be formed out of a given number of things, the *combinations* of the ſaid given things; and the parcels conſiſting of three things each, which may be formed out of the ſame given number of things, the *conternations* of the ſaid given things; and the parcels conſiſting of four things each, which may be formed out of the ſame given number of things, the *conquaternations* of the ſaid given things. But this degree of accuracy in our expreſſions would evidently lead to the compoſition of an immenſe number of new words, in order to expreſs the variety of conjunctions that may be made of the things given in parcels of different ſorts, ſuch as parcels conſiſting of two things, parcels conſiſting of three things, parcels conſiſting of four things, parcels conſiſting of five things, parcels conſiſting of ſix things, and the like; the uſe of which multitude of new words

might be found inconvenient. And therefore other perſons, who were apprehenſive of this inconvenience, and yet were deſirous of avoiding the inaccuracy of employing the word *combinations* for parcels conſiſting of more than two things, have propoſed to make uſe of the more general words *complications* or *complexions* (derived from the Latin verb *complicare*, which ſignifies *to fold up together)* for parcels conſiſting of three things, or four things, or five, or more, things, each, made out of a given number of things ſucceſſively ſo united together: and ſome authors, with great ſagacity and judgment, have recommended the word *elections* to be uſed on this occaſion, in order to comprehend thoſe methods of reckoning and claſſing the things under conſideration by which the things when taken ſeparately, or one by one, are admitted as one ſpecies of combinations of them; and even, when nothings, or cyphers, are taken in their ſtead, thoſe nothings, or cyphers, are admitted as another ſpecies of their combinations, or elections. But the generality of writers who have treated of this ſubject, make uſe of the word *combinations* to denote all the different parcels of things, whether conſiſting of two things, or of three things, or of four things, or of any greater number of things, which can be formed out of a given parcel of things; and even to denote the given things, when taken ſingly or ſeparately; and alſo the nothings, or cyphers, which are ſet down inſtead of them, when they are not taken at all: nor does there ſeem to be any neceſſity for inventing new words on the occaſion.

Theſe definitions of the words that will occur moſt frequently in this doctrine of combinations being premiſed, I now proceed to conſider the doctrine itſelf.

27. Now when we are enquiring into the number of combinations of a given number of things, the ſaid things may either be all unlike and clearly diſtinguiſhable from each other, or ſome of them may be exactly like others of them, ſo as not to be diſtinguiſhable from them. And the ſaid things may either be ſo combined together that no one thing ſhall be contained oftener in any of the propoſed combinations than it occurs in the original number of things which are

propoſed to be combined together; or they may be combined together without this reſtriction, or ſo that in ſome of the propoſed combinations the ſame thing may occur oftener than it does in the original number of things which are propoſed to be combined together, to wit, by being combined with itſelf *. And different ſuppoſitions may be made, and different

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* I am not quite certain that this laſt ſentence is a faithful tranſlation of the original, which I ſhall therefore here ſubjoin for the reader's attentive conſideration. *Eæque vel ita combinari debent, ut in nullâ combinatione res eadem ſæpiùs contineatur quàm ipſa reperitur in toto rerum numero; vel ſic, ut in eâdem combinatione res eadem etiàm ſæpiùs recurrere, hoc eſt, ut ſecum ipſâ quoque combinari, poſſit.* The meaning of this obſcure ſentence (as far as I can underſtand it) may be illuſtrated by the two following examples.

In the firſt place, let us ſuppoſe that the things that are to be combined together are ſix in number, all clearly diſtinguiſhable from each other, and denoted by the ſix letters *a*, *b*, *c*, *d*, *e*, *f*. And let us ſuppoſe that theſe ſix letters are to be combined together in quaternions, or quadruplets, or parcels conſiſting of four letters each. Then, ſays the author, theſe quadruplets may be either reſtrained to thoſe only which conſiſt of four different letters, or in which the ſame letter does not occur oftener than once, or than it occurs in the original enumeration of the ſix things, *a*, *b*, *c*, *d*, *c*, *f*, out of which theſe quadruplets are to be formed, ſuch as the quadruplets *abcd*, *acde*, *adef*, &c; or theſe quadruplets may be formed without this reſtriction, ſo as to admit the ſame letter to be contained in them, more than once, or to be, as it were, combined with itſelf, as happens in the quadruplets *aabc*, *aacd*, *aade*, *aaef*, *aabd*, *aabe*, *aabf*, *aace*, *aaab*, *aaac*, *aaad*, &c.

In the ſecond place, let us ſuppoſe that the things that are to be combined together are, as before, ſix in number, but that two of them are exactly like each other, and are therefore denoted by the ſame letter *a*, and that three of them are alſo exactly like each other, and therefore denoted by the ſame letter *b*, and the ſixth is different both from thoſe of the firſt ſet, and from thoſe of the ſecond ſet, and is therefore denoted by the letter *c*; ſo that the ſix things that are to be combined together, are denoted by the letters *a*, *a*, *b*, *b*, *b*, *c*. And let it be required to combine theſe ſix things, or letters, together in quaternions, or quadruplets, or parcels conſiſting of four letters each. Then, ſays the author, theſe quadruplets may be either ſo reſtrained that they ſhall not contain either of the three letters *a*, *b*, and *c* oftener than it is contained in the original enumeration of them, to wit, *a*, *a*, *b*, *b*, *b*, *c*, that is, that no quadruplet ſhall contain the letter *a* oftener than twice, or the letter *b* oftener than three times, or the letter *c* oftener than once; as is the caſe with the quadruplets *abbc*, *aabc*, *aabb*, *abbb*, *bbbc*; or the ſaid quadruplets may be formed without this reſtriction, ſo as to admit the letter *a* to be repeated more than twice, and the letter *b* to be repeated more than three times, and the letter *c* to be repeated more than different queſtions conſequently may be propoſed, concerning the manner in which the quantities are combined together. For it may either be required to find the number of all the poſſible combinations of a given number of things, by taking them firſt ſingly, then in couplets or parcels of two, and then in triplets or parcels of three, and then in quadruplets or parcels of four, and ſo on according to all the exponents they will admit of; or it may be required only to find the number of all the combinations that may be made of the ſame given number of things according to one, or more, of their exponents, ſeparately; as, for inſtance, by diſpoſing them in parcels of two things, or in parcels of three things, or in parcels of four things, each. In each of theſe ways of combining the things under conſideration, a great variety of queſtions and problems may be propoſed concerning them, the full diſcuſſion of which would lead us into a very ample field of ſpeculation. But of theſe we ſhall only ſelect a few of the moſt curious and important, which we conceive to be neceſſary to the ſolution of the queſtions concerning the doctrine of chances, or the art of forming reaſonable conjectures concerning future events depending on chance, which will be conſidered in the ſubſequent part of this treatiſe.

28. Let it then be required, in the firſt place, to find the number of all the poſſible combinations that can be formed of a given number of things according to all the exponents they will admit of, upon a ſuppoſition that all the things that are to be combined together, are unlike to, and clearly diſtinguiſhable from, each other, and conſequently are denoted by different letters.

Let the things that are to be thus combined be denoted by the ſeveral ſmall letters *a*, *b*, *c*, *d*, *e*, &c. Let theſe

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than once; as is the caſe in the quadruplets *aaab*, *aaac*, *aaaa*, *bbbb*, *aacc*, *accc*, &c.

This is the only meaning that I can find for the foregoing paſſage; but I cannot help entertaining ſome doubt whether it is the true one. *Ideò quære.*

letters be ſet down in ſeparate lines, or rows, one under another, in the manner following.

In the firſt line we muſt place the firſt letter *a*, by itſelf.

In the ſecond line we muſt place the letter *b*; firſt, by itſelf; and then in conjunction with *a*, ſo as to form the combination, or couplet, *a b*, or *b a*. For *a b* and *b a* are, in this doctrine of combinations, to be conſidered as only one combination, becauſe in this doctrine no regard is to be had to the order in which the letters are placed, as there was in the doctrine of permutations.

In the third line we muſt place the third letter *c*; firſt, by itſelf; and then in conjunction with the preceeding letters *a* and *b*, ſo as to form the binions, or couplets, *a c*, *b c*; and, laſtly, with the preceeding couplet *a b*, ſo as to form the triplet *a b c*.

a.

b, ab.

c, ac, bc, abc.

d, ad, bd, cd, abd, acd, bcd, abcd.

e, ae, be, ce, de, abe, ace, bce, ade, bde, cde, abce, abde, acde, bcde, abcde.

In the fourth line we muſt place the fourth letter *d*; firſt, by itſelf; and, 2dly, in conjunction with each of the three preceeding letters *a*, *b*, and *c*, ſo as to form the three pairs or couplets of letters, *a d*, *b d*, and *c d*; and, 3dly, in conjunction with each of the three foregoing couplets, *a b*, *a c*, *b c*, ſo as to form the three triplets *a b d*, *a c d*, and *b c d*; and, 4thly, in conjunction with the foregoing triplet, *a b c*, ſo as to form the quadruplet, *a b c d*.

And in like manner the fifth letter *e* muſt be placed in the beginning of the fifth line; firſt, by itſelf; and, 2dly, in conjunction with each of the four preceeding letters *a*, *b*, *c*, *d*, ſo as to form the four pairs, or couplets, *a e*, *b e*, *c e*, and *d e*; and, 3dly, in conjunction with each of the ſix foregoing couplets, *ab*, *ac*, *bc*, *ad*, *bd*, *cd*, ſo as to form the ſix triplets *abe*, *ace*, *bce*, *ade*, *bde*, and *cde*; and, 4thly, in conjunction with each of the foregoing triplets, *abc*, *abd*, *acd*, *bcd*, ſo as to form the four quadruplets *abce*, *abde*, *acde*, and *bcde*; and, 5thly, in conjunction with the foregoing quadruplet *abcd*, ſo as to form the quintuplet, *abcde*.

And in the ſame manner muſt every following letter, *f*, *g*, *h*, &c, be combined with each of the preceeding letters, and with every preceeding combination of them, if the number of things, or letters, to be combined together, was more than five.

29. And from this manner of combining any given number of things, or letters, together, it is plain that we ſhall thereby obtain all the poſſible combinations of them, ſo that no combination of them whatſoever can be formed, or conceived, that will not be contained in one or other of the ſucceſſive lines, or rows, of quantities ſo generated from each other: and likewiſe it is plain that each of the combinations ſo obtained will be different from every other, or that no combination will occur in the ſaid lines more than once. And conſequently the ſum-total of all the combinations ſet down in the lines, or rows, of quantities ſo formed out of any given number of quantities, will be the number of all the poſſible combinations which the ſaid given number of quantities will admit of. We muſt therefore endeavour to find the number of all the combinations of a given number of quantities that will be contained in an equal number of lines, or rows, of quantities formed, or generated, from each other in the manner above deſcribed.

30. Now, in order to diſcover the number of the combinations contained in a given number of the foregoing lines, or rows, of quantities, it will be proper to obſerve, "that every new line, or row, muſt contain as many combinations as all the preceeding lines, or rows, added together, and one combination over;" and for this reaſon, to wit, becauſe the letter which is at the beginning of every new line, is placed

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there, firſt, by itſelf, and afterwards in conjunction with all the letters and their ſeveral combinations in all the preceeding lines. Thus, for example, the letter *e* is placed in the beginning of the fifth line, firſt by itſelf, and afterwards in conjunction with each of the foregoing letters *a*, *b*, *c*, *d*, and with every combination of them in couplets, triplets, and quadruplets, contained in all the four foregoing lines; and conſequently the number of combinations contained in this fifth line (reckoning the letter *e* by itſelf for one of them) will be equal to the number of all the combinations contained in all the four preceeding lines, beginning with the letters *a*, *b*, *c*, and *d*, and one combination over. This obſervation is of great uſe. For from it we may deduce the whole number of combinations contained in any given number of theſe lines, or which may be made with any given number of letters, by reaſoning in the manner following.

A LEMMA.

31. If in the increaſing geometrical progreſſion 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c, of which the firſt term is 1, and the common ratio is that of 1 to 2, we take any number of terms whatever, as, for example, n terms, and call the ſum of the ſaid n terms S, and afterwards add another term to the ſaid ſeries, the ſaid new term will be equal to S + 1, or to the ſum of all the former n terms together with an unit.

DEMONSTRATION.

Since the terms in the ſeries 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c, increaſe continually in the proportion of 1 to 2, it is evident that all the terms of it after the firſt term 1 will be the ſeveral powers of 2 in their natural order, to wit, $\overline{2}|^{1}$, $\overline{2}|^{2}$, $\overline{2}|^{3}$, $\overline{2}|^{4}$, $\overline{2}|^{5}$, $\overline{2}|^{6}$, $\overline{2}|^{7}$, &c, and conſequently that the *n*th, or laſt, term of it will be $\overline{2}|^{n-1}$. If therefore to theſe *n* terms we add another term, the ſaid new term will be $= 2 \times \overline{2}|^{n-1}$, or $\overline{2}|^{n}$. We are therefore to ſhew that $\overline{2}|^{n}$ is $= S + 1$.

Now, ſince S is $= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 +$ &c $+ \overline{2}|^{n-1}$, $2S$ will be $= 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 +$ &c $+ \overline{2}|^{n}$; which ſeries conſiſts of *n* terms as well as the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 +$ &c $+ \overline{2}|^{n-1}$, or S. Therefore (adding 1 to both ſides) $2S + 1$ will be equal to the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 +$ &c $+ \overline{2}|^{n}$, that is, to the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 +$ &c $+ \overline{2}|^{n-1}$ together with the new term $\overline{2}|^{n}$, or to $S + \overline{2}|^{n}$. Therefore (ſubtracting the ſeries S from both ſides) the new term $\overline{2}|^{n}$ will be equal to $S + 1$, or to the ſum of all the *n* terms of the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 +$ &c $+ \overline{2}|^{n-1}$, together with 1. Q. E. D.

COROLL. 1. It follows therefore that the ſecond and third and other following terms of the increaſing geometrical progreſſion $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 +$ &c, may be generated, or derived, from the firſt term 1, not only by doubling it continually, but by the application of the property that has juſt now been ſhewn to belong to the terms of ſuch a ſeries, to wit, by adding together all the preceeding terms, and increaſing their ſum by an unit. Thus, $1 + 1$ will be $= 2$, which is the ſecond term; and $1 + 2 + 1$, will be $(= 3 + 1) = 4$, which is the third term; and $1 + 2 + 4 + 1$, will be $(= 3 + 4 + 1 = 7 + 1) = 8$, which is the fourth term; and $1 + 2 + 4 + 8 + 1$ will be $(= 7 + 8 + 1 = 15 + 1) = 16$, which is the fifth term; and $1 + 2 + 4 + 8 + 16 + 1$ will be $(= 15 + 16 + 1 = 31 + 1) = 32$, which is the ſixth term; and $1 +$

2 + 4 + 8 + 16 + 32 + 1 will be (= 31 + 32 + 1 = 63 + 1) = 64, which is the ſeventh term; and 1 + 2 + 4 + 8 + 16 + 32 + 64 + 1 will be (= 63 + 64 + 1 = 127 + 1) = 128, which is the eighth term. And in the ſame manner may all the following terms of this progreſſion, however numerous, be generated from thoſe that preceed them by means of the aforeſaid property.

Coroll. 2. And hence it follows, *è converſo*, that, if there be a ſeries of terms beginning from 1, the terms of which are generated one from the other by means of the foregoing property, or by adding together all the preceeding terms, and increaſing their ſum by an unit, the ſaid ſeries will be the geometrical progreſſion 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c.

32. Now it has been ſhewn above in art. 28, that the numbers of quantities, or combinations, contained in the above-mentioned lines, or rows, beginning with the letters *a*, *b*, *c*, *d*, *e*, &c, are generated from the firſt quantity *a*, and from each other in the manner juſt now deſcribed, or that the number of quantities in every new line is equal to the ſum of the numbers of all the quantities in all the preceeding lines, together with an unit. It follows therefore, from coroll. 2, of the foregoing lemma, that the numbers of quantities contained in the ſaid ſeveral lines muſt conſtitute the geometrical progreſſion 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c, or 1 + $\overline{2}|^{1}$, + $\overline{2}|^{2}$, + $\overline{2}|^{3}$, + $\overline{2}|^{4}$, + $\overline{2}|^{5}$, + $\overline{2}|^{6}$, + $\overline{2}|^{7}$, + &c + $\overline{2}|^{n-1}$, ſuppoſing the number of lines to be *n*. Therefore the ſum of the numbers of quantities contained in all the *n* lines beginning with the letters *a*, *b*, *c*, *d*, *e*, &c, will be equal to the ſum of the firſt *n* terms of the increaſing geometrical progreſſion 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c, or to the ſeries 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c + $\overline{2}|^{n-1}$, or 1 + $\overline{2}|^{1}$, + $\overline{2}|^{2}$, + $\overline{2}|^{3}$, + $\overline{2}|^{4}$, + $\overline{2}|^{5}$, + $\overline{2}|^{6}$, + $\overline{2}|^{7}$, + &c + $\overline{2}|^{n-1}$. But, by the foregoing lemma, this ſeries together with an unit is

equal to $\overline{2}|^{n}$. Therefore this ſeries alone is equal to $\overline{2}|^{n} - 1$. And conſequently the number of all the quantities contained in all the ſaid n lines, or the number of all the poſſible combinations of the n letters a, b, c, d, e, &c, (reckoning the ſaid letters, when taken ſingly, among the ſaid combinations) will be $= \overline{2}|^{n} - 1$. And hence ariſes the following

Rule for finding the number of all the poſſible different combinations of a given number of things according to all their different exponents.

33. Raiſe the number 2 to the power of which n or the given number of things, that are to be combined together, is the index; and ſubtract 1 from the ſaid power. The remainder $\overline{2}|^{n} - 1$ will be the number of combinations that was required,

34. Coroll. 1. From this rule it follows, in the firſt place, that, if we conſider the total omiſſion of all the n letters as one way of combining them, the number of all the poſſible combinations will be greater than it was before by an unit, and therefore will be $= \overline{2}|^{n}$; and it follows, in the ſecond place, that, if we exclude this caſe of the omiſſion of all the letters (which may be called the *nullion*, or the combination of them by *nones*), and likewiſe exclude the ſeveral letters when taken ſingly, or ſeparately, (which are not in ſtrictneſs *combinations* of them), the number of the remaining combinations of n different things, or letters, in binions and ternions and quaternions, or in couplets, and triplets, and quadruplets, and in parcels of more than four letters in each, will be $\overline{2}|^{n} - 1 - n$, or $\overline{2}|^{n} - n - 1$.

Thus, for example, the number of all the different conjunctions, or combinations, that may be made of the ſeven planets, to wit, Saturn, Jupiter, Mars, Venus, Mercury, the Earth, and the Moon, (taking the word *combinations* in the extent given to it in the foregoing rule) will be $= \overline{2}|^{7} - 1$ ($= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 - 1 = 128 - 1$) $= 127$; from which if we ſubtract 7, which is the number of the planets taken ſingly, or ſeparately, (in which caſes there are not properly any conjunctions of them), the remaining number $\overline{2}|^{7} - 1 - 7$, or $127 - 7$, or 120, will be the number of all the poſſible conjunctions, or combinations, of them by conjoining two together, or three together, or four together, or five together, or ſix together, or all the ſeven together; or, it will be the number of all the poſſible conjunctions of them properly ſo called. And the twelve regiſters, as they are called, or rows of pipes in a muſical organ, by means of which the ſound of it is made to change ſo remarkably from a ſoft and gentle ſound to a very loud and ſolemn one, may be made to undergo $\overline{2}|^{12} - 1$, or $4096 - 1$, or 4095 combinations, or variations.

35. Corroll. 2. If we examine the number of combinations of the letters a, b, c, d, and e, in the ſeveral lines, or rows of quantities, ſet down above in art. 28, we ſhall find that the number of combinations that have even numbers for their exponents contained in each of the ſaid lines after the firſt line (which contains only the ſingle quantity a) is equal to the number of combinations that have odd numbers for their exponents contained in the ſame line. Thus, in the ſecond line, which begins with the letter b, there is one quantity, namely b, with an odd number, to wit, 1, for its exponent, and one quantity, namely, ab, with an even number, to wit, 2, for its exponent. And in the third line beginning with the letter c, there are two quantities, namely, c and abc, with odd numbers, to wit, 1 and 3, for their exponents, and two quantities, namely, ac and bc, with an even number, to wit, 2, for their exponent. And in the fourth line beginning with the letter d, there are four quantities, namely, d, abd, acd, and bcd, which have the odd numbers,

numbers 1 and 3 for their exponents; and there are four other quantities, namely, *a d*, *b d*, *c d*, and *a b c d*, which have the even numbers 2 and 4 for their exponents. And in the fifth line, beginning with the letter *e*, there are eight quantities, namely, *e*, *a b e*, *a c e*, *b c e*, *a d e*, *b d e*, *c d e*, and *a b c d e*, which have the odd numbers 1, 3, and 5 for their exponents; and eight other quantities, namely, *a e*, *b e*, *c e*, *d e*, and *a b c e*, *a b d e*, *a c d e*, *b c d e*, which have the even numbers 2 and 4 for their exponents. And the ſame thing muſt happen in the ſixth line of quantities beginning with the letter *f*, and in every following line, becauſe every new line is formed by ſetting down the new letter firſt by itſelf, and then combining it with the firſt letter *a*, and afterwards with all the quantities contained in the ſecond, third, and other following lines that preceed the new line. Now the combination of the new letter with each of the quantities in the ſecond, third, and other following lines, will turn all the quantities that have odd numbers for their exponents into quantities that have even numbers for their exponents, and all the quantities that have even numbers for their exponents into quantities that have odd numbers for their exponents. And therefore, as the number of quantities with odd numbers for their exponents in each of the ſaid ſecond, third, and other following lines, was equal to the number of quantities with even numbers for their exponents, it follows that of the new quantities in the new line ariſing from the combination of the new letter with all the quantities contained in the ſecond, third, and other following lines, there will be as many that have odd numbers for their exponents as there will be that have even numbers for their exponents. And, if we add to theſe quantities the new letter itſelf, which is to be placed in the beginning of the new line, and of which the exponent is 1, and the combination of the new letter with the firſt letter *a*, of which combination the exponent is the even number 2, whereby we ſhall obtain all the quantities ſet down in the new line, it is evident that the addition of theſe two quantities (of which the firſt has the odd number 1, and the ſecond has the even number 2, for its exponent) will not alter the equality of the numbers of combinations,

binations, or quantities, of each kind, but that the number of quantities in the new line that have odd numbers for their exponents will ſtill be equal to the number of quantities in the ſame line that have even numbers for their exponents.

36. Coroll. 3. If therefore we add all the quantities contained in all the lines except the firſt line (which contains only the ſingle quantity a) together, it is evident that the number of quantities in ſuch ſum that will have odd numbers for their exponents will be equal to the number of quantities that will have even numbers for their exponents.

37. Coroll. 4. And, if we add together all the quantities contained in all the lines, including the firſt line, which contains the ſingle quantity a (the exponent of which is the odd number 1), the number of quantities in ſuch ſum that will have odd numbers for their exponents will exceed by an unit the number of quantities that will have even numbers for their exponents.

38. Coroll. 5. And, if to all the quantities contained in all the lines together we add, as another combination, the caſe denoted by a cypher 0, or the caſe of the omiſſion of all the letters, which we have above in art. 34, called the combination by *nones*, or the *nullion*, and conſider 0, or the exponent of this combination as an even number, the number of quantities in the ſaid ſum that will have odd numbers for their exponents will be exactly equal to the number of quantities that will have even numbers for their exponents.

39. Coroll. 6. It has been ſhewn in Coroll. 1, that the number of all the quantities in all the n lines taken together, and the caſe of the nullion is equal to $\overline{2}|^{n}$. It follows therefore from Coroll. 5, that the number of quantities in this ſum that will have odd numbers for their exponents will be equal to half of $\overline{2}|^{n}$, or to $\frac{\overline{2}|^{n}}{2}$, or $\overline{2}|^{n-1}$, and the number of

quantities in the ſame ſum that will have even numbers for their exponents, including the nullion, will likewiſe be $\overline{2}|^{n-1}$; and conſequently the number of quantities in the ſaid ſum that will have even numbers for their exponents, without including the nullion, will be $\overline{2}|^{n-1} - 1$. The ſame thing will be demonſtrated in another manner here below in the 6th Corollary of Chapter 4.

CHAPTER III.

Of the numbers of combinations that may be made of a given number of things in parcels conſiſting of two things, or of three things, or of four things, or of any other particular number of things, each; and of the numbers known by the name of the figurate numbers, *and their properties, with which the inveſtigation of the ſaid combinations is connected.*

40. FROM an attentive conſideration of the five lines, or rows, of quantities in art. 28, of the foregoing chapter, beginning with the letters *a*, *b*, *c*, *d*, and *e*, and which exhibit all the different combinations that can be formed out of thoſe five letters, it will be evident, that the *binions*, or *couplets*, or parcels conſiſting of two letters, in every new line of quantities, are formed by combining the ſingle letter which is placed in the beginning of ſuch new line, with each of the ſingle letters contained in all the foregoing lines of quantities; and that the *ternions* or *triplets*, or parcels conſiſting of three letters, in ſuch new line, are formed by combining the ſingle letter which is placed in the beginning of ſuch new line, with each of the *binions* or *couplets*, contained in all the foregoing lines; and the *quaternions*, or *quadruplets*, or parcels conſiſting of four letters,

in

in ſuch new line, are formed by combining the ſingle letter which is placed in the beginning of ſuch new line, with each of the *ternions*, or *triplets*, contained in all the foregoing lines; and, in like manner, in all higher combinations than *quaternions*, the combinations denoted by any exponent *m* in ſuch new line, are formed by combining the ſaid ſingle letter which is placed in the beginning of ſuch new line, with all the combinations denoted by the next lower exponent $m - 1$ contained in all the foregoing lines. It follows therefore that the number of *binions*, or *couplets*, of letters in every new line will be equal to the number of all the ſingle letters in all the foregoing lines taken together; and that the number of ternions, or triplets, of letters in ſuch new line, will be equal to the number of all the binions, or couplets, of letters in all the foregoing lines taken together; and the number of quaternions, or quadruplets, of letters in ſuch new line, will be equal to the number of all the ternions, or triplets, of letters in all the foregoing lines taken together; and in like manner, that the number of combinations of any higher order than quaternions, denoted by the exponent *m*, in ſuch new line, will be equal to the number of combinations of the next lower order, which is denoted by the exponent $m - 1$, in all the foregoing lines taken together. From theſe obſervations we may derive the following concluſions:

41. *Firſt Concluſion.* As there is only one ſingle letter in each of the ſaid lines, or rows, of quantities, to wit, the letter in the beginning of the line, the ſingle letters in all the lines ſucceſſively will exhibit a ſet of units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, &c, which are the figurate numbers of the firſt order.

42. *Second Concluſion.* As there is no binion, or couplet, of letters in the firſt line (which contains only the letter *a*), and there is only one binion in the ſecond line, to wit, *a b*; and two binions in the third line, to wit, *ac*, *bc*; and three binions in the fourth line, to wit, *ad*, *bd*, *cd*; and, in general, as the number of binions in every new line is

equal to the number of the ſingle letters in all the preceeding lines taken together; it follows that the numbers of binions, or couplets, of letters in the firſt, ſecond, third, and fourth, and other following lines, will be 0, 1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1; &c, or 0, 1, 2, 3, 4, 5, &c, or the ſeries of numbers 1, 2, 3, 4, 5, &c, in their natural order, with a cypher, 0, prefixed to them.

Theſe numbers form an arithmetical progreſſion, in which the common difference of the terms is 1; and they are often called *the natural numbers*, or a ſeries of *lateral* numbers, or the figurate numbers of the ſecond order.

43. *Third Concluſion.* As there are no *ternions*, or *triplets*, of letters in the two firſt lines; and there is one ternion, or triplet, to wit, *a b c*, in the third line; and 1 + 2, or 3 ternions, to wit, *a b d*, *a c d*, and *b c d*, in the fourth line; and 1+2+3, or 6 ternions, to wit, *a b e*, *a c e*, *b c e*, *a d e*, *b d e*, *c d e*, in the fifth line; and, in general, there are as many ternions in every new line as there are binions in all the foregoing lines together; it follows that the numbers of ternions, or triplets, of letters in the firſt, ſecond, third, fourth, fifth, and other following lines, of rows, of quantities, will be 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, which are formed by the continual addition of the numbers of the binions contained in the ſaid lines, or of the terms of the ſeries 0, 1, 2, 3, 4, 5, 6, 7, &c, with a new cypher, 0, prefixed to them.

Theſe numbers 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, are often called the *trigonal*, or *triangular*, numbers, or the figurate numbers of the third order.

44. *Fourth Concluſion.* As there are no quaternions, or quadruplets, in the three firſt lines; and there is one quaternion, or quadruplet, to wit, *a b c d*, in the fourth line; and 1+3, or 4, quaternions, to wit, *a b c e*, *a b d e*, *a c d e*, and *b c d e*, in the fifth line; and as, in general, there are as many quaternions in every new line as there are ternions in all the foregoing lines together; it follows that the number of quaternions, or quadruplets, of letters in the firſt, ſecond,

third, fourth, fifth, and other following lines, or rows, of quantities, will be 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, reſpectively; which numbers are formed by the continual addition of the ternions, or triplets, contained in the ſaid lines, or of the terms of the laſt preceeding ſeries 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, with a new cypher, 0, prefixed to them.

Theſe numbers 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, are often called the *pyramidal numbers*, or the figurate numbers of the fourth order.

45. *Fifth Concluſion.* In like manner the numbers of the quinions, or quintuplets, of letters contained in the ſeveral ſucceſſive lines, or rows, of quantities beginning with the letters *a*, *b*, *c*, *d*, *e*, &c, will be 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, reſpectively; which are formed by the continual addition of the quaternions, or quadruplets, contained in the ſaid lines, or of the terms of the laſt preceeding ſeries 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, with a new cypher, 0, prefixed to them.

Theſe numbers 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, are often called the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, or the figurate numbers of the fifth order. And they are alſo ſometimes called the *triangulo-triangular*, or *trigono-trigonal*, numbers.

46. *Sixth Concluſion.* And, in like manner, the numbers of the *ſenions*, or *ſextuplets*, of letters contained in the ſaid ſeveral ſucceſſive lines, or rows, of quantities beginning with the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, &c, will be 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, reſpectively; which are formed by the continual addition of the quinions, or quintuplets, contained in the ſaid lines, or of the terms of the laſt preceeding ſeries 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, with a new cypher, 0, prefixed to them.

Theſe numbers 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, are often called the *pyramido-pyramidal* numbers, and ſometimes the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, or the figurate numbers of the ſixth order.

47. And in the ſame manner the numbers of the *ſeptenaries*, or *ſeptuplets*, and *octonaries*, or *octuplets*, and other higher combinations of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, &c, contained in the ſaid ſeveral ſucceſſive lines, or rows, of quantities beginning with the ſaid letters reſpectively, will form the ſeventh and eighth and other following higher orders of the figurate numbers reſpectively.

48. And thus we have unexpectedly been led by the conſideration of the nature of combinations to the contemplation of the *figurate* numbers, or of the numbers that are formed from a ſeries of equal numbers, or units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c, by the continual addition of all the terms, and by the like addition of all the terms of every following ſeries ſo obtained. For theſe are the numbers to which arithmeticians have given the name of the *figurate numbers*.

49. In order to repreſent the ſeveral orders, or ſerieſes, of theſe figurate numbers in one view, and thereby to render what I have further to obſerve concerning them more eaſy to be apprehended, I have ſubjoined the following table of them, containing the firſt twelve terms of the firſt twelve orders, or ſerieſes, of the ſaid numbers; which the reader, if he chooſes it, may eaſily continue to a greater extent, both downwards, or towards the bottom of the page, and ſideways towards the right hand. In this table the Indian, or Arabian, figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (that are placed on the left ſide of the table in a direction parallel to the ſide of the page, and ſeparated from the table by a double black line) expreſs the places, or numbers, of the ſeveral horizontal rows of numbers to which they are adjacent reſpectively, and alſo the numbers of letters, or things that are to be combined together. And the capital Roman figures I, II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII, (that are placed directly over the table, parallel to the top of the page) expreſs the places, or numbers, of the ſeveral vertical columns, and are likewiſe the exponents of the combinations of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, &c,

which are repreſented by the ſaid vertical columns reſpectively. And the ſaid vertical columns themſelves are the ſeveral orders, or ſerieſes, of figurate numbers, or of the ſeveral combinations of the firſt, ſecond, third, and other following orders, as far as the twelfth order, of which the Roman numerals I, II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII, at the top of the table, are the exponents, with the proper number of cyphers, 0, prefixed to them. Thus, the firſt vertical column on the left-hand ſide of the table, under the Roman numeral I, is a ſeries of units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, or the firſt order of figurate numbers, which Dr. Wallis calls *Monadicks*, and repreſents the numbers of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, that occur ſingly, or without being joined with any other letter, in the ſeveral lines, or rows, of quantities that are ſet down in chap. 2, art. 28, and which are ſuppoſed to be continued to the twelfth line; and the ſecond vertical column on the left-hand ſide of the table, under the Roman numeral II, is the ſeries 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or the ſecond order of figurate numbers called *the natural*, or *lateral*, numbers *, with a cypher, 0, prefixed to them, and repreſents the numbers of binions, or couplets, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by two in a parcel, that occur in the ſaid twelve ſucceſſive lines, or rows, of quantities; and the third vertical column, under the Roman numeral III, is the ſeries 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, or the third order of figurate numbers, called the *trigonal*, or *triangular*, numbers, with two cyphers prefixed to them, and repreſents the numbers of ternions, or triplets, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by three in a parcel, that occur in the ſaid twelve ſucceſſive lines: and the fourth vertical column, under the Roman numeral IV, is the ſeries 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, 120, 165, or the fourth order of figurate numbers (called the *pyramidal numbers*) with three cyphers prefixed to them, and repreſents the numbers of quaternions, or quadruplets, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by four in a parcel, that occur in the ſaid twelve ſucceſſive lines: and the fifth vertical column, under the Roman numeral V, is the ſeries 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 252, 462, or the fifth order of figurate numbers (called the *triangulo-pyramidal*, or *trigono-pyramidal*, *numbers*) with four cyphers prefixed to them, and repreſents the numbers of quintuplets, or quinions, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by five in a parcel, that occur in the ſaid twelve ſucceſſive lines: and the ſixth vertical column, under the Roman numeral VI, is the ſeries 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, or the ſixth order of figurate numbers (called the *pyramido-pyramidal numbers*) with five cyphers prefixed to them, and repreſents the numbers of ſextuplets, or ſenions, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by ſix in a parcel, that occur in the ſaid twelve ſucceſſive lines *. And, in like manner, the ſix following vertical columns, under the Roman numerals VII, VIII, IX, X, XI, and XII, contain the ſeventh, eighth, ninth, tenth, eleventh, and twelfth orders of figurate numbers, with ſix, ſeven, eight, nine, ten, and eleven, cyphers prefixed to them reſpectively, and repreſent the numbers of ſeptuplets, octuplets, noncuplets, decuplets, undecuplets, and duodecuplets, or of ſeptenions, octonions, novenions, denions, undenions, and duodenions, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, reſpectively, or combinations of them by ſeven in a parcel, and by eight in a parcel, and by nine in a parcel, and by ten in a parcel, and by eleven in a parcel, and by twelve in a parcel, that occur in the ſaid twelve ſucceſſive lines.

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* See Dr. John Wallis's *Diſcourſe of Combinations, Alterations, and Aliquot Parts*, bound up with his Algebra, page 109.

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* Dr. John Wallis, in his *Diſcourſe of Combinations, Alternations, and Aliquot Parts*, bound up with his Algebra, page 109, and Mr. Nicholas Mercator, in his *Logarithmotechnia*, publiſhed in the firſt volume of this collection of tracts, called *Scriptores Logarithmici*, page 178, call the 5th order of figurate numbers, to wit, 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, 330, &c, the *triangulo-triangular*, or *trigono-trigonal*, numbers, and the 6th order of figurate numbers, to wit, 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, inſtead of calling the former the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, and the latter the *pyramido-pyramidal* numbers, as our author calls them. So that there appears to be a variation amongſt different writers on this ſubject with reſpect to the names to be given to the figurate numbers of the 5th and 6th, and other higher orders. And therefore, to avoid ambiguity, it ſeems to be moſt convenient to denote the figurate numbers of the 5th and 6th, and all higher, orders, only by the numbers or exponents, of their orders, calling them the figurate numbers of the fifth, and the 6th, and the 7th, and the 8th, and other following higher orders.

A Table of the firſt twelve orders, or ſerieſes, of figurate numbers, or of all the different combinations that may be made of twelve different letters a, b, c, d, e, f, g, h, i, k, l, *and* m, *by taking them, firſt, ſingly, and then combining them with each other in parcels conſiſting of two letters, of three letters, of four letters, of five letters, of ſix letters, of ſeven letters, of eight letters, of nine letters, of ten letters, of eleven letters, and of twelve letters.*

The Roman numerals at the top of the table are the exponents of the combinations of the letters a, b, c, d, e, f, g, h, i, k, l, m, *exhibited by the ſeveral vertical columns in it.*

lett.	*num*	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
a.	1.	1	0	0	0	0	0	0	0	0	0	0	0
b.	2.	1	1	0	0	0	0	0	0	0	0	0	0
c.	3.	1	2	1	0	0	0	0	0	0	0	0	0
d.	4.	1	3	3	1	0	0	0	0	0	0	0	0
e.	5.	1	4	6	4	1	0	0	0	0	0	0	0
f.	6.	1	5	10	10	5	1	0	0	0	0	0	0
g.	7.	1	6	15	20	15	6	1	0	0	0	0	0
h.	8.	1	7	21	35	35	21	7	1	0	0	0	0
i.	9.	1	8	28	56	70	56	28	8	1	0	0	0
k.	10.	1	9	36	84	126	126	84	36	9	1	0	0
l.	11.	1	10	45	120	210	252	210	120	45	10	1	0
m.	12.	1	11	55	165	330	462	462	330	165	55	11	1

The firſt vertical column on the left hand contains the letters that are to be combined together.

The ſecond column expreſſes the numbers of letters that are to be combined together.

50. The properties of the numbers exhibited in the foregoing table are truly curious and ſurpriſing. For it not only contains in it (as we have ſeen in the foregoing pages) the clue to the myſterious doctrine of combinations, but it is alſo the ground, or foundation, of moſt of the important and abſtruſe diſcoveries that have been made in the other branches of the mathematicks, as is well known to thoſe perſons who are ſkilled in the higher parts of geometry. We ſhall here give a ſlight ſketch, and but a ſlight one, of ſome of the ſaid properties, without a formal demonſtration of any of them except the twelfth and laſt, which is that which is moſt immediately connected with the ſubject of combinations which we are here inquiring into; the other eleven properties being either eaſy conſequences of the ſaid 12th property, or being ſufficiently evident from the manner in which the foregoing table was conſtructed, or the ſeveral orders of figurate numbers were generated from each other.

Some wonderful properties of the foregoing table of combinations.

51. The firſt property. The ſecond of the vertical columns of numbers in the ſaid table, or that which is placed under the exponent II, begins with one cypher; the third column begins with two cyphers; the fourth column with three cyphers; and, in general, every column with as many cyphers, wanting one, as there are units in the exponent of the combinations repreſented by it, ſo that, if the exponent

of the column is *c*, the number of cyphers in the beginning of the column will be $c-1$.

This property is too evident to need, or, perhaps, to admit of, any proof.

52. The ſecond property. The firſt ſignificant terms of the ſeveral vertical columns, taken in their order in a ſlanting line downwards from the top of the table on the left hand to the bottom of it on the right hand, are the ſame with the ſignificant terms of the firſt vertical column; and the ſecond ſignificant terms of the ſeveral vertical columns, taken in the ſame manner, are the ſame with the ſignificant terms of the ſecond vertical column; and the third ſignificant terms of the ſeveral vertical columns, taken in the ſame manner, are the ſame with the ſignificant terms of the third vertical column; and, in like manner, the fourth, fifth, ſixth, and other following, ſignificant terms of the ſeveral vertical columns are the ſame with the ſignificant terms of the fourth, fifth, ſixth, and other following, vertical columns, reſpectively: ſo that the firſt of thoſe oblique lines of terms conſtitutes a ſeries of units, or 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, or the firſt order of figurate numbers; and the ſecond of thoſe oblique lines of terms conſtitutes a ſeries of the natural, or lateral, numbers, or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or the ſecond order of figurate numbers; and the third of thoſe oblique lines of terms conſtitutes a ſeries of the trigonal or triangular numbers, to wit, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, or the third order of figurate numbers; and, in like manner, the fourth, fifth, ſixth, and other following oblique lines of terms, conſtitute the fourth, fifth, ſixth, and other following orders of figurate numbers reſpectively.

53. The third property. The ſecond term in every vertical column, beginning from 1, is the ſame number as the exponent of the combinations exhibited by the ſaid column, or as the number denoted by a Roman numeral at the top of the column, which denotes its place or order in the table. Thus, the ſecond term of the fourth vertical column 1, 4, 10, 20, 35, 56, 84, 120, 165, reckoning from 1, is 4;

and the ſecond term of the fifth vertical column is 5, and the ſecond term of the ſixth vertical column is 6; and the ſame thing may be obſerved in all the following columns.

This property is too evident to need a proof.

54. The fourth property. Every term in the foregoing table is equal to the ſum of all the terms that ſtand above it in the next preceeding vertical column. Thus, for example, 56 (which is the ſixth ſignificant term in the fourth vertical column) is equal to the ſum of 1, 3, 6, 10, 15, and 21, which are the firſt ſix ſignificant terms of the next preceeding, or third, vertical column, and which all ſtand above the ſaid term 56, or above the term 28 in the third vertical column, which ſtands even with the ſaid term 56 in the fourth column.

This property is manifeſt from art. 40, 41, 42, &c. —— 48.

55. The fifth property. Every term in the table, after the firſt term 1 in each horizontal row of terms, is equal to the ſum of the two terms that ſtand immediately above it in the ſame vertical column and in the next preceeding vertical column. Thus, for example, 56 (which is the 9th term in the fourth vertical column, including the cyphers, or the ſixth term excluſive of the cyphers) is equal to the ſum of 35, which is the term next above 56 in the ſame vertical column, and 21, which is the term next above 56 in the next preceeding or third vertical column. And, from the manner in which the table is formed, the ſame thing is evident of every other term in the table.

56. The ſixth property. The terms of every tranſverſe, or horizontal, column increaſe gradually from 1 to a certain magnitude, and then decreaſe again by the ſame degrees to 1, ſo as to make the terms that are equidiſtant from the beginning and the end of the column be equal to each other. Thus, for example, the terms of the 7th tranſverſe, or horizontal column are 1, 6, 15, 20, 15, 6, 1, in which the firſt and the laſt term are, both of them, 1, the ſecond

term and the laſt term but one, are, both of them, 6, the third term and the laſt term but two are, both of them, 15; and the middle term is 20, which is greater than any of the others. And the ſame thing may be obſerved in all the other tranſverſe or horizontal columns.

57. This property may be ſhewn to be general or to extend to all the tranſverſe, or horizontal columns, how numerous ſoever, to which we may ſuppoſe the table to be continued, as well as to the twelve columns ſet down in the foregoing table, by proving that, if it is true in any one horizontal column, (as we have ſeen that it is in all the horizontal columns ſet down in the foregoing table), it will alſo be true in the next following horizontal column. Now this may be proved in the manner following.

It appears from the fifth property already mentioned, that the ſecond and other following terms of every new tranſverſe, or horizontal, column are equal to the ſums of the two terms that ſtand immediately above them in the ſame vertical column, and in the next preceeding vertical column, reſpectively, or to the ſucceſſive ſums of the terms of the next preceeding horizontal column, taken two by two. Thus, for example, the ſecond, third, and other following terms of the 8th horizontal column are 7, 21, 35, 35, 21, 7, and 1; of which 7 is = 1 + 6, or the ſum of the firſt and ſecond term of the next preceeding, or 7th horizontal column 1, 6, 15, 20, 15, 6, 1; and 21 is = 6 + 15, or the ſum of the ſecond and third terms of the ſaid ſeventh horizontal column; and 35 is = 15 + 20, or the ſum of the third and fourth terms of the ſaid ſeventh horizontal column; and the ſecond 35 is = 20 + 15, or the ſum of the fourth and fifth terms of the ſaid ſeventh horizontal column; and the ſecond 21 is = 15 + 6, or the ſum of the fifth and ſixth terms of the ſaid ſeventh horizontal column; and the ſecond 7 is = 6 + 1, or the ſum of the ſixth and ſeventh terms of the ſaid ſeventh horizontal column; and the ſecond 1 is = 1 + 0, or the ſum of the ſeventh and eighth terms of the ſaid 7th horizontal column. So that the terms of the ſaid 8th horizontal column, 1, 7, 21, 35, 35, 21, 7, 1, may be derived from the terms of the next preceeding, or 7th, horizontal column, 1, 6, 15, 20, 15, 6, 1, by ſetting down the latter terms twice following in two parallel horizontal rows, with the terms in the lower row advanced one ſtep to the right hand beyond thoſe in the upper row, ſo that the firſt term of the ſecond row ſhall be immediately under the ſecond term of the firſt row, and the ſecond term of the ſecond row under the third term of the firſt row, and the third term of the ſecond row under the fourth term of the firſt row, and every following term of the ſecond row under the next higher term of the firſt row, and then adding the correſpondent terms, or the terms which ſtand in the ſame vertical lines, together, in the manner following.

$$\begin{array}{cccccccc} 1, & 6, & 15, & 20, & 15, & 6, & 1 & \\ & 1, & 6, & 15, & 20, & 15, & 6, & 1 \\ \hline 1, & 7, & 21, & 35, & 35, & 21, & 7, & 1 \end{array}$$

Now from this manner of deriving the terms of the 8th horizontal row of numbers from the 7th horizontal row of numbers, it is manifeſt that, ſince the terms of the ſeventh row that are equidiſtant from the two extreme terms 1 and 1, are equal to each other, the terms of the 8th row which are equidiſtant from the two extreme terms 1 and 1, muſt likewiſe be equal to each other, being the ſums of equal numbers that are added together in an oppoſite order, to wit, 6 + 1 and 1 + 6, 15 + 6 and 6 + 15, and 20 + 15 and 15 + 20. And this method of reaſoning will prove in like manner that, ſince the terms of the 8th horizontal row of numbers that are equidiſtant from the two extreme terms 1 and 1 are equal to each other, the terms of the 9th horizontal row of numbers that are equidiſtant from the two extreme terms 1 and 1 will alſo be equal to each other; and conſequently that, to whatever extent the table be ſuppoſed to be continued, the terms of every following horizontal row of numbers that are equidiſtant from the two extreme terms 1 and 1 will be equal to each other.

Q. E. D.

58. The ſeventh property. If we take a certain number of vertical columns of numbers in the foregoing table, and continue the terms in each column till they are as many as there are columns, and then add up the ſeveral numbers in each column, and place the ſums thereby obtained in a new horizontal line, or row, at the bottom of the ſaid columns, the firſt of theſe ſums will be equal to the laſt but one, the ſecond of them to the laſt but two, the third of them to the laſt but three, and in general, the *m*th term to the laſt but *m*. Thus, for example, if we take the firſt eight vertical columns, and continue them to eight terms each (including the cyphers in the beginning of all but the firſt column), and then add the numbers in each ſeparate column into one ſum, the ſums thereby obtained will be 8, 28, 56, 70, 56, 28, 8, and 1; of which the firſt ſum 8 is equal to the laſt but one, the ſecond ſum 28 is equal to the laſt but two, and the third ſum 56 is equal to the laſt but three, and the fourth term 70 is itſelf the laſt term but four.

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	2	1	0	0	0	0	0
1	3	3	1	0	0	0	0
1	4	6	4	1	0	0	0
1	5	10	10	5	1	0	0
1	6	15	20	15	6	1	0
1	7	21	35	35	21	7	1
8	28	56	70	56	28	8	1

59. This property follows from the fourth property, which is mentioned above in art. 54. For, by that property, each of the ſaid ſums is equal to the next following term of the next horizontal row of numbers, that is, in the example here given, of the ninth horizontal row of numbers in the foregoing table, which is 1, 8, 28, 56, 70, 56, 28, 8, and 1; to wit, the ſum of the units in the firſt vertical column is equal to the ſecond term 8 of the ſaid ninth horizontal row of numbers, and the ſum of the numbers 1, 2, 3, 4, 5, 6, 7, in the ſecond vertical column is equal to the third

term 28 of the ſaid ninth horizontal row of numbers; and the ſum of the numbers 1, 3, 6, 10, 15, 21, in the third vertical column is equal to the fourth term 56 of the ſaid ninth horizontal row of numbers; and in like manner, the ſums of the numbers 1, 4, 10, 20, 35, in the fourth vertical column, and of the numbers 1, 5, 15, 35, in the fifth vertical column, and of the numbers in the ſixth, ſeventh, and eighth vertical columns, are reſpectively equal to the numbers 70, 56, 28, 8, and 1, of the fifth, ſixth, ſeventh, eighth, and ninth, terms of the ſaid ninth horizontal row of numbers. But, by the ſixth property (which has been mentioned above in art. 56, and demonſtrated in art. 57) the terms of the ſaid ninth horizontal row of numbers, 1, 8, 28, 56, 70, 56, 28, 8, and 1, that are equidiſtant from the two extreme terms 1 and 1, are equal to each other, and muſt be ſo from the manner in which they are generated. Therefore the ſaid ſums of the numbers contained in the ſaid eight vertical columns, being equal to the ſecond, third, fourth, and other following terms of the ſaid ninth horizontal row of numbers, muſt be ſuch that, if an unit be prefixed to them (whereby their number will be increaſed to nine terms), the terms that are equidiſtant from the two extreme terms 1 and 1 will be equal to each other. And conſequently, if an unit be not prefixed to them, the firſt of thoſe ſums will be equal to the laſt of them but one, and the ſecond of them will be equal to the laſt but two, and the third of them will be equal to the laſt but three, and the mth of them will be equal to the laſt but m. And this, it is evident, will be true, if inſtead of eight vertical columns continued to eight terms each, we were to take any other number of vertical columns, how great ſoever, and continue them till the number of the terms in each column (including the cyphers) was equal to the number of the columns. And therefore it is true univerſally. Q. E. D.

60. The eighth property. The horizontal rows of numbers in the foregoing table of combinations, beginning with the ſecond row, exhibit the co-efficients of the ſeveral ſucceſſive powers of a binomial quantity, as $a+b$. Thus, the

numbers

numbers in the ſecond horizontal row, to wit, 1 and 1, are the co-efficients of the two members a and b of the ſaid binomial quantity $a + b$ itſelf, or (as it is ſometimes called) of the firſt, or ſimple, power of the ſaid binomial quantity. The numbers in the third horizontal row, to wit, 1, 2, and 1, are the co-efficients of the ſeveral terms aa, $2ab$, and bb, of the compound quantity $aa + 2ab + bb$, which is the ſquare, or ſecond power, of the ſaid binomial quantity. The numbers in the 4th horizontal row, to wit, 1, 3, 3, and 1, are the co-efficients of the ſeveral terms of the compound quantity $a^3 + 3a^2b + 3ab^2 + b^3$, which is the cube, or third power, of the ſaid binomial quantity. The numbers in the 5th horizontal row, to wit, 1, 4, 6, 4, and 1, are the co-efficients of the ſeveral terms of the compound quantity $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which is the 4th power of the ſaid binomial quantity. And, in like manner, the numbers in the 6th, 7th, 8th, 9th, and every following horizontal row to the nth row (n being any whole number whatſoever) will be the co-efficients of the terms of the 5th, 6th, 7th, 8th, and every following power of the ſaid binomial quantity $a + b$, to the $\overline{n - 1|}$th power, reſpectively; and the numbers in the $\overline{n + 1|}$th horizontal row of terms in the ſaid table will be the co-efficients of the terms of the nth power of the ſaid binomial quantity.

61. This property will appear from the manner in which the powers of the binomial quantity $a + b$ are generated from each other by multiplication, which is as follows:

$$
\begin{array}{l}
a+b \\
a+b \\
\hline
aa+ab \\
\quad +ab \quad +bb \\
\hline
aa+2ab \quad +bb = \overline{a+b}|^{2} \\
\qquad\quad a+b \\
\hline
a^{3}+2a^{2}b+abb \\
\quad +a^{2}b+2abb \quad +b^{3} \\
\hline
a^{3}+3a^{2}b+3ab^{2} \quad +b^{3} = \overline{a+b}|^{3} \\
\qquad\qquad\quad a+b \\
\hline
a^{4}+3a^{3}b+3a^{2}b^{2}+ab^{3} \\
\quad +a^{3}b+3a^{2}b^{2}+3ab^{3}+b^{4} \\
\hline
a^{4}+4a^{3}b+6a^{2}b^{2}+4ab^{3}+b^{4} = \overline{a+b}|^{4}.
\end{array}
$$

Or, if, for brevity's ſake, we ſubſtitute $1+1$ inſtead of $a+b$, the multiplication will be as follows:

$$
\begin{array}{l}
1+1 \\
1+1 \\
\hline
1+1 \\
\quad +1+1 \\
\hline
1+2+1 = \overline{1+1}|^{2} \\
\quad 1+1 \\
\hline
1+2+1 \\
\quad +1+2+1 \\
\hline
1+3+3+1 = \overline{1+1}|^{3} \\
\qquad 1+1 \\
\hline
1+3+3+1 \\
\quad +1+3+3+1 \\
\hline
1+4+6+4+1 = \overline{1+1}|^{4}.
\end{array}
$$

Here we ſee that every new power of the binomial quantity 1 + 1 is formed by adding together the terms of the next preceeding power of it by two at a time, or in ſetting down the terms of the preceeding power twice following in two parallel horizontal rows, with the terms in the lower row advanced one ſtep to the right hand beyond thoſe in the upper row; which is the manner in which the ſeveral horizontal rows of numbers in the foregoing table of combinations are, or may be, derived from each other, as appears from the 5th and 6th properties above-mentioned in art. 55, 56, and 57. Therefore, ſince theſe co-efficients of the terms of the powers of the binomial quantity 1+1 are generated from 1 + 1, in the ſame manner as the 3d, 4th, 5th, and other following rows of numbers in the foregoing table of combinations are generated from the ſame numbers 1 and 1 in the ſecond horizontal row, it follows that the numbers contained in the 3d, 4th, 5th, and other ſubſequent horizontal rows muſt coincide with, or be the ſame with, the co-efficients of the terms of the ſquare, cube, fourth power, and other following correſpondent powers of the ſaid binomial quantity. Q. E. D.

62. The ninth property. The ſums of the numbers contained in the ſeveral ſucceſſive horizontal rows in the foregoing table of combinations increaſe continually in the proportion of 1 to 2; or the ſum of the numbers in every new horizontal row is double of the ſum of the numbers in the next preceeding horizontal row. Theſe ſums are as follows:

In the 1ſt horizontal row 1+0+0 &c are = 1.
In the 2d horizontal row 1 + 1 + 0 + 0 &c are = 2.
In the 3d row 1 + 2 + 1 + 0 + 0 &c are = 4.
In the 4th row 1 + 3 + 3 + 1 + 0 + 0 &c are = 8.
In the 5th row 1 + 4 + 6 + 4 + 1 + 0 + 0 &c are = 16.
In the 6th row 1 + 5 + 10 + 10 + 5 + 1 &c are = 32.
In the 7th row 1 + 6 + 15 + 20 + 15 + 6 + 1 are = 64.
In the 8th row 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 are = 128.

Each of theſe ſums is double of the ſum immediately preceeding it. And the ſame thing is true of the ſums of the terms of the four following horizontal rows in the foregoing table, and of the ſums of the terms of all the following horizontal rows that would belong to it, if it were continued to any greater number of vertical columns and horizontal rows whatſoever.

63. This property follows from what has been ſhewn above in art. 57, to wit, that every new horizontal row of numbers in the ſaid table may he derived from the next preceeding horizontal row of numbers by ſetting down the numbers of the ſaid preceeding row twice following in two parallel horizontal rows, with the terms in the lower row advanced one ſtep further to the right hand than the terms in the upper row, and then adding the terms of the two rows that ſtand in the ſame vertical lines together. For the ſum of the numbers contained in the new horizontal line ariſing from the addition of the ſaid two lines together, muſt evidently be double of the ſum of the numbers in only one of the lines ſo added. Thus, if we ſet down the numbers of the 7th horizontal row of numbers, to wit, 1, 6, 15, 20, 15, 6, 1, twice following in two parallel rows one under the other, as follows,

1, 6, 15, 20, 15, 6, 1
  1, 6, 15, 20, 15, 6, 1,

and then add the two rows together, ſo as to make a new line of numbers, to wit,

1, 7, 21, 35, 35, 21, 7, 1,

it is evident that the ſum of the numbers contained in this new line muſt be double of the ſum of the numbers contained in either of the two former lines. Q. E. D.

64. This property may alſo be derived from the laſt or 8th property, ſet forth and proved in art. 60 and 61. For, ſince every new power of the binomial quantity 1 + 1 muſt be greater than the next preceeding power of it in the proportion of 1 + 1, or 2, to 1; and it has been ſhewn in art. 61, that the ſeveral horizontal rows of numbers in the fore-

going table exhibit the members of the ſeveral ſucceſſive powers of the binomial quantity $1 + 1$; it follows that the ſaid horizontal rows of numbers muſt be greater, one than the other, in the ſame proportion of $1+1$, or 2, to 1.

Q. E. D.

65. The tenth property. If the ſums of the numbers contained in the ſeveral horizontal rows of the foregoing table of combinations be continually added to each other, the new ſums thence ariſing, or the ſums of the former ſums, will form a ſeries of numbers, which will be equal to the ſeveral powers of 2, with an unit ſubtracted from them, or to $2 - 1$, $\overline{2}|^{2} - 1$, $\overline{2}|^{3} - 1$, $\overline{2}|^{4} - 1$, $\overline{2}|^{5} - 1$, $\overline{2}|^{6} - 1$, $\overline{2}|^{7} - 1$, &c, or to $2 - 1$, $4 - 1$, $8 - 1$, $16 - 1$, $32 - 1$, $64 - 1$, $128 - 1$, &c, or 1, 3, 7, 15, 31, 63, 127, &c.

Thus, for example, the ſums of the eight firſt horizontal rows of numbers are 1, 2, 4, 8, 16, 32, 64, and 128, reſpectively, as we have ſeen in art. 62. Now, if theſe ſums are added together, we ſhall have

$1 = 1 = 2 - 1$

$1 + 2 = 3 = 4 - 1 = \overline{2}|^{2} - 1$,

$1 + 2 + 4 = 7 = 8 - 1 = \overline{2}|^{3} - 1$,

$1 + 2 + 4 + 8 = 15 = 16 - 1 = \overline{2}|^{4} - 1$,

$1 + 2 + 4 + 8 + 16 = 31 = 32 - 1 = \overline{2}|^{5} - 1$,

$1 + 2 + 4 + 8 + 16 + 32 = 63 = 64 - 1 = \overline{2}|^{6} - 1$,

$1 + 2 + 4 + 8 + 16 + 32 + 64 = 127 = 128 - 1 = \overline{2}|^{7} - 1$, and

$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 = 255 = 256 - 1 = \overline{2}|^{8} - 1$.

And univerſally, if the number of the horizontal rows be n, the ſum of the ſums of all the numbers contained in them will be $\overline{2}|^{n} - 1$.

66. This is evident from art. 33. For the ſum of all theſe ſums is the number of all the poſſible combinations of n letters; which is ſhewn in art. 33 to be $= \overline{2}|^{n} - 1$. Therefore the ſucceſſive ſums of the ſums of all the numbers con-

tained in the ſaid horizontal rows will form the ſeries. 1, $2-1$, $\overline{2}|^{2}-1$, $\overline{2}|^{3}-1$, $\overline{2}|^{4}-1$, $\overline{2}|^{5}-1$, $\overline{2}|^{6}-1$, $\overline{2}|^{7}-1$, $\overline{2}|^{8}-1$, &c .... $\overline{2}|^{n-1}$. Q. E. D.

67. The eleventh property. If we divide the terms of the ſecond, or any other of the vertical columns of the foregoing table of combinations by the correſponding terms (or terms ſituated in the ſame horizontal line) of the next preceding vertical column, the ſeveral quotients thence ariſing will be equal to the terms of an arithmetical progreſſion conſiſting of fractions, of which the common difference is a fraction of which an unit is the numerator, and the number which is the exponent of the firſt of the ſaid vertical columns (by the terms of which the terms of the other vertical column are divided) is the denominator.

Thus, for example, if we divide each of the terms of the ſecond vertical column 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 (omitting the firſt term, which is a cypher, or 0), by the correſponding term of the firſt vertical column 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, the quotients will be $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, $\frac{4}{1}$, $\frac{5}{1}$, $\frac{6}{1}$, $\frac{7}{1}$, $\frac{8}{1}$, $\frac{9}{1}$, $\frac{10}{1}$, $\frac{11}{1}$, which differ from each other by the common difference $\frac{1}{1}$.

And, if we divide each of the terms of the third vertical column 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 (omitting the two cyphers) by the correſponding term of the ſecond vertical column, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, the quotients will be $\frac{1}{2}$, $\frac{3}{3}$, $\frac{6}{4}$, $\frac{10}{5}$, $\frac{15}{6}$, $\frac{21}{7}$, $\frac{28}{8}$, $\frac{36}{9}$, $\frac{45}{10}$, and $\frac{55}{11}$, which are reſpectively equal to $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, $\frac{6}{2}$, $\frac{7}{2}$, $\frac{8}{2}$, $\frac{9}{2}$, and $\frac{10}{2}$, which differ from each by the fraction $\frac{1}{2}$, of which 1 is the numerator, and 2, or the exponent of the ſecond vertical column 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, (by the terms of which the terms of the third vertical column are divided), is the denominator.

If we divide the terms of the fourth vertical column (omitting the cyphers at the beginning) by the correſponding terms of the third column, the quotients will be as follows; to wit,

$$\frac{1}{3}=\frac{1}{3}$$
$$\frac{4}{6}=\frac{2}{3}$$
$$\frac{10}{10}=\frac{3}{3}$$
$$\frac{20}{15}=\frac{4}{3}$$
$$\frac{35}{21}=\frac{5}{3}$$
$$\frac{56}{28}=\frac{6}{3}$$
$$\frac{84}{36}=\frac{7}{3}$$
$$\frac{120}{45}=\frac{8}{3}$$

and $\frac{165}{55}=\frac{9}{3}$, which differ from each other by the fraction $\frac{1}{3}$, of which 1 is the numerator, and 3, or the exponent of the third vertical column (by the terms of which the terms of the fourth vertical column are divided), is the denominator.

In like manner the quotients that ariſe by dividing the terms of the fifth vertical column by the correſponding terms of the fourth vertical column are the following, to wit; $\frac{1}{4}$, $\frac{5}{10}$, $\frac{15}{20}$, $\frac{35}{35}$, $\frac{70}{56}$, $\frac{126}{84}$, $\frac{210}{120}$, and $\frac{330}{165}$, which are reſpectively equal to $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, $\frac{6}{4}$, $\frac{7}{4}$, and $\frac{8}{4}$.

And the quotients that ariſe by dividing the terms of the ſixth vertical column by the correſponding terms of the fifth vertical column are $\frac{1}{5}, \frac{6}{15}, \frac{21}{35}, \frac{56}{70}, \frac{126}{126}, \frac{252}{210}$, and $\frac{462}{330}$, which are reſpectively equal to $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \frac{6}{5}$, and $\frac{7}{5}$.

The quotients that ariſe by dividing the terms of the ſeventh vertical column by the correſponding terms of the ſixth column are $\frac{1}{6}, \frac{7}{21}, \frac{28}{56}, \frac{84}{126}, \frac{210}{252}$, and $\frac{462}{462}$, which are reſpectively equal to $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$, and $\frac{6}{6}$.

The quotients that ariſe by dividing the terms of the eighth vertical column by the correſpondent terms of the ſeventh column are $\frac{1}{7}, \frac{8}{28}, \frac{36}{84}, \frac{120}{210}$, and $\frac{330}{462}$, which are reſpectively equal to $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}$, and $\frac{5}{7}$.

The quotients that ariſe by dividing the terms of the ninth vertical column by the correſponding terms of the eighth column are $\frac{1}{8}, \frac{9}{36}, \frac{45}{120}$, and $\frac{165}{330}$, which are reſpectively equal to $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}$, and $\frac{4}{8}$.

The quotients that ariſe by dividing the terms of the tenth vertical column by the correſpondent terms of the ninth column are $\frac{1}{9}, \frac{10}{45}, \frac{55}{165}$, which are reſpectively equal to $\frac{1}{9}$, $\frac{2}{9}$, and $\frac{3}{9}$.

And the quotients that ariſe by dividing the terms of the eleventh vertical column by the correſpondent terms of the tenth column are $\frac{1}{10}$ and $\frac{11}{55}$, which are reſpectively equal to $\frac{1}{10}$ and $\frac{2}{10}$.

68. This property of theſe numbers might, if it were neceſſary to the main object of this Treatiſe, be derived from

the following, or 12th, property of the ſaid numbers, which we will now proceed to ſet forth and to demonſtrate.

69. The twelfth property. The ſum of all the numbers contained in any one of the vertical columns of the foregoing table of combinations, is to the ſum of the like number of terms that ſhould be all equal to the laſt term of the column, in the ſame proportion as 1 to the exponent of the column, or the number which denotes the place of the column, and which is marked by a Roman numeral figure at the top of it.

Thus, in column 1ſt, which conſiſts entirely of units, the ſum of all the twelve terms is 12, which is to the ſum of twelve terms all equal to the laſt term as 1 to 1, or the exponent of the firſt column. This propoſition is ſelf-evident.

In the 2d column the twelve terms are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11; the ſum of which is 66. And the ſum of twelve terms equal to the laſt, or greateſt, term 11, is 132. Now 66 is to 132 as 1 is to 2, or the exponent of this ſecond column.

In the 3d column the twelve terms are 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55; the ſum of which is 220. And the ſum of twelve terms equal to the laſt, or greateſt, term 55, is 660. Now 220 is to 660 as 1 is to 3, or the exponent of the third column.

In the 4th column the twelve terms are 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, 120, and 165; the ſum of which is 495. And the ſum of twelve terms equal to the laſt, or greateſt, term 165, is 1980. Now 495 is to 1980 as 1 is to 4, or the exponent of the fourth column.

In the 5th column the twelve terms are 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, and 330; the ſum of which is 792. And the ſum of twelve terms equal to the laſt, or greateſt, term 330, is 3960. Now 792 is to 3960 as 1 is to 5, or the exponent of the fifth column.

In the 6th column the twelve terms are 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, and 462; the ſum of which is 924.

And the ſum of twelve terms equal to the laſt, or greateſt, term 462, is 5544. Now 924 is to 5544 as 1 is to 6, or the exponent of the ſixth column.

In the 7th column the twelve terms are 0, 0, 0, 0, 0, 0, 1, 7, 28, 84, 210, and 462; the ſum of which is 792. And the ſum of twelve terms equal to the laſt, or greateſt, term 462, is 5544. Now 792 is to 5544 as 1 is to 7, or the exponent of the ſeventh column.

In the 8th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 1, 8, 36, 120, and 330; the ſum of which is 495. And the ſum of twelve terms equal to the laſt, or greateſt, term 330, is 3960. Now 495 is to 3960 as 1 is to 8, or the exponent of the eighth column.

In the 9th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 1, 9, 45, and 165; the ſum of which is 220. And the ſum of twelve terms equal to the laſt, or greateſt, term 165, is 1980. Now 220 is to 1980 as 1 is to 9, or the exponent of the ſaid ninth column.

In the 10th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 10, and 55; the ſum of which is 66. And the ſum of twelve terms all equal to the laſt, or greateſt, term 55, is 660. Now 66 is to 660 as 1 is to 10, or the exponent of the ſaid tenth column.

In the 11th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, and 11; the ſum of which is 12. And the ſum of twelve terms all equal to the laſt, or greateſt, term 11, is 132. Now 12 is to 132 as 1 is to 11, or the exponent of the ſaid eleventh column.

In the 12th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, and 1; the ſum of which is 1. And the ſum of twelve terms all equal to the laſt, or greateſt, term 1, is 12. Now 1 is to 12 as 1 is to the exponent of the ſaid twelfth column, that exponent being 12.

And the ſame thing will be found to be true, if, inſtead of taking twelve terms in each of the ſaid vertical columns, we were to take any leſſer number, as five, or ſix, or ſeven, terms, or any greater number of terms whatſoever, as fifteen, or twenty, or a hundred terms, continuing the table both downwards and ſideways for that purpoſe, namely, that, if the exponent of any one of the vertical columns be called *c*, the ſum of all the terms in the ſaid column, continued to any number of terms whatſoever, will be to the ſum of as many terms all equal to the laſt, or greateſt, term of the ſaid column, as 1 is to *c* [*].

70. This

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[*] The attentive reader may perhaps have obſerved, in reading the foregoing tranſlation of the twelve ſurpriſing properties of the numbers contained in the table of combinations, exhibited in page 74, (which properties are ſet forth in pages 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, and 91) that the 7th property (which is ſet forth in art. 58, page 80) is not the ſame with the 7th property in the author's original, (which is contained in page 19), but anſwers to the 8th property ſet forth in the ſaid original; and that there is no property in the tranſlation that exactly anſwers to the ſaid 7th property in the original. The reaſon of this omiſſion is, that the 7th property ſet forth in the author's original, ſeemed to me to be the ſame with the next preceeding, or 6th, property, and therefore to be an unneceſſary repetition. But, whether it is ſo, or not, muſt be referred to the reader's judgment. And therefore I will here ſet down both the 6th and the 7th property, as they are expreſſed in the original.

The ſixth property is expreſſed in theſe words. *Columnæ cujuſvis tranſverſæ termini ab unitate aliquouſque creſcunt, deinde per eaſdem gradus rurſùm decreſcunt. Idem intellige de ſummis columnarum verticalium æquè-altarum, ceu terminis ſequentis columnæ tranſverſæ, per quartam proprietatem.*

And the ſeventh property is expreſſed in theſe words. *Columnarum verticalium æquè-altarum baſes, ſive termini columnæ tranſverſæ cujuſlibet, primus quidem et ultimus ſignificativus perpetuò inter ſe æquantur, ut et ſecundus et penultimus, tertius et antepenultimus, atque ita porrò, ſi columna pluribus terminis ſignificativis conſtet.*

Now this 7th property ſeems to me to be a mere repetition of the foregoing 6th property, and particularly of the firſt ſentence of it, to wit, *Columnæ cujuſvis tranſverſæ termini ab unitate aliquouſque creſcunt, deinde per eoſdem gradus rurſùm decreſcunt.* Theſe words, "*ab unitate creſcunt, deinde per eoſdem gradus rurſùm decreſcunt*" ſeem only to be paraphraſed, or more fully explained, by the words of the 7th property, to wit, *primus quidem et ultimus perpetuò inter ſe æquantur, ut et ſecundus et penultimus, tertius et antepenultimus, atque ita porrò* Therefore, as I could find no new meaning to the words of the 7th property, whereby it could be diſtinguiſhed from the 6th property, I thought it better to omit it.

Yet we may obſerve that there are twelve properties of the figurate numbers, or numbers contained in the foregoing table of combinations, ſet down in the tranſlation as well as in the author's original. This is owing to my having divided the 10th property of theſe numbers mentioned in the author's

original

70. This is the moſt important property belonging to the figurate numbers, and that which will be of moſt uſe to us in treating of the doctrine of chances, or the art of forming probable conjectures concerning events that depend on chance, which is the ſubject of this treatiſe. I ſhall therefore now endeavour to demonſtrate this property of the ſaid numbers in a ſcientific and ſatisfactory manner, and ſo as to convince my readers that it muſt be true in all caſes whatſoever (however great the number of vertical columns, and that of the terms in each vertical column, may be ſuppoſed to be taken) as well as in the ſmall number of caſes exhibited in the foregoing table of combinations. And in order to this I ſhall proceed to lay down the four following preliminary propoſitions, or lemmas, as the ground-work of the following demonſtration.

LEMMA I.

71. The ſum of any number of terms whatſoever in the firſt vertical column in the foregoing table of combinations, is equal to the ſum of an equal number of terms that are all equal to the laſt term.

original (which conſiſts of two branches) into two ſeparate properties, calling the firſt branch of the ſaid 10th property in the original (which is expreſſed in theſe words, "*Summæ ſerierum tranſverſarum progrediuntur in continuâ ratione duplâ*") the 9th property in the tranſlation; and the ſecond branch of it (which is expreſſed in theſe words, "*Summarum verò ſummæ, ab initio collectæ, terminos conſtituunt progreſſionis duplæ unitate multatos*") the 10th property in the tranſlation.

The differences therefore between the properties of theſe numbers, as expreſſed in the tranſlation, and as expreſſed in the original, are as follows. The ſix firſt properties of theſe numbers in the tranſlation anſwer to the ſix firſt properties of them in the original reſpectively; the 7th property in the tranſlation anſwers to the 8th property in the original: the 8th property in the tranſlation anſwers to the 9th property in the original: the 9th and 10th properties in the tranſlation anſwer to the firſt and ſecond branches of the 10th property in the original: and the 11th and 12th properties in the tranſlation anſwer to the 11th and 12th properties in the original, reſpectively.

DEMONSTRATION.

This is evident, becauſe all the terms in the firſt column are units, or equal to the laſt term. Therefore the ſum of all the ſaid terms is the ſum of the ſame number of terms equal to the laſt term. Q. E. D.

LEMMA II.

72. If in any one of the vertical columns of numbers in the foregoing table of combinations, after the firſt column, we take as many terms (including the cyphers in the beginning of the column) as there are units in the exponent of the column, the ſum of all the ſaid terms will be to the ſum of the ſame number of terms that are all equal to the laſt of them in the ſame proportion as 1 is to the exponent of the ſaid column.

DEMONSTRATION.

By the firſt property of theſe figurate numbers, ſet forth in art. 51, the number of cyphers at the beginning of each of the ſaid vertical columns is leſs by an unit than the exponent of the ſaid column. And, by the ſecond property of theſe numbers, ſet forth in art. 52, the firſt term in every column after the cyphers is an unit. Therefore the ſum of all the terms of the vertical column that are ſuppoſed in this lemma to be taken (which are only as many as there are units in the exponent of the column) will be the ſum of ſome cyphers and an unit, and conſequently will be equal only to an unit. And the ſum of the ſame number of terms all equal to the laſt will be equal to the ſum of the ſame number of terms all equal to an unit, or will be equal to the exponent of the column. Therefore the ſum of all the terms in the ſaid vertical column will be to the ſum of as many terms all equal to the laſt term (which is an unit) in the ſame proportion as 1 is to the exponent of the column.

Q. E. D.

Thus, for example, in the 4th vertical column, if we take only the four firſt terms 0, 0, 0, 1, the ſum of theſe terms will be = 1, and the ſum of four terms all equal to the laſt term, which is 1, will be (= 1 + 1 + 1 + 1) = 4. And therefore the former ſum is to the latter as 1 is to 4, which is the exponent of the ſaid column. And, in general, if the exponent of the column be c, and we take c terms in it, the $c - 1$ terms will be all cyphers, and the cth term will be 1. Therefore the ſum of the ſaid c terms will be 1. And the ſum of c terms all equal to the laſt term (which is 1) will be c. Therefore the former ſum will be to the latter ſum as 1 is to c.

This lemma is the ſame with the general propoſition hereafter to be proved, or the 12th property of the figurate numbers, in the caſe of taking only the firſt ſignificant term in each of the vertical columns, which firſt term is always an unit.

LEMMA III.

73. If the above deſcribed 12th property of the figurate numbers (which we are preparing to demonſtrate the truth of) ſhould be found to be true in any one of the vertical columns of numbers contained in the foregoing table of combinations, or the ſum of any number of terms taken in the ſaid column ſhould be to the ſum of the ſame number of terms all equal to the laſt, or greateſt, term, always in the ſame conſtant proportion, whatever be the number of terms ſo taken; and this proportion be that of 1 to a certain number denoted by r, ſo that the ſum of the terms ſo taken ſhall always be equal to the quotient that ariſes by dividing the latter ſum (or the ſum of the ſame number of terms all equal to the laſt term) by the number r; the exceſs

of the number of the terms ſo taken above the number r will be to the exceſs of the number of terms ſo taken above 1 in the ſame proportion as the laſt term but one of the terms ſo taken to the laſt term of all.

DEMONSTRATION.

Let the terms ſuppoſed to be taken in the ſaid vertical column be A, B, C, D, &c .... K, L, of which L is the laſt, and K the laſt but one. And let the number of the terms ſo taken be n. We are then to prove that $n-r$ will be to $n-1$ as K is to L.

Now, ſince the ſum of all the n terms A, B, C, D, &c .... K, L is ſuppoſed to be to the ſum of n terms all equal to the laſt term L, or to $n \times L$, in the proportion of 1 to r; and the ſum of all the terms A, B, C, D, &c .... K, L, except the laſt term L, or the ſum of all the $n-1$ terms A, B, C, D, &c .... K, is alſo ſuppoſed to be to the ſum of $n-1$ terms all equal to the laſt term K, or to $\overline{n-1}| \times K$, in the ſame proportion of 1 to r; it follows that the ſum of all the n terms A, B, C, D, &c .... K, L will be $= \frac{n \times L}{r}$, and the ſum of all the $n-1$ terms A, B, C, D, &c, K, will be $= \frac{\overline{n-1}| \times K}{r}$. But this latter ſum, or A + B + C + D + &c + K, is leſs than the former ſum, or A + B + C + D + &c K + L, by L. Therefore $\frac{\overline{n-1} \times K}{r}$ is $= \frac{n \times L}{r} - L = \frac{n \times L - r \times L}{r} = \frac{\overline{n-r} \times L}{r}$, and conſequently $\overline{n-1}| \times K$ is $= \overline{n-r}| \times L$. Therefore $n-r$ is to $n-1$ as K is to L.

Q. E. D.

LEMMA IV.

74. If in the foregoing table of combinations, or figurate numbers, we take two contiguous vertical columns; and the numbers in the firſt of the two columns are found to have the twelfth property above-deſcribed, or the ſum of any number of terms of it is to the ſum of as many terms all equal to the laſt, or greateſt, term, in the ſame proportion as the ſum of any other number of its terms is to the ſum of as many terms all equal to the laſt, or greateſt, of this latter number of terms; and the ſaid proportion is that of 1 to the number r; and in the ſecond of the ſaid two contiguous vertical columns it be found that for a certain number of terms the numbers in the ſaid column are likewiſe poſſeſſed of the ſame 12th property, and that the ſum of the ſaid number of terms is to the ſum of as many terms all equal to the laſt, or greateſt, term in the proportion of 1 to $r+1$, and that the ſum of any leſſer number of its terms is to the ſum of as many terms all equal to the laſt, or greateſt, of the ſaid leſſer number of terms, in the ſame proportion of 1 to $r+1$; I ſay, then, that, if we take another term in the ſaid ſecond vertical column above the number before taken, and in which the ſaid 12th property has been found to take place, the ſaid 12th property will take place likewiſe with reſpect to the numbers in the ſaid ſecond column, when increaſed by the ſaid new term, and the ſum of all the terms in the ſaid column, including the ſaid new term, will be to the ſum of as many terms all equal to the ſaid new term, in the ſame proportion of 1 to $r+1$.

DEMONSTRATION.

Let n be the number of terms that are taken in the ſecond of the two vertical columns; and let the ſame number of terms be taken in the firſt of them. Let the terms in the ſaid firſt column be A, B, C, D, &c . . . . K, L, and thoſe in the ſecond column be a, b, c, d, *&c* . . . , k, l. Then, by the ſuppoſition, the ſum of the n terms A, B, C, D, &c . . . .

. . . . K, L, will be to the ſum of *n* terms, all equal to the laſt term L, or to $n \times L$, in the proportion of 1 to *r*; and the ſum of all the *n* terms *a*, *b*, *c*, *d*, *&c* . . . . *k*, *l*, will be to the ſum of *n* terms all equal to *l*, or to $n \times l$, in the proportion of 1 to $r+1$. Now let another term *m* be added to the former terms *a*, *b*, *c*, *d*, *&c* . . . . *k*, *l*, of the ſecond of the ſaid two vertical columns. We are then to prove that the ſum of all the terms *a*, *b*, *c*, *d*, *&c* . . . . *k*, *l*, and *m* (the number of which is $n+1$) will be to the ſum of as many terms all equal to the laſt term *m*, or to $\overline{n+1} \times m$, in the ſame proportion of 1 to $r+1$.

Now, by the 4th property of the figurate numbers above ſet forth in art. 54, it is manifeſt that *l*, or the *n*th term of the ſecond of the two vertical columns, will be equal to the ſum of all the terms in the preceeding vertical column except the laſt term L, or to the ſum of the $n-1$ terms A, B, C, D, &c . . . . K. But, by the ſuppoſition, the ſum of theſe terms is leſs than the ſum of as many terms equal to the laſt term K in the proportion of 1 to *r*, or is equal to $\frac{\overline{n-1} \times K}{r}$. Therefore *l* is $= \frac{\overline{n-1} \times K}{r}$.

But, becauſe the above-deſcribed 12th property is ſuppoſed to belong to the numbers of the firſt of the ſaid two vertical columns, to wit, A, B, C, D, &c . . . . K, L, and the ſum of any number of terms in the ſaid column is ſuppoſed to be to the ſum of as many terms all equal to its laſt term in the conſtant proportion of 1 to *r*, it follows from lemma 3d, art. 73, that $n-r$ will be to $n-1$ as K is to L. Therefore $\overline{n-1} \times K$ will be $= \overline{n-r} \times L$; and conſequently $\frac{\overline{n-1} \times K}{r}$ will be $= \frac{\overline{n-r} \times L}{r}$.

Therefore *l* (which has been ſhewn to be $= \frac{\overline{n-1} \times K}{r}$) will be $= \frac{\overline{n-r} \times L}{r}$; and conſequently $n-r$ will be to *l* as *r* is to L.

But, by the ſuppoſition, the ſum of the terms A, B, C, D, &c . . . K, L, is to the ſum of the ſame number of terms all equal to the laſt term L, or to $n \times L$, in the proportion of 1 to r. Therefore $r \times \overline{A+B+C+D+\&c+K+L}$ is $= n \times L$; and conſequently r is to L as n is to $A+B+C+D+\&c+K+L$.

Therefore $n-r$ will be to l as n is to $A+B+C+D+\&c+K+L$.

But, by the 4th property of the figurate numbers above ſet forth in art. 54, m (which is the $n+1$th term of the ſecond vertical column) is equal to the ſum of A, B, C, D, &c, K, L, or the n firſt terms of the preceeding column. Therefore $n-r$ will be to l as n is to m; and conſequently $\overline{n-r} \times m$ will be $= n \times l$.

But, by the ſuppoſition concerning the numbers in the ſecond vertical column, the ſum of the firſt n terms of it, to wit, $a+b+c+d+\&c+k+l$, is to the ſum of as many terms all equal to the laſt term l, or to $n \times l$, in the proportion of 1 to $r+1$. Therefore $\overline{r+1} \times \overline{a+b+c+d+\&c+k+l}$ is $= n \times l$; and conſequently $\overline{n-r} \times m$ (which is equal to $n \times l$) will be $= \overline{r+1} \times \overline{a+b+c+d+\&c+k+l}$. Therefore $n-r : r+1 :: a+b+c+d+\&c+k+l : m$. Therefore, *componendo*, we ſhall have ($n-r+r+1$, or) $n+1 : r+1 :: a+b+c+d+\&c+k+l+m : m$; and *permutando*, $n+1 : a+b+c+d+\&c+k+l+m :: r+1 : m$; and, *invertendo*, $a+b+c+d+\&c+k+l+m : n+1 :: m : r+1$. But $n+1$ is to $\overline{n+1} \times m :: \overline{r+1} : \overline{r+1} \times m$. Therefore, *ex æquo*, $a+b+c+d+\&c+k+l+m : \overline{n+1} \times m :: m : \overline{r+1} \times m :: 1 : r+1$; that is, the ſum of the firſt $n+1$ terms of the ſecond vertical column will be to $n+1$ times the laſt, or $\overline{n+1}$th, term, m, of the ſaid column in the ſame proportion of 1 to $r+1$ in which the ſum of the firſt n terms of it was to n times the laſt, or nth, term. Q. E. D.

Coroll. It follows from this lemma, that, if the number of terms in the ſecond of the two vertical columns be increaſed from n terms to any other number of terms whatſoever denoted by $n+p$, it will be true with reſpect to the column, when ſo increaſed, that the ſum of all its terms will be to $n+p$ times the laſt, or greateſt, term of it in the ſame proportion of 1 to $r+1$. For the lemma may be ſucceſſively extended from a column conſiſting of $n+1$ terms, to a column conſiſting of $n+2$ terms, and to a column conſiſting of $n+3$ terms, and to a column conſiſting of $n+4$ terms, and ſo on till we come to the column of $n+p$ terms; the reaſonings being exactly the ſame in this extenſion of it to theſe ſeveral columns of $n+2$ terms, $n+3$ terms, $n+4$ terms, &c, as in the lemma itſelf, in which, upon a ſuppoſition that the ſum of n terms of the column is to n times the laſt, or greateſt, or nth, term of it in the proportion of 1 to $r+1$, it is ſhewn that the ſum of $n+1$ terms of it will be to $n+1$ times the laſt, or greateſt, or $\overline{n+1}|$th, term of it in the ſame proportion of 1 to $r+1$.

A demonſtration of the 2d, 3d, and 4th foregoing Lemmas, contained in art. 72, 73, and 74, by Mr. John Bernoulli, the author's brother.

75. Many years ago, when I communicated the foregoing propoſitions concerning the figurate numbers to my brother, Mr. John Bernoulli, he obſerved to me that the demonſtrations of them might be made ſhorter and more elegant by uniting the three laſt of the four preceeding lemmas into one, in the manner following.

A LEMMA.

If in a table of the figurate numbers (ſuch as the foregoing table of combinations, in page 71, art. 49), it be the property of the terms of any one of the vertical columns that, if we take, 1ſt, any number of ſucceſſive terms in it, and, 2dly, the ſame number of terms, all equal to the laſt, or greateſt, of the ſaid ſucceſſive terms, the ſum of the ſaid ſucceſſive terms ſhall be to the ſum of the ſame number of terms, all equal to their laſt, or greateſt, term, in the conſtant proportion of 1 to a certain number denoted by the letter r; then it will follow that, if in the next higher vertical column of the ſaid table of figurate numbers we take a number of ſucceſſive terms greater by an unit than the number of ſucceſſive terms taken in the former vertical column, the ſum of theſe ſucceſſive terms in this ſecond column will be to the ſum of the ſame number of terms, all equal to the laſt, or greateſt, of the ſaid ſucceſſive terms, in the proportion of 1 to $r+1$.

DEMONSTRATION.

Let the terms of the former of the two vertical columns be a, b, c, d, e, and f, of which the number is 6, or in general, n; and let the terms of the next higher vertical column be 0, g, h, i, l, p, q, of which the number is $n+1$.

The upper term of this ſecond column is a cypher, 0, becauſe every new vertical column of terms muſt have one more cypher preceeding its ſignificant terms than the column immediately preceeding it.

Theſe two vertical columns of terms will be as follows:

$$n\left|\begin{array}{c|c} a & 0 \\ b & g \\ c & h \\ d & i \\ e & l \\ f & p \\ & q \end{array}\right| n+1,$$

If theſe columns are the 1ſt and 2d columns of the table, the terms a, b, c, d, e, and f will, each of them, be equal to 1, and g, h, i, l, p, q, will be 1, 2, 3, 4, 5, 6. If theſe columns are the 2d and 3d columns, a will be 0, and b, c, d, e, f will be 1, 2, 3, 4, 5, and g, h, i, l, p, q will be 0, 1, 3, 6, 10, 15. If theſe columns are he 3d and 4th columns, a will be 0, and b will alſo be 0, and c, d, e, f will be 1, 3, 6, 10, and g, h, i, k, p, q will be 0, 0, 1, 4, 10, 20. And in like manner more of the upper terms of both theſe vertical columns will be cyphers, or 0, the farther the columns are taken to the right hand in the table in page 71. But, wherever the columns are taken, the number of terms in them muſt be ſo great as to reach below the cyphers, and take in ſome of the ſignificant terms. Theſe things being premiſed, the demonſtration of this lemma will be as follows.

By the 4th property of the figurate numbers ſet forth above in art. 54, we ſhall have $q = a+b+c+d+e+f$,
and $p = a+b+c+d+e$,
and $l = a+b+c+d$,
and $i = a+b+c$,
and $h = a+b$,
and $g = a$.

And, by the ſuppoſition, the ſum of the n terms a, b, c, d, e, f will be to n times the laſt, or greateſt, term f, as 1 is to r; and the ſum of the $n-1$ terms a, b, c, d, e will be to $n-1$ times the laſt, or greateſt, term e, in the ſame proportion of 1 to r; and the ſum of the $n-2$ terms a, b, c, d will be to $n-2$ times the laſt, or greateſt, term d, in the ſame proportion of 1 to r; and, in like manner, $a+b+c$ will be to $n-3$ times c as 1 to r; and $a+b$ will be to $n-4$ times b as 1 to r; and a will be to $n-5$ times a as 1 to r.

Therefore $a+b+c+d+e+f$ will be $= \frac{nf}{r}$, and $a+b+c+d+e$ will be $= \frac{\overline{n-1}\times e}{r}$, and $a+b+c+d$ will be $= \frac{\overline{n-2}\times d}{r}$, and $a+b+c$ will be $= \frac{\overline{n-3}\times c}{r}$, and $a+b$ will be $= \frac{\overline{n-4}|\times b}{r}$, and a will be $= \frac{\overline{n-5}|\times a}{r}$.

Therefore q (which is equal to $a+b+c+d+e+f$) will be $= \frac{nf}{r}$; and p (which is equal to $a+b+c+d+e$) will be $= \frac{\overline{n-1}|\times e}{r}$; and l (which is equal to $a+b+c+d$) will be $= \frac{\overline{n-2}|\times d}{r}$; and i (which is equal to $a+b+c$) will be $= \frac{\overline{n-3}|\times c}{r}$; and b (which is equal to $a+b$) will be $= \frac{\overline{n-4}|\times b}{r}$; and g (which is equal to a) will be $= \frac{\overline{n-5}|\times a}{r}$.

Therefore $q+p+l+i+b+g$ will be $= \frac{nf}{r} + \frac{\overline{n-1}|\times e}{r} + \frac{\overline{n-2}|\times d}{r} + \frac{\overline{n-3}|\times c}{r} + \frac{\overline{n-4}|\times b}{r} + \frac{\overline{n-5}|\times a}{r} = \frac{nf}{r} + \frac{ne-e}{r} + \frac{nd-2d}{r} + \frac{nc-3c}{r} + \frac{nb-4b}{r} + \frac{na-5a}{r} = \frac{nf+ne+nd+nc+nb+na}{r} - \frac{e}{r} - \frac{2d}{r} - \frac{3c}{r} - \frac{4b}{r} - \frac{5a}{r} = \frac{n\times\overline{f+e+d+e+b+a}}{r} \frac{-e-2d-3c-4b-5a}{r} = \frac{n\times q}{r} \frac{-e-2d-3c-4b-5a}{r} = \frac{n\times q}{r} \frac{-e-d-c-b-a}{r} \frac{-d-e-b-a}{r} \frac{-c-b-a}{r} \frac{-b-a}{r} - \frac{a}{r} = \frac{n\times q}{r} - \frac{p}{r} - \frac{l}{r} - \frac{i}{r} - \frac{b}{r} - \frac{g}{r} = \frac{n\times q}{r} \frac{-p-l-i-b-g}{r} = \frac{nq-p-l-i-b-g}{r}$.

Therefore (if we multiply both ſides by r) we ſhall have $r \times \overline{q+p+l+i+b+g} = nq-p-l-i-b-g$, or $rq + r \times \overline{p+l+i+b+g} = nq-p-l-i-b-g$. And conſequently (adding $p+l+i+b+g$ to both ſides) we ſhall have $rq + r\times\overline{p+l+i+b+g} + p+l+i+b+g = nq$, or $rq + \overline{r+1}|\times\overline{p+l+i+b+g} = nq$; and (ſubtracting rq from both ſides) $\overline{r+1}|\times\overline{p+l+i+b+g} = nq-rq$.

Therefore (dividing both ſides by $r+1$) we ſhall have $p+l+i+h+g=\frac{nq-rq}{r+1}$; and (adding q to both ſides) $q+p+l+i+h+g=\frac{nq-rq}{r+1}+q=\frac{nq-rq}{r+1}+\frac{\overline{r+1}\times q}{r+1}$ $=\frac{nq-rq}{r+1}+\frac{rq+q}{r+1}=\frac{nq+q}{r+1}=\frac{\overline{n+1}\times q}{r+1}$. And conſequently $q+p+l+i+h+g$, or $g+h+i+l+p+q$, or $0+g+h+i+l+p+q$, will be to $\overline{n+1}\times q$ as 1 is to $r+1$, or the ſum of the $n+1$ ſucceſſive terms 0, g, h, i, l, p, q, of the ſecond vertical column of terms will be to the ſum of $n+1$ times the laſt term q, or the ſame number of terms, all equal to the laſt, or greateſt, term q, in the proportion of 1 to $r+1$ *. Q. E. D.

The principal Propoſition, or the 12th property above-deſcribed of the figurate numbers, or numbers contained in the foregoing table of combinations, is as follows.

76. The ſum of any number of terms in any of the vertical columns contained in the foregoing table of combinations is to the ſum of the ſame number of terms all equal to the laſt term of them, in the proportion of 1 to the exponent of the ſaid column, or to the number which denotes, or expreſſes, its place in the ſaid table.

Thus, in the firſt column, of which the exponent is 1, the ſum of any number of terms of it denoted by n will be to n times the laſt term of it in the proportion of 1 to 1, or a proportion of equality. In the ſecond column, of which the exponent is 2, the ſum of *n* terms of it will be to the ſum of *n* terms all equal to the laſt or greateſt term, in the proportion of 1 to 2. In the third column, of which the exponent is 3, the ſum of *n* terms of it will be to the ſum of *n* terms all equal to the laſt, or greateſt, term, in the proportion of 1 to 3. In the fourth column, of which the exponent is 4, the ſum of *n* terms of it will be to the ſum of *n* terms all equal to the laſt, or greateſt, term, in the proportion of 1 to 4. And, in general, in the *c*th column, or that of which the exponent is *c*, the ſum of *n* terms of it will be to the ſum of *n* terms all equal to the laſt, or greateſt, or *n*th, term, in the proportion of 1 to *c*.

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* See upon this ſubject the works of Mr. John Bernoulli himſelf, publiſhed at Lauſanne, in Switzerland, in the year 1742, in four volumes quarto, vol. iii. page 521, in the 47th lecture on the doctrine of the Integration of infinitely ſmall differences, or the Inverſe method of differences.

DEMONSTRATION.

77. The truth of this propoſition with reſpect to the 1ſt vertical column (which conſiſts wholly of units) is ſhewn above in lemma 1, art. 71, and indeed is almoſt ſelf-evident. And with reſpect to the terms of the ſecond vertical column, to wit, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, &c, it may be proved by means of the ſecond and fourth of the foregoing lemmas in the manner following. Since in the firſt vertical column the ſum of any number of terms 1, 1, 1, 1, 1, &c. denoted by *n*, is to the ſum of as many terms, all equal to the laſt term 1, as 1 is to 1; and in the ſecond vertical column the ſum of the two firſt terms 0 and 1 is to the ſum of two terms, both equal to the laſt term 1, as 1 is to 1 + 1, as is ſhewn in lemma 2, art. 72; it follows from lemma 4, art. 74, that in the ſame ſecond vertical column the ſum of the three firſt terms 0, 1, and 2, will be to the ſum of three terms all equal to the laſt term 2, in the ſame proportion of 1 to 1 + 1, or 2, and conſequently that the ſum of the four firſt terms 0, 1, 2, and 3, will be to the ſum of four terms all equal to the laſt term 3, and the ſum of the five firſt terms 0, 1, 2, 3, and 4, will be to the ſum of five terms all equal to the laſt term 4, and, in general, the ſum of any number of its terms denoted

by n will be to the ſum of n terms all equal to the laſt of them, in the ſame proportion of 1 to 1 + 1, or 2.

Q. E. D.

78. This may likewiſe be proved of the numbers contained in the ſaid ſecond vertical column, independently of the foregoing lemmas, in the manner following.

The numbers contained in the ſaid ſecond vertical column are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. Now, if we ſet down theſe numbers twice over in two horizontal lines, one under the other, but in contrary orders, ſo that in the ſecond line the laſt term of the firſt line ſhall be placed firſt, and the laſt term but one of the firſt line ſhall be placed ſecond, and ſo on, as in theſe two lines,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,
11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0,

it is evident that the ſum of every two numbers ſtanding in the ſame vertical line, will be equal to 11, or, in general, to the laſt, or greateſt, term of the ſeries, or (if the ſeries conſiſt of n terms, and conſequently the laſt term be $n-1$) to $n-1$. Therefore the ſum of both ſerieſes will be equal to a ſeries conſiſting of the ſame number of terms, or n terms, all equal to the greateſt term $n-1$; and conſequently the upper ſeries $0+1+2+3+4+5+6+7+8+9+10+11+$ &c. . $n-1$ alone will be equal to only half of the ſeries of n terms all equal to $n-1$, or will be to it in the proportion of 1 to 2. Q. E. D.

79. To demonſtrate the ſaid 12th property with reſpect to the numbers contained in the third and fourth and other following vertical columns of the foregoing table of combinations, we muſt have recourſe to the ſecond and fourth lemmas, as in the firſt demonſtration juſt now given of the ſaid property with reſpect to the numbers in the ſecond vertical column. This may be done in the manner following.

By lemma 2d it appears that this property takes place in all the vertical columns, if we continue the terms of each column only till their number is equal to the exponent of the column, or ſo as to take in only the firſt ſignificant term of the column, which is always an unit. Therefore in the third vertical column, continued only to the three terms 0, 0, 1, the ſum of the ſaid three terms is to the ſum of three terms all equal to the laſt term 1, in the proportion of 1 to 3, or 1 to 2+1. But it has been ſhewn that in the ſecond vertical column 0, 1, 2, 3, 4, 5, 6, 7, &c. it is true univerſally that, whatever be the number we take of its terms, the ſum of the ſaid terms will be to the ſum of as many terms all equal to the laſt term in the proportion of 1 to 2. Here therefore we have the caſe of lemma 4, to wit, that of two contiguous vertical columns, the ſecond and the third, in the former of which the ſum of any number of terms denoted by n is to the ſum of the ſame number of terms all equal to the laſt term in the proportion of 1 to a certain number, which we there denoted by r, and which here is 2, and in the latter of which the ſum of the three firſt terms 0, 0, 1, is to the ſum of three terms all equal to the laſt term 1 as 1 is to $r+1$, or 2+1. It follows therefore from lemma 4th, that, if we take the next term 3 of the ſaid latter vertical column, or continue the ſaid column to four terms, the ſum of the ſaid four terms 0, 0, 1, 3, will alſo be to the ſum of four terms all equal to the laſt term 3 in the ſame proportion of 1 to 2+1. And, for the ſame reaſon, the ſum of the five firſt terms of the ſaid third vertical column will be to the ſum of five terms all equal to the fifth term of it, and the ſum of the ſix firſt terms of it will be to the ſum of ſix terms all equal to the ſixth term of it, and, in general, the ſum of any greater number of its terms, denoted by n, will be to the ſum of n terms all equal to the nth term of it, in the ſame proportion of 1 to 2+1, or of 1 to 3. Q. E. D.

In like manner we may prove that in the 4th vertical column the ſum of any number n of its terms is to the ſum of n terms all equal to its nth term in the proportion of 1 to 3+1, or 1 to 4. For, ſince we have proved that the proportion of theſe two ſums in the third column is that of 1 to 3; and by lemma 2 it appears that, if we take only the four firſt terms of the 4th column, to wit, 0, 0, 0, 1, the proportion of theſe ſums will be that of 1 to 3 + 1, or 4; it follows from lemma 4th and its corollary, that, if we take five terms of this fourth column, or ſix terms of it, or ſeven terms of it, or, in general, *n* terms of it, the proportion of the ſum of the terms ſo taken to the ſum of the ſame number of terms all equal to the laſt term will always be the ſame proportion of 1 to 3 + 1, or of 1 to 4. Q. E. D.

And by proceeding to apply lemma 2 and lemma 4 in the ſame manner to the fifth, and ſixth, and ſeventh, and other following vertical columns, it may be ſhewn that the proportion of the ſum of any number of terms denoted by *n* to the ſum of *n* terms all equal to the laſt, or greateſt term, will be in the fifth column that of 1 to 4 + 1, or 5, and in the ſixth column that of 1 to 5 + 1, or 6, and, in the ſeventh column that of 1 to 6 + 1, or 7, and, in general, in the *c*th column that of 1 to *c*. Q. E. D.

80. COROLL. 1. In each of the aforeſaid vertical columns of numbers the ſum of any number of the terms beginning with 1, or the firſt ſignificant term of the column, and not reckoning the cyphers that preceed it, as we have hitherto done, will be to the ſum of the ſame number of terms all equal to the next term in the ſaid column after the terms ſo ſummed, in the proportion of 1 to *c*, or the exponent of the column.

Let the terms in the propoſed vertical column, whereof we are to ſum up the ſignificant terms be *a*, *b*, *c*, *d*, &c, *k*, and *l*, including the cyphers, ſo that ſome of the firſt letters *a*, *b*, *c*, &c, ſhall ſtand for cyphers, agreeably to the notation in lemma 4; and let the whole number of theſe terms, including the cyphers, be *n*, agreeably to the ſame notation. And let *m* be the term that comes immediately after *l* the laſt term of the ſet whoſe ſum we are to examine; or, in other words, let *m* be the $\overline{n+1|}$th term of the propoſed vertical column, including the cyphers. Alſo let *c* be the exponent of the ſaid column, and *r* be $= c - 1$. Then by the firſt property of the figurate numbers ſet forth above in art. 51, r will be the number of cyphers in the beginning of the ſaid vertical column, and conſequently $n-r$ will be the number of ſignificant terms in the ſaid column, without the cyphers. We are therefore to prove that the ſum of all the $n-r$ ſignificant terms of the ſaid column a, b, c, d, &c, k and l is to the ſum of $n-r$ terms all equal to the next term m in the proportion of 1 to c or of 1 to $r+1$.

Now the ſum of all the $n-r$ ſignificant terms of the ſaid vertical column is equal to the ſum of all the n terms of the ſaid column, including the cyphers, becauſe the cyphers are all equal to nothing. And it is ſhewn in the latter part of the demonſtration of lemma 4, that the ſum of the n terms a, b, c, d, &c, k and l is to the next term m in the ſame proportion as $n-r$ is to $r+1$. Therefore the ſum of all the $n-r$ ſignificant terms of the ſaid column will be to the next term m in the ſame proportion of $n-r$ to $r+1$. But m is to $\overline{n-r} \times m$ in the ſame proportion as $r+1$ is to $\overline{n-r} \times \overline{r+1}$. Therefore, *ex æquo*, the ſum of all the $n-r$ ſignificant terms of the ſaid column will be to $\overline{n-r} \times m$, or to the ſum of $n-r$ terms all equal to the next term m, as $n-r$ is to $\overline{n-r} \times \overline{r+1}$, and conſequently as 1 is to $r+1$, or as 1 is to c. Q. E. D.

81. Coroll. 2. By the help of the foregoing corollary we may find the ſum of any given number of terms in any of the vertical columns of the foregoing table of combinations, without actually adding the terms together, by proceeding in the following manner.

Let the number of terms to which the ſeveral vertical columns are continued, be n. Then, as there is one cypher prefixed to the ſignificant terms in the ſecond column, and two cyphers in the third column, and three cyphers in the fourth column, and, in general, $c-1$ cyphers in the cth column; it is evident that the number of terms in the ſecond column, without the cyphers, will be $n-1$; and that of the terms in the third column, without the cyphers, will be $n-2$; and that of the terms in the fourth column, without the cyphers, will be $n-3$; and, in general, that of the terms in the cth column, without the cyphers, will be $n-\overline{c-1}$, or $n-c+1$. The ſums of the terms in theſe ſeveral columns may therefore be thus determined.

In the firſt place, the ſum of the n ſignificant terms in the firſt column, which are all units, will be $n\times 1$, or n.

2dly, The ſum of the $n-1$ ſignificant terms in the ſecond column will, by the foregoing corollary, be to $n-1$ times the next following, or $\overline{n+1}$th, term of the ſecond column as 1 is to 2. But, by the 4th property of the figurate numbers above ſet forth in art. 54, the $\overline{n+1}$th term of the ſecond column is equal to the ſum of the firſt n terms of the firſt column, that is, to n. Therefore the ſum of the $n-1$ ſignificant terms of the ſecond column will be to $n-1$ times n, or to $n\times\overline{n-1}$, as 1 is to 2, and conſequently will be equal to $\frac{n\times\overline{n-1}}{2}$. Q. E. I.

3dly, The ſum of the $n-2$ ſignificant terms in the third column will, by the foregoing corollary, be to $n-2$ times the next following, or $\overline{n+1}$th, term of the ſame third column as 1 is to 3. But, by the 4th property of the figurate numbers above ſet forth, the $\overline{n+1}$th term of the third column is equal to the ſum of the firſt n terms of the ſecond column, including the cyphers, or (which comes to the ſame thing) to the ſum of the firſt $n-1$ ſignificant terms of the ſaid ſecond column; which has juſt now been ſhewn to be equal to $\frac{n\times\overline{n-1}}{2}$. Therefore the ſum of the $n-2$ ſignificant terms in the 3d column will be to $n-2$ times $\frac{n\times\overline{n-1}}{2}$, or to $\frac{n\times\overline{n-1}\times\overline{n-2}}{2}$, as 1 is to 3, and conſequently will be equal to $\frac{n\times\overline{n-1}\times\overline{n-2}}{2\times 3}$. Q. E. I.

4thly, The ſum of the $n-3$ ſignificant terms in the fourth column will, by the foregoing corollary, be to $n-3$ times the next following, or $\overline{n+1}$th, term of the ſame fourth column as 1 is to 4. But, by the fourth property of the figurate numbers, the $\overline{n+1}$th term of the fourth column is equal to the ſum of the firſt n terms of the third column, including the cyphers, or of the firſt $n-2$ ſignificant terms of the ſaid third column: which has juſt now been ſhewn to be equal to $\frac{n\times\overline{n-1}\times\overline{n-2}}{2\times3}$. Therefore the ſum of the $n-3$ ſignificant terms in the 4th column will be to $n-3$ times $\frac{n\times\overline{n-1}\times\overline{n-2}}{2\times3}$, or to $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3}$, as 1 is to 4, and conſequently will be $=\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3\times4}$. Q. E. I.

5thly, In like manner the ſum of the $n-4$ ſignificant terms in the fifth column will, by the foregoing corollary, be to $n-4$ times the next following, or $\overline{n+1}$th, term of the ſame column as 1 is to 5. But, by the 4th property of the figurate numbers above ſet forth, the $\overline{n+1}$th term of the fifth column is equal to the ſum of the firſt n terms of the fourth column, including the cyphers, or to the firſt $n-3$ ſignificant terms of it; which has juſt now been ſhewn to be $=\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3\times4}$. Therefore the ſum of the $n-4$ ſignificant terms in the fifth column will be to $n-4$ times $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3\times4}$, or to $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times3\times4}$ as 1 is to 5, and conſequently will be $=$ $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times3\times4\times5}$. Q. E. I.

And in like manner it is evident that the ſum of the n firſt terms of the cth column, including the $c-1$ cyphers in the beginning of it, or the ſum of the $n-\overline{c-1}$, or $n-c+1$, firſt ſignificant terms of the ſaid *c*th column, will be equal to the fraction $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \&c \times \overline{n-c+1}}{2 \times 3 \times 4 \times 5 \times \&c \times c}$, in the numerator of which the laſt factor is $n - \overline{c-1}$, or $n - c + 1$, and in the denominator of which the laſt factor is *c*.

Q. E. I.

82. Coroll. 3. Since, by the 4th property of the figurate numbers above ſet forth in art. 54, the $\overline{n+1}|$th terms of the ſecond, third, fourth, and other following vertical columns of numbers in the foregoing table of combinations, are reſpectively equal to the ſums of the *n* firſt terms of the firſt, ſecond, third, and other following vertical columns, which ſums have been ſhewn to be, reſpectively, equal to n, $\frac{n \times \overline{n-1}}{2}$, $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, &c, it follows that the $\overline{n+1}|$th terms of the ſecond, third, fourth, fifth, ſixth, and other following vertical columns will be n, $\frac{n \times \overline{n-1}}{2}$, $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$, and $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5}$, &c; and conſequently, as every term in the firſt vertical column is $= 1$, the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns will be 1, n, $\frac{n \times \overline{n-1}}{2}$, $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$, $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5}$, &c, and, in general, the $\overline{n+1}|$th term of the *c*th column will be $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \&c \times \overline{n-c+2}}{2 \times 3 \times 4 \times 5 \, \&c \times \overline{c-1}}$.

83. Coroll. 4. Since the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns of the foregoing table of combinations are 1,

n,

n, $\frac{n\times\overline{n-1}}{2}$, $\frac{n\times\overline{n-1}\times\overline{n-2}}{2\times3}$, $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3\times4}$,
$\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times3\times4\times5}$, &c,
$\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\text{\&c}\times\overline{n-c+2}}{2\times3\times4\times5\times\text{\&c}\times\overline{c-1}}$, it follows that the nth terms of the ſaid vertical columns will be ſuch as ariſe by ſubſtituting $n-1$ inſtead of n in the foregoing values of the $\overline{n+1}$th terms, and conſequently will be as follows, to wit, 1, $n-1$, $\frac{\overline{n-1}\times\overline{n-2}}{2}$, $\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3}$, $\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times3\times4}$,
$\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}}{2\times3\times4\times5}$, &c,
$\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\text{\&c}\,\overline{n-c+1}}{2\times3\times4\times5\times\text{\&c}\times\overline{c-1}}$.

An Example of the truth of Coroll. 4.

84. As an example of the truth of this corollary we will derive in this manner the numbers that form the loweſt horizontal row of terms in the foregoing table of combinations, or the twelfth terms of the ſeveral vertical columns of the ſaid table.

Now in this caſe n, or the number of terms in the ſeveral vertical columns, is $= 12$. Therefore $n-1$, $n-2$, $n-3$, $n-4$, $n-5$, $n-6$, $n-7$, $n-8$, $n-9$, $n-10$, and $n-11$, are reſpectively equal to 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, and 1. Therefore $n-1$, or the twelfth term of the ſecond vertical column will be $= 11$; and $\frac{\overline{n-1}\times\overline{n-2}}{2}$, or the 12th term

term of the third vertical column, will be ($= \overline{n-1} \times \frac{n-2}{2} = 11 \times \frac{n-2}{2} = 11 \times \frac{10}{2}) = 11 \times 5 = 55$; and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3}$, or the 12th term of the fourth vertical column, will be ($= \frac{\overline{n-1} \times \overline{n-2}}{2} \times \frac{n-3}{3} = 55 \times \frac{n-3}{3} = 55 \times \frac{9}{3} = 55 \times 3) = 165$; and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4}$, or the 12th term of the 5th vertical column, will be ($= \frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3} \times \frac{n-4}{4} = 165 \times \frac{n-4}{4} = 165 \times \frac{8}{4} = 165 \times 2) = 330$; and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}}{2 \times 3 \times 4 \times 5}$, or the 12th term of the 6th vertical column, will be ($= \frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4} \times \frac{n-5}{5} = 330 \times \frac{n-5}{5} = 330 \times \frac{7}{5} = 66 \times 7) = 462$; and, in like manner, the 12th term of the ſeventh vertical column will be ($= 462 \times \frac{n-6}{6} = 462 \times \frac{6}{6} = 462 \times 1) = 462$; and the 12th term of the eighth vertical column will be ($= 462 \times \frac{n-7}{7} = 462 \times \frac{5}{7} = 66 \times 5) = 330$; and the 12th term of the ninth vertical column will be ($= 330 \times \frac{n-8}{8} = 330 \times \frac{4}{8} = 330 \times \frac{1}{2}) = 165$; and the 12th term of the tenth vertical column will be ($= 165 \times \frac{n-9}{9} = 165 \times \frac{3}{9} = 165 \times \frac{1}{3} = \frac{165}{3}) = 55$; and the 12th term of the eleventh vertical column will be ($= 55 \times \frac{n-10}{10} = 55 \times \frac{2}{10} = 55 \times \frac{1}{5} = \frac{55}{5}) = 11$; and the 12th term of the twelfth, or laſt, vertical column will be ($= 11 \times \frac{n-11}{11} = 11 \times \frac{1}{11}) = 1$. Therefore the 12th terms of the ſaid twelve vertical columns will be as follows, to wit, 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, and 1; which are the numbers ſet down in the foregoing table.

83. Coroll. 5. It has been ſhewn above in art. 60 and 61, that the horizontal rows of numbers in the foregoing table of combinations, beginning with the ſecond row, exhibit the co-efficients of the terms of the ſeveral ſucceſſive powers of a binomial quantity, ſuch as $a+b$, every nth horizontal row of numbers being the co-efficients of the terms of the $\overline{n-1}|$th power of the ſaid binomial quantity; whence it follows that the numbers contained in every $\overline{n+1}|$th horizontal row of numbers in the ſaid table will be the co-efficients of the terms of the nth power of the ſaid binomial quantity. But it is evident that the numbers contained in every $\overline{n+1}|$th horizontal row of terms in the ſaid table are the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns of terms in the ſaid table, reckoning the terms from the top of the ſaid table, and including the cyphers at the tops of all the ſeveral vertical columns, except the firſt. Therefore the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns of terms in the ſaid table will be the co-efficients of the terms of the nth power of the ſaid binomial quantity. But it has been ſhewn above in art. 82, coroll. 3, that the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, and ſixth, and other following vertical columns of terms in the ſaid table are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c. Therefore the co-efficients of the terms of the nth power of the ſaid binomial quantity $a+b$ will alſo be 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c; and conſequently the quantity $\overline{a+b}|^n$, or the ſaid nth power itſelf of the ſaid binomial quantity $a+b$, will be equal to the

ſeries $a^n + na^{n-1}b + n \times \frac{n-1}{2} a^{n-2}b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}b^3 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4}b^4 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{n-5}b^5$ + &c, or (if we put A for 1, and B for n and C for $n \times \frac{n-1}{2}$, and D for $n + \frac{n-1}{2} \times \frac{n-2}{3}$, and E for $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and F for $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, and G, H, I, K, L, &c, for the numeral co-efficients of the ſeventh, eighth, ninth, tenth, eleventh, and other following terms of the ſeries reſpectively) to the ſeries $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8} Ha^{n-8}b^8 + \frac{n-8}{9} Ia^{n-9}b^9 + \frac{n-9}{10} Ka^{n-10}b^{10} + \frac{n-10}{11} La^{n-11}b^{11}$ + &c.; which ſeries will continue till the numerator of the generating fraction becomes $n - n$, or 0, and conſequently the ſaid fraction itſelf becomes equal to 0 likewiſe, and therefore the term in which the ſaid fraction enters as a factor, will alſo be equal to 0, as will alſo all the following terms of the ſeries, which would be derived from the ſaid term by continual multiplications. The ſeries therefore will break off, or end with the term next preceeding the ſaid term which is equal to 0 *.

We will now proceed to illuſtrate and confirm the truth of this corollary, by applying the foregoing ſeries to the com-

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* This is the famous binomial theorem invented by Sir Iſaac Newton, but of which he has no where given a demonſtration. And the demonſtration here given of it by Mr. James Bernoulli, is that to which I alluded in the firſt volume of the Collection of Tracts, in two volumes, quarto, called *Scriptores Logarithmici*, page 349, art. 4, and in the ſecond volume of the ſame Collection, page 157, art. 9.

computation of the terms of ſome of the loweſt powers of the binomial quantity $a+b$, ſo as to produce by means of it all the numbers contained in the foregoing table of combinations, in page 74.

Examples of the application of the foregoing ſeries to the computation of the terms of the powers of the binomial quantity a + b.

86. In the firſt place let us ſuppoſe n to be $= 1$. Then we ſhall have the ſeries $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \&c$, $(= a^1 + \frac{1}{1} \times 1 \times a^{1-1} b + \frac{1-1}{2} B a^{1-2} b^2 + \frac{1-2}{3} \times C a^{1-3} b^3 + \&c, = a^1 + a^0 \times b + \frac{0}{2} \times B a^{1-2} b^2 + \frac{1-2}{3} \times C \times a^{1-3} b^3 + \&c, = a^1 + 1 \times b + 0 + 0 + \&c,) = a + b$; or the ſaid ſeries is in this caſe equal to the binomial quantity $a + b$ itſelf; as it ought to be.

87. Now let n be $= 2$.

Then we ſhall have the ſeries $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \&c$, $(= a^2 + \frac{2}{1} \times 1 \times a^{2-1} b + \frac{2-1}{2} B a^{2-2} b^2 + \frac{2-2}{3} C a^{2-3} b^3 + \&c, = a^2 + 2a^1 b + \frac{1}{2} \times 2 \times a^0 b^2 + \frac{0}{3} \times C a^{2-3} b^3 + \&c, = a^2 + 2ab + \frac{2}{2} \times 1 \times b^2 + 0 + \&c) = a^2 + 2ab + b^2$; or the ſaid ſeries will in this caſe be equal to the trinomial quantity $a^2 + 2ab + b^2$; as it ought to be, becauſe that quantity is the ſquare of the binomial quantity $a + b$.

88. If n is $= 3$, we ſhall have the ſeries $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 +$ &c $(= a^3 + \frac{3}{1} \times 1 \times a^{3-1}b + \frac{3-1}{2} \times \frac{3}{1} \times 1 \times a^{3-2}b^2 + \frac{3-2}{3} Ca^{3-3}b^3 + \frac{3-3}{4} Da^{3-4}b^4 +$ &c, $= a^3 + 3a^2b + \frac{2}{2} \times 3 \times a^1b^2 + \frac{1}{3} C \times a^0b^3 + \frac{0}{4} D \times a^{3-4}b^4 +$ &c $= a^3 + 3a^2b + 3ab^2 + \frac{1}{3} \times 3 \times 1 \times b^3 + 0 +$ &c$) = a^3 + 3a^2b + 3ab^2 + b^3$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^3$, or the cube of the binomial quantity $a + b$, will be equal to the quadrinomial quantity $a^3 + 3a^2b + 3ab^2 + b^3$; as it will be found to be upon trial.

89. If n is $= 4$, we ſhall have the ſeries $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 +$ &c $(= a^4 + \frac{4}{1} \times 1 \times a^{4-1}b + \frac{4-1}{2} Ba^{4-2}b^2 + \frac{4-2}{3} Ca^{4-3}b^3 + \frac{4-3}{4} Da^{4-4}b^4 + \frac{4-4}{5} Ea^{4-5}b^5 +$ &c $= a^4 + 4a^3b + \frac{3}{2} \times 4 \times a^2b^2 + \frac{2}{3} \times C \times a^1b^3 + \frac{1}{4} D \times a^0 \times b^4 + \frac{0}{5} E \times a^{4-5}b^5 +$ &c $= a^4 + 4a^3b + 6a^2b^2 + \frac{2}{3} \times 6 \times ab^3 + \frac{1}{4} D \times 1 \times b^4 + 0 +$ &c. $= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + \frac{1}{4} \times 4 \times b^4 + 0 +$ &c$) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^4$, or the fourth power of the binomial quantity $a + b$, will be

equal to the quinquinomial quantity $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$; as it will be found to be upon trial.

90. If n is $= 5$, we ſhall have $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \&c$, $(= a^5 + \frac{5}{1} \times 1 \times a^{5-1}b + \frac{5-1}{2} Ba^{5-2}b^2 + \frac{5-2}{3} Ca^{5-3}b^3 + \frac{5-3}{4} Da^{5-4}b^4 + \frac{5-4}{5} Ea^{5-5}b^5 + \frac{5-5}{6} Fa^{5-6}b^6 + \&c = a^5 + 5a^4b + \frac{4}{2} Ba^3b^2 + \frac{3}{3} Ca^2b^3 + \frac{2}{4} Da^1b^4 + \frac{1}{5} Ea^0b^5 + \frac{0}{6} \times Fa^{5-6}b^6 + \&c = a^5 + 5a^4b + 2Ba^3b^2 + Ca^2b^3 + \frac{1}{2} Da^1b^4 + \frac{1}{5} E \times 1 \times b^5 + 0 + \&c) = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^5$, or the fifth power of the binomial quantity $a+b$, will be equal to the ſextinomial quantity $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$; as, upon trial, it will be found to be.

91. If n is $= 6$, we ſhall have the ſeries $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \&c$ $(= a^6 + \frac{6}{1} \times 1 \times a^{6-1}b + \frac{6-1}{2} Ba^{6-2}b^2 + \frac{6-2}{3} Ca^{6-3}b^3 + \frac{6-3}{4} Da^{6-4}b^4 + \frac{6-4}{5} Ea^{6-5}b^5 + \frac{6-5}{6} Fa^{6-6}b^6 + \frac{6-6}{7} Ga^{6-7}b^7 + \&c = a^6 + 6a^5b + \frac{5}{2} Ba^4b^2 + \frac{4}{3} Ca^3b^3 + \frac{3}{4} Da^2b^4 + \frac{2}{5} Ea^1b^5 + \frac{1}{6} Fa^0b^6 + \frac{0}{7} Ga^{6-7}b^7 + \&c) = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. Therefore

fore $\overline{a+b}|^{n}$, or $\overline{a+b}|^{6}$, or the ſixth power of the binomial quantity $a+b$, will be equal to the ſeptinomial quantity $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. And ſo, upon trial, it will be found to be.

92. If n is $= 7$, we ſhall have the ſeries $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8} Ha^{n-8}b^8 +$ &c $(= a^7 + \frac{7}{1} \times 1 \times a^6b + \frac{6}{2} Ba^5b^2 + \frac{5}{3} Ca^4b^3 + \frac{4}{4} Da^3b^4 + \frac{3}{5} Ea^2b^5 + \frac{2}{6} Fa^1b^6 + \frac{1}{7} Ga^0b^7 + \frac{0}{8} Ha^{7-8}b^8 +$ &c $= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + 1 \times b^7 + 0 +$ &c$) = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$. Therefore $\overline{a+b}|^{n}$, or $\overline{a+b}|^{7}$, or the ſeventh power of the binomial quantity $a + b$, will be equal to the octinomial quantity $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$. And ſo, upon trial, it will be found to be.

93. If n is $= 8$, we ſhall have the ſeries $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8} H a^{n-8}b^8 + \frac{n-8}{9} I a^{n-9}b^9 +$ &c $(= a^8 + \frac{8}{1} \times 1 \times a^7b + \frac{7}{2} Ba^6b^2 + \frac{6}{3} Ca^5b^3 + \frac{5}{4} Da^4b^4 + \frac{4}{5} Ea^3b^5 + \frac{3}{6} Fa^2b^6 + \frac{2}{7} Ga^1b^7 + \frac{1}{8} Ha^0b^8 + \frac{0}{9} I a^{8-9}b^9 +$ &c $= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + 1 \times b^8 + 0 +$ &c$) = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4$

$56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$. Therefore $\overline{a+b}\rvert^{n}$, or $\overline{a+b}\rvert^{8}$, or the eighth power of the binomial quantity $a+b$, will be equal to the compound quantity $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$. And ſo, upon trial, it will be found to be.

94. If n is $= 9$, we ſhall have the ſeries $a^n + \frac{n}{1}Aa^{n-1}b + \frac{n-1}{2}Ba^{n-2}b^2 + \frac{n-2}{3}Ca^{n-3}b^3 + \frac{n-3}{4}Da^{n-4}b^4 + \frac{n-4}{5}Ea^{n-5}b^5 + \frac{n-5}{6}Fa^{n-6}b^6 + \frac{n-6}{7}Ga^{n-7}b^7 + \frac{n-7}{8}Ha^{n-8}b^8 + \frac{n-8}{9}Ia^{n-9}b^9 + \frac{n-9}{10}Ka^{n-10}b^{10} +$ &c ($= a^9 + \frac{9}{1} \times 1 \times a^8b + \frac{8}{2}Ba^7b^2 + \frac{7}{3}Ca^6b^3 + \frac{6}{4}Da^5b^4 + \frac{5}{5}Ea^4b^5 + \frac{4}{6}Fa^3b^6 + \frac{3}{7}Ga^2b^7 + \frac{2}{8}Ha^1b^8 + \frac{1}{9}Ia^0b^9 + \frac{0}{10}Ka^{9-10}b^{10} +$ &c $= a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + 1 \times b^9 + 0 +$ &c) $= a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$. Therefore $\overline{a+b}\rvert^{n}$, or $\overline{a+b}\rvert^{9}$, or the ninth power of the binomial quantity $a+b$, will be equal to the compound quantity $a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$. And ſo, upon trial, it will be found to be.

95. If n is $= 10$, we ſhall have the ſeries $a^n + \frac{n}{1}Aa^{n-1}b + \frac{n-1}{2}Ba^{n-2}b^2 + \frac{n-2}{3}Ca^{n-3}b^3 + \frac{n-3}{4}Da^{n-4}b^4 + \frac{n-4}{5}Ea^{n-5}b^5 + \frac{n-5}{6}Fa^{n-6}b^6 + \frac{n-6}{7}Ga^{n-7}b^7 + \frac{n-7}{8}Ha^{n-8}b^8 + \frac{n-8}{9}Ia^{n-9}b^9 + \frac{n-9}{10}Ka^{n-10}b^{10} + \frac{n-10}{11}$

$La^{n-11}b^{11}$ + &c $(= a^{10} + \frac{10}{1} \times 1 \times a^9b + \frac{9}{2} Ba^8b^2 + \frac{8}{3} Ca^7b^3 + \frac{7}{4} Da^6b^4 + \frac{6}{5} Ea^5b^5 + \frac{5}{6} Fa^4b^6 + \frac{4}{7} Ga^3b^7 + \frac{3}{8} Ha^2b^8 + \frac{2}{9} Ia^1b^9 + \frac{1}{10} Ka^0b^{10} + \frac{0}{11} La^{10-11}b^{11}$ + &c $= a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10a^1b^9 + 1 \times b^{10} + 0 +$ &c$) = a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^{10}$, or the tenth power of the binomial quantity $a + b$, will be equal to the compound quantity $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$. And ſo, upon trial, it will be found to be.

96. Laſtly, let n be $= 11$. Then will the ſeries $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8} Ha^{n-8}b^8 + \frac{n-8}{9} Ia^{n-9}b^9 + \frac{n-9}{10} Ka^{n-10}b^{10} + \frac{n-10}{11} La^{n-11}b^{11} + \frac{n-11}{12} Ma^{n-12}b^{12}$ + &c be $(= a^{11} + \frac{11}{1} \times 1 \times a^{10}b + \frac{10}{2} Ba^9b^2 + \frac{9}{3} Ca^8b^3 + \frac{8}{4} Da^7b^4 + \frac{7}{5} Ea^6b^5 + \frac{6}{6} Fa^5b^6 + \frac{5}{7} Ga^4b^7 + \frac{4}{8} Ha^3b^8 + \frac{3}{9} Ia^2b^9 + \frac{2}{10} Ka^1b^{10} + \frac{1}{11} La^0b^{11} + \frac{0}{12} Ma^{11-12}b^{12}$ + &c $= a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5 + 462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + 1 \times b^{11} + 0 +$ &c$) = a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5 + 462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + b^{11}$. Therefore

fore $\overline{a+b}|^{n}$, or $\overline{a+b}|^{11}$, or the eleventh power of the binomial quantity $a+b$, will be equal to the compound quantity $a^{11} + 11a^{10}b + 55a^{9}b^{2} + 165a^{8}b^{3} + 330a^{7}b^{4} + 462a^{6}b^{5}$ $462a^{5}b^{6} + 330a^{4}b^{7} + 165a^{3}b^{8} + 55a^{2}b^{9} + 11ab^{10} + b^{11}$. And ſo, upon trial, it will be found to be.

97. It appears therefore that the ſeries $a^{n} + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^{2} + \frac{n-2}{3} C a^{n-3} b^{3} + \frac{n-3}{4} D a^{n-4} b^{4} + \frac{n-4}{5} E a^{n-5} b^{5} + \frac{n-5}{6} F a^{n-6} b^{6}$ + &c, which has been obtained in the foregoing 5th corollary for the value of the quantity $\overline{a+b}|^{n}$, or the nth power of the binomial quantity $a+b$, does truly exhibit the value of the ſaid power when the index n is equal to either 1, or 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, or 10, or 11; in which caſes the co-efficients of the terms of the ſaid ſeries are equal to the numbers contained in the ſeveral ſucceſſive horizontal rows of terms in the foregoing table of combinations, in page 74, beginning with the ſecond row.

Additional Corollaries, not contained in the original text of Mr. James Bernoulli.

98. To theſe five corollaries, which are contained in Mr. James Bernoulli's original text, it may not be amiſs to add the following corollaries, which are eaſily deducible from Mr. Bernoulli's propoſitions, and which will enable us to find a general expreſſion for the terms of any of the verti-

cal columns in the foregoing table of combinations, or, in other words, for the figurate numbers of any propoſed order.

99. Coroll. 6. It has been ſhewn in art. 82, corol. 3, page 112, that the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns of terms in the foregoing table of combinations are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c. But the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns of terms in the ſaid table are the firſt, ſecond, third, fourth, fifth, ſixth, and other following terms of the $\overline{n+1}|$th horizontal row of terms in the ſaid table. Therefore the firſt, ſecond, third, fourth, fifth, ſixth, and other following terms of the $\overline{n+1}|$th horizontal row of terms in the ſaid table are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c.

100. Coroll. 7. Since the ſeveral terms of the $\overline{n+1}|$th horizontal row are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c, it follows that, if p be any whole number greater than n, as, for example, $n+1$, or $n+2$, or $n+3$, or $n+4$, &c, the ſeveral terms of the $\overline{p+1}|$th horizontal row will be 1, p, $p \times \frac{p-1}{2}$, $p \times \frac{p-1}{2} \times \frac{p-2}{3}$, $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}$, and $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}$, &c. And conſequently the third term of the $\overline{p+1}|$th horizontal row of terms, when p is equal to $n+1$, or the third term of the $\overline{n+2}|$th horizontal

row of terms, will be the quantity which ariſes by ſubſtituting $n+1$ inſtead of p in the third term, $p\times\frac{p-1}{2}$, of the laſt-mentioned ſeries, that is, $\overline{n+1}\times\frac{n+1-1}{2}$, or $\overline{n+1}\times\frac{n}{2}$, or $n\times\frac{n+1}{2}$; and the fourth term of the $\overline{p+1}$th horizontal row of terms when p is $=n+2$, or the fourth term of the $\overline{n+3}$th horizontal row of terms will be that which ariſes by ſubſtituting $n+2$ inſtead of p in the fourth term, $p\times\frac{p-1}{2}\times\frac{p-2}{3}$, of the laſt ſeries, that is, $\overline{n+2}\times\frac{n+2-1}{2}\times\frac{n+2-2}{3}$, or $\overline{n+2}\times\frac{n+1}{2}\times\frac{n}{3}$, or $n\times\frac{n+1}{2}\times\frac{n+2}{3}$.

And, in like manner, if we ſubſtitute $n+3$ inſtead of p in the 5th term $p\times\frac{p-1}{2}\times\frac{p-2}{3}\times\frac{p-3}{4}$ of the laſt ſeries, we ſhall have $\overline{n+3}\times\frac{n+3-1}{2}\times\frac{n+3-2}{3}\times\frac{n+3-3}{4}$, or $\overline{n+3}\times\frac{n+2}{2}\times\frac{n+1}{3}\times\frac{n}{4}$, or $n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$, for the 5th term of the $\overline{n+4}$th horizontal row of terms; and, if we ſubſtitute $n+4$ inſtead of p in the 6th term, $p\times\frac{p-1}{2}\times\frac{p-2}{3}\times\frac{p-3}{4}\times\frac{p-4}{5}$, of the laſt ſeries, we ſhall have $\overline{n+4}\times\frac{n+4-1}{2}\times\frac{n+4-2}{3}\times\frac{n+4-3}{4}\times\frac{n+4-4}{5}$, or $\overline{n+4}\times\frac{n+3}{2}\times\frac{n+2}{3}\times\frac{n+1}{4}\times\frac{n}{5}$, or $n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\times\frac{n+4}{5}$, for the 6th term of the $\overline{n+5}$th horizontal row of terms. So that the 3d term of the $\overline{n+2}$th horizontal row of terms, and the 4th term of the $\overline{n+3}$th horizontal row, and the 5th term of the $\overline{n+4}$th horizontal row, and the 6th term of the $\overline{n+5}$th horizontal row, will be $n\times\frac{n+1}{2}$, $n\times$

$\frac{n+1}{2} \times \frac{n+2}{3}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, and $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, reſpectively.

101. Coroll. 8. Theſe four terms $n \times \frac{n+1}{2}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, and $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, are derived from the number n by the continual multiplication of the fractions $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, and $\frac{n+4}{5}$, the numerators and denominators of which both increaſe continually by an unit. Therefore, if we put C for the firſt, D for the ſecond, E for the third, and F for the fourth of theſe terms, we ſhall have $C = \frac{n+1}{2} \times n$, and $D = \frac{n+2}{3} \times C$, and $E = \frac{n+3}{4} \times D$, and $F = \frac{n+4}{5} \times E$.

And, from the manner in which theſe four terms were derived from the 3d, 4th, 5th, and 6th terms of the ſeries $1, p, p \times \frac{p-1}{2}, p \times \frac{p-1}{2} \times \frac{p-2}{3}, p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}, p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}$, &c, in the laſt corollary, to wit, by ſubſtituting $n+1$, $n+2$, $n+3$, and $n+4$, inſtead of p in the ſaid 3d, 4th, 5th, and 6th terms, reſpectively, it is eaſy to ſee that the 7th term of the $\overline{n+6|}$th horizontal row of terms, and the 8th term of the $\overline{n+7|}$th horizontal row, and the 9th term of the $\overline{n+8|}$th horizontal row, and the 10th term of the $\overline{n+9|}$th horizontal row, and the 11th, 12th, 13th, 14th, 15th, and other following terms of the $\overline{n+10|}$th, $\overline{n+11|}$th, $\overline{n+12|}$th, $\overline{n+13|}$th, $\overline{n+14|}$th, and other following horizontal rows of terms in the ſaid table, reſpectively, will be equal to $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$,

× H, $\frac{n+8}{9}$ × I, $\frac{n+9}{10}$ × K, $\frac{n+10}{11}$ × L, $\frac{n+11}{12}$ × M, $\frac{n+12}{13}$ × N, $\frac{n+13}{14}$ × O, &c, in which the capital letters G, H, I, K, L, M, N, O, &c, denote the 7th, 8th, 9th, 10th, 11th, 12th, 13th, 14th, 15th, and other next following terms, of the $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, $\overline{n+10}|$th, and other following horizontal rows of terms, reſpectively, as they ariſe, and the generating fractions $\frac{n+5}{6}$, $\frac{n+6}{7}$, $\frac{n+7}{8}$, $\frac{n+8}{9}$, $\frac{n+9}{10}$, $\frac{n+10}{11}$, $\frac{n+11}{12}$, $\frac{n+12}{13}$, $\frac{n+13}{14}$, &c, are a continuation of the foregoing generating fractions $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, and are derived from them by the continual addition of an unit to both their numerators and denominators.

102. Coroll. 9. It is ſhewn above in the 6th property of the numbers contained in the foregoing table of combinations, art. 56 and 57, pages 77 and 78, that in every horizontal row of terms in the ſaid table of combinations, the firſt and laſt term are, each of them, an unit, and the terms that are equidiſtant from the firſt and laſt terms are equal to each other. It follows therefore that the 3d term of the $\overline{n+2}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 3d term of it reckoned from the beginning, or from the left hand to the right; and that the 4th term of the $\overline{n+3}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 4th term of it reckoned from the beginning, or from the left hand to the right; and that the 5th term of the $\overline{n+4}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 5th term of it, reckoned from the beginning, or from the left hand to the right; and that the 6th term of the $\overline{n+5}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 6th term of it, reckoned from the beginning, or from the left hand to the right; and, in like manner, that the

7th, and 8th, and 9th, and 10th, and other following terms of the $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, and other following horizontal rows of terms, reſpectively, reckoned from the ends of the ſaid rows, or from the right hand to the left, will be equal to the 7th, and 8th, and 9th, and 10th, and other following correſponding terms of the ſame horizontal rows, reſpectively, reckoned from the beginnings of the ſaid rows, or from the left hand to the right. But it was ſhewn in corollary 7th, that the 3d term of the $\overline{n+2}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2}$; and that the 4th term of the $\overline{n+3}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3}$; and that the 5th term of the $\overline{n+4}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$; and that the ſixth term of the $\overline{n+5}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$: or that, if the ſaid third term of the $\overline{n+2}|$th horizontal row of terms, reckoned from the beginning of it, be called C, and the ſaid 4th term of the $\overline{n+3}|$th horizontal row of terms, reckoned from the beginning of it, be called D, and the ſaid 5th term of the $\overline{n+4}|$th horizontal row of terms, reckoned from the beginning of it, be called E, and the ſaid 6th term of the $\overline{n+5}|$th horizontal row of terms, reckoned from the beginning of it, be called F, we ſhall have $C = n \times \frac{n+1}{2}$, and $D = \frac{n+2}{3} \times C$, and $E = \frac{n+3}{4} \times D$, and $F = \frac{n+4}{5} \times E$. And it is obſerved in the laſt, or 8th, corollary, that, if the 7th term of the $\overline{n+6}|$th horizontal row of terms be called G, and the 8th term of the $\overline{n+7}|$th horizontal row be called H, and the 9th term of the $\overline{n+8}|$th horizontal row, and the 10th term of the $\overline{n+9}|$th horizontal row, and the 11th term of the $\overline{n+10}|$th horizontal row, and the 12th term of the $\overline{n+11}|$th horizontal row, and the next following terms of the next following horizontal rows, all reckoned from the beginnings of thoſe ſeveral horizontal rows, or from the left hand to the right, be called I, K, L, and M, &c, reſpectively, we ſhall have G $= \frac{n+5}{6} \times$ F, and H $= \frac{n+6}{7} \times$ G, and I $= \frac{n+7}{8} \times$ H, and K $= \frac{n+8}{9} \times$ I, and L $= \frac{n+9}{10} \times$ K, and M $= \frac{n+10}{11} \times$ L, and conſequently that the 3d term of the $\overline{n+2}|$th horizontal row of terms, and the 4th term of the $\overline{n+3}|$th horizontal row of terms, and the 5th, 6th, 7th, 8th, 9th, 10th, 11th, and 12th, and other next following terms of the $\overline{n+4}|$th, $\overline{n+5}|$th, $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, $\overline{n+10}|$th, and $\overline{n+11}|$th, and other next following horizontal rows of terms, reckoned from the beginnings of thoſe ſeveral horizontal rows, or from the left hand to the right, will be equal to $n \times \frac{n+1}{2}$, or C, and $\frac{n+2}{3} \times$ C, $\frac{n+3}{4} \times$ D, $\frac{n+4}{5} \times$ E, $\frac{n+5}{6} \times$ F, $\frac{n+6}{7} \times$ G, $\frac{n+7}{8} \times$ H, $\frac{n+8}{9} \times$ I, $\frac{n+9}{10} \times$ K, and $\frac{n+10}{11} \times$ L, &c, reſpectively. It follows therefore that the 3d term of the $\overline{n+2}|$th horizontal row of terms, and the 4th term of the $\overline{n+3}|$th horizontal row of terms, and the 5th, 6th, 7th, 8th, 9th, 10th, 11th, and 12th, and other next following terms of the $\overline{n+4}|$th, $\overline{n+5}|$th, $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, $\overline{n+10}|$th, and $\overline{n+11}|$th, and other next following horizontal rows of terms, reckoned from the ends of thoſe ſeveral horizontal rows, or from the right hand to the left, will alſo be reſpectively equal to $n \times \frac{n+1}{2}$, or C, and $\frac{n+2}{3} \times C$, $\frac{n+3}{4} \times D$, $\frac{n\times 4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$, $\frac{n+8}{9} \times I$, $\frac{n+9}{10} \times K$, and $\frac{n+10}{11} \times L$, &c.

Of the figurate numbers, or the ſignificant terms of the vertical columns of terms in the foregoing table of combinations, page 74.

103. Coroll. 10. We come now to conſider the vertical columns of terms in the foregoing table of combinations.

Now it is evident, in the firſt place, that the firſt ſignificant term in every vertical column of terms in the ſaid table is an unit, and that the ſecond ſignificant term is the number which is the exponent of the column; as has been obſerved above in art. 53, page 76. So that, if the whole number *n* be the exponent of the column, the two firſt ſignificant terms of the ſaid column, immediately following the cyphers at the top of it, will be 1 and *n*. It remains that we find the values of the following terms in the ſaid column, after the terms 1 and *n*. Now this may be done by means of the foregoing corollaries, in the manner following.

In the foregoing table of combinations the number of cyphers at the top of the vertical column of which the exponent is *n*, is $n-1$; as is obſerved above in art. 51, page 75. And conſequently the firſt ſignificant term in the ſaid vertical column, to wit, 1, will be the *n*th term of it, and conſequently will be ſituated in the *n*th horizontal row of terms in the ſaid table; and the ſecond ſignificant term in the ſaid vertical column of terms, to wit, *n*, will be ſituated in the $\overline{n+1}$th horizontal row of terms; and the 3d ſignificant term in the ſaid vertical column will be ſituated in the $\overline{n+2}$th horizontal row of terms; and the 4th ſignificant figure in the ſaid vertical column will be ſituated in the $\overline{n+3}$th horizontal row of terms; and, in like manner, the 5th, and 6th, and 7th, and 8th, and 9th, and 10th, and other following ſignificant terms in the ſaid *n*th vertical column will be ſituated in the $\overline{n+4}$th, and $\overline{n+5}$th, and $\overline{n+6}$th, and $\overline{n+7}$th, and $\overline{n+8}$th, and $\overline{n+9}$th, and other following horizontal rows of terms reſpectively.

And, further, the firſt ſignificant term, to wit, 1, in the ſaid *n*th vertical column of terms, is likewiſe the firſt term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and the ſecond ſignificant term in the ſaid *n*th vertical column of terms, to wit, *n*, is likewiſe the ſecond term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and the 3d ſignificant term in the ſaid *n*th vertical column is likewiſe the third term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and the 4th ſignificant term of the ſaid *n*th vertical column is likewiſe the 4th term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and, in like manner, the 5th, 6th, 7th, 8th, and other following ſignificant terms of the ſaid *n*th vertical column of terms, are likewiſe the 5th, 6th, 7th, 8th, and other following terms, reckoned from the right hand to the left, of the ſeveral horizontal rows of terms in which they are ſituated, reſpectively.

But it has been ſhewn that the 1ſt, 2d, 3d, 4th, 5th, 6th, and other following ſignificant terms in the *n*th vertical column of terms are ſituated in the *n*th, $\overline{n+1}$th, $\overline{n+2}$th, $\overline{n+3}$th, $\overline{n+4}$th, $\overline{n+5}$th, and other next following horizontal rows of terms, reſpectively.

Therefore the 1ſt, 2d, 3d, 4th, 5th, 6th, and other following ſignificant terms of the *n*th vertical column of terms are likewiſe the 1ſt, 2d, 3d, 4th, 5th, 6th, and other next following terms, reckoned from the right hand to the left, of the nth, $\overline{n+1}$th, $\overline{n+2}$th, $\overline{n+3}$th, $\overline{n+4}$th, $\overline{n+5}$th, and other next following horizontal rows of terms, reſpectively.

But it has been ſhewn in coroll. 9, that the 3d term of the $\overline{n+2}$th horizontal row of terms, reckoned from the right hand to the left, is equal to $n \times \frac{n+1}{2}$, or $\frac{n+1}{2} \times n$, or $\frac{n+1}{2} \times B$; and that the 4th term of the $\overline{n+3}$th horizontal row of terms, reckoned from the right hand to the left, is $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, or $\frac{n+2}{3} \times C$; and that the 5th term of the $\overline{n+4}$th horizontal row of terms, and the 6th term of the $\overline{n+5}$th horizontal row of terms, and the 7th, 8th, 9th, 10th, and other next following terms of the $\overline{n+6}$th, $\overline{n+7}$th, $\overline{n+8}$th, $\overline{n+9}$th, and other next following horizontal rows of terms, reſpectively, all reckoned from the right hand to the left, are equal to $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, &c.

Therefore the 3d ſignificant term of the nth vertical column of terms will be equal to $n \times \frac{n+1}{2}$, or $\frac{n+1}{2} \times n$, or $\frac{n+1}{2} \times B$; and the 4th ſignificant term of the ſame vertical column will be equal to $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, or $\frac{n+2}{3} \times C$; and the 5th, and 6th, and 7th, and 8th, and 9th, and 10th, and other following ſignificant terms of the ſame vertical column will be equal to $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$, $\frac{n+8}{9} \times I$, &c.; and conſequently the whole of the ſaid nth vertical column of terms, including the two firſt ſignificant terms 1 and n, or, in other words, the whole ſeries of figurate numbers of the nth order, will be 1, n, or $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, $\frac{n+10}{11}$ L, $\frac{n+11}{12}$ M, $\frac{n+12}{13}$ N, $\frac{n+13}{14}$ O, $\frac{n+14}{15}$ P, &c, *ad infinitum.* Q. E. I.

Examples of the application of the foregoing ſeries to the computation of the figurate numbers of ſeveral ſucceſſive orders.

104. In the firſt place we will ſuppoſe the letter n to denote 1.

Then we ſhall have $n+1$ $(= 1 + 1) = 2$, and $n+2$ $(= 1 + 2) = 3$, and $n+3$ $(= 1 + 3) = 4$, and $n+4$ $(= 1 + 4) = 5$, and $n+5$, $n+6$, $n+7$, $n+8$, $n+9$, $n+10$, &c, $=$ 6, 7, 8, 9, 10, 11, &c, reſpectively. Therefore the ſeveral terms 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, $\frac{n+10}{11}$ L, &c, will, in this caſe, be equal to 1, $\frac{1}{1}$ A, $\frac{2}{2}$ B, $\frac{3}{3}$ C, $\frac{4}{4}$ D, $\frac{5}{5}$ E, $\frac{6}{6}$ F, $\frac{7}{7}$ G, $\frac{8}{8}$ H, $\frac{9}{9}$ I, $\frac{10}{10}$ K, $\frac{11}{11}$ L, &c, reſpectively, or to 1, 1 A, 1 B, 1 C, 1 D, 1 E, 1 F, 1 G, 1 H, 1 I, 1 K, 1 L, &c, or to 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c; and therefore the twelve firſt terms in the firſt vertical column in the foregoing table of combinations, or the twelve firſt figurate numbers of the 1ſt order, obtained by means of the foregoing ſeries, will be 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, and 1, or a ſet of units; as they ought to be, and as they are in the foregoing table of combinations.

105. In the next place we will ſuppoſe the exponent *n* to be equal to 2, in order to obtain, by means of the foregoing ſeries, the ſignificant terms in the ſecond vertical column in the foregoing table of combinations, or the figurate numbers of the ſecond order.

Now, if *n* is = 2, the terms of the ſeries 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, &c, will be reſpectively equal to 1, $\frac{2}{1}$ A, $\frac{2+1}{2}$ B, $\frac{2+2}{3}$ C, $\frac{2+3}{4}$ D, $\frac{2+4}{5}$ E, $\frac{2+5}{6}$ F, $\frac{2+6}{7}$ G, $\frac{2+7}{8}$ H, $\frac{2+8}{9}$ I, $\frac{2+9}{10}$ K, &c, or 1, $\frac{2}{1}$ A, $\frac{3}{2}$ B, $\frac{4}{3}$ C, $\frac{5}{4}$ D, $\frac{6}{5}$ E, $\frac{7}{6}$ F, $\frac{8}{7}$ G, $\frac{9}{8}$ H, $\frac{10}{9}$ I, $\frac{11}{10}$ K, &c, or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. Therefore the firſt eleven ſignificant terms in the ſecond vertical column of terms in the foregoing table of combinations, or the firſt eleven figurate numbers of the ſecond order, obtained by means of the foregoing ſeries, will be the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11; as they ought to be.

106. In the 3d place we will ſuppoſe the exponent *n* to be equal to 3, in order to obtain, by means of the foregoing ſeries, the ſignificant terms in the 3d vertical column of terms in the foregoing table of combinations, or the figurate numbers of the third order, or (as they are often called) the *triangular* numbers.

Now, if *n* is = 3, the terms of the ſeries 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I,

&c,

&c, will be reſpectively equal to 1, $\frac{3}{1}$ A, $\frac{3+1}{2}$ B, $\frac{3+2}{3}$ C, $\frac{3+3}{4}$ D, $\frac{3+4}{5}$ E, $\frac{3+5}{6}$ F, $\frac{3+6}{7}$ G, $\frac{3+7}{8}$ H, $\frac{3+8}{9}$ I, &c, or 1, $\frac{3}{1}$ A, $\frac{4}{2}$ B, $\frac{5}{3}$ C, $\frac{6}{4}$ D, $\frac{7}{5}$ E, $\frac{8}{6}$ F, $\frac{9}{7}$ G, $\frac{10}{8}$ H, $\frac{11}{9}$ I, &c, or 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, &c. Therefore the firſt ten ſignificant terms in the third vertical column of terms in the foregoing table of combinations, or the firſt ten figurate numbers of the third order, or the firſt ten triangular numbers, obtained by means of the foregoing ſeries, are 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55; which are the ſame numbers with thoſe ſet down above in page 74, in the third vertical column of the foregoing table of combinations.

107. In the 4th place we will ſuppoſe the exponent n to be = 4, in order to obtain, by means of the foregoing ſeries, the ſignificant terms in the 4th vertical column in the foregoing table of combinations, or the figurate numbers of the 4th order, or (as they are often called) the *pyramidal* numbers.

Now, if n is = 4, the terms of the ſeries 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, &c, will be reſpectively equal to 1, $\frac{4}{1}$ A, $\frac{4+1}{2}$ B, $\frac{4+2}{3}$ C, $\frac{4+3}{4}$ D, $\frac{4+4}{5}$ E, $\frac{4+5}{6}$ F, $\frac{4+6}{7}$ G, $\frac{4+7}{8}$ H, &c, or 1, $\frac{4}{1}$ A, $\frac{5}{2}$ B, $\frac{6}{3}$ C, $\frac{7}{4}$ D, $\frac{8}{5}$ E, $\frac{9}{6}$ F, $\frac{10}{7}$ G, $\frac{11}{8}$ H, &c, or 1, 4, 10, 20, 35, 56, 84, 120, 165, &c. Therefore the firſt nine ſignificant terms in the fourth vertical column of terms in the foregoing table of combinations, or the firſt nine figurate numbers of the 4th order, or the firſt nine pyramidal numbers, obtained by means of the foregoing ſeries, are 1, 4, 10, 20, 35, 56, 84, 120, and 165; which are the ſame numbers with thoſe ſet down above in page 74, in the fourth vertical column of the foregoing table of combinations.

108. In the 5th place we will ſuppoſe the exponent n to be = 5, in order to obtain, by means of the foregoing ſeries, the ſignificant terms in the 5th vertical column of terms in the foregoing table of combinations, or the figurate number of the 5th order.

Now, if n is = 5, the terms of the ſeries 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, &c, will be reſpectively equal to 1, $\frac{5}{1}$ A, $\frac{5+1}{2}$ B, $\frac{5+2}{3}$ C, $\frac{5+3}{4}$ D, $\frac{5+4}{5}$ E, $\frac{5+5}{6}$ F, $\frac{5+6}{7}$ G, &c, or 1, $\frac{5}{1}$ A, $\frac{6}{2}$ B, $\frac{7}{3}$ C, $\frac{8}{4}$ D, $\frac{9}{5}$ E, $\frac{10}{6}$ F, $\frac{11}{7}$ G, &c, or 1, 5, 15, 35, 70, 126, 210, 330, &c. Therefore the firſt eight ſignificant terms in the 5th vertical column of terms in the foregoing table of combinations, or the firſt eight figurate numbers of the 5th order, are 1, 5, 15, 35, 70, 126, 210, and 330; which are the ſame numbers with thoſe ſet down above in page 74 in the 5th vertical column of the foregoing table of combinations.

109. In like manner, if the exponent n is = 6, the terms of the ſeries 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, &c, will be reſpectively equal to 1, $\frac{6}{1}$ A, $\frac{7}{2}$ B, $\frac{8}{3}$ C, $\frac{9}{4}$ D, $\frac{10}{5}$ E, $\frac{11}{6}$ F, &c, or 1, 6, 21, 56, 126, 252, 462, &c. and conſequently the firſt ſeven ſignificant terms of the 6th vertical column of terms in the foregoing table, or the firſt ſeven figurate numbers of the 6th order, will be 1, 6, 21, 56, 126, 252, and 462; which are the ſame numbers with thoſe ſet down above in page 74 in the 6th vertical column of the foregoing table of combinations.

110. And, if the exponent n is $=7$, the terms of the ſeries $1, \frac{n}{1}A, \frac{n+1}{2}B, \frac{n+2}{3}C, \frac{n+3}{4}D, \frac{n+4}{5}E$, &c, will be reſpectively equal to $1, \frac{7}{1}A, \frac{8}{2}B, \frac{9}{3}C, \frac{10}{4}D, \frac{11}{5}E$, &c, or 1, 7, 28, 84, 210, 462, &c; and conſequently the firſt ſix ſignificant terms of the 7th vertical column of terms in the foregoing table of combinations, or the firſt ſix figurate numbers of the 7th order, will be 1, 7, 28, 84, 210, and 462; which are the ſame numbers with thoſe ſet down above, in page 74, in the 7th vertical column of the foregoing table of combinations.

111. And, if the exponent n is $=8$, the terms of the ſeries $1, \frac{n}{1}A, \frac{n+1}{2}B, \frac{n+2}{3}C, \frac{n+3}{4}D$, &c, will be reſpectively equal to $1, \frac{8}{1}A, \frac{9}{2}B, \frac{10}{3}C, \frac{11}{4}D$, &c, or 1, 8, 36, 120, 330, &c; and conſequently the firſt five ſignificant terms of the 8th vertical column of terms in the foregoing table of combinations, or the firſt five figurate numbers of the 8th order, will be 1, 8, 36, 120, and 330; which are the ſame numbers with thoſe ſet down above in page 74, in the 8th vertical column of the foregoing table of combinations.

112. And, if the exponent n is $=9$, the terms of the ſeries $1, \frac{n}{1}A, \frac{n+1}{2}B, \frac{n+2}{3}C$, &c, will be reſpectively equal to $1, \frac{9}{1}A, \frac{10}{2}B, \frac{11}{3}C$, &c, or 1, 9, 45, 165, &c; and conſequently the firſt four ſignificant terms in the 9th vertical column of terms in the foregoing table of combinations, or the firſt four figurate numbers of the 9th order, will be 1, 9, 45, and 165; which are the ſame numbers with thoſe ſet down above in page 74, in the 9th vertical column of the foregoing table of combinations.

113. And, if the exponent n is $= 10$, the terms 1, $\frac{n}{1}$A, $\frac{n+1}{2}$B, &c, will be reſpectively equal to 1, $\frac{10}{1}$A, $\frac{11}{2}$B, &c, or 1, 10, 55, &c; and conſequently the three firſt ſignificant terms of the 10th vertical column of terms in the foregoing table of combinations, or the three firſt figurate numbers of the 10th order, will be 1, 10, and 55; which are the ſame numbers with thoſe ſet down above in page 74, in the 10th vertical column of the ſaid table of combinations.

114. It appears, therefore, that all the numbers ſet down above in page 74, in the ten firſt vertical columns of the foregoing table of combinations, may be obtained by the application of the general ſeries 1, $\frac{n}{1}$A, $\frac{n+1}{2}$B, $\frac{n+2}{3}$C, $\frac{n+3}{4}$D, $\frac{n+4}{5}$E, $\frac{n+5}{6}$F, $\frac{n+6}{7}$G, $\frac{n+7}{8}$H, $\frac{n+8}{9}$I, $\frac{n+9}{10}$K, $\frac{n+10}{11}$L, &c; which is a confirmation of the truth of the ſaid ſeries, and of the reaſonings by which it was obtained.

A general

A general expreſſion of the value of the fraction $\frac{1}{\overline{a+b}|^{n}}$, *or the reciprocal of any integral power of the binomial quantity* a + b, *in an infinite ſeries.*

115. Coroll. 11. From the foregoing corollary we may derive a general expreſſion for the value of the quantity $\overline{a+b}|^{-n}$, or $\frac{1}{\overline{a+b}|^{n}}$, in an infinite ſeries of terms, when the index n is any whole number whatſoever.

For the quantity $\frac{1}{\overline{a+b}|^{n}}$ is equal to the ſeries which reſults from the diviſion of the numerator 1 by the binomial quantity $a+b$ as many times as there are units in the index n. And the quotients that ariſe from theſe diviſions are a ſet of infinite ſerieſes conſiſting of terms marked alternately with the ſign — and the ſign +, and of which the numeral coefficients will be the figurate numbers of the ſeveral ſucceſſive orders. This will appear by making a few of theſe diviſions; which I ſhall therefore now proceed to make: but, in order to render the operations ſomewhat ſhorter and eaſier than they otherwiſe would be, I ſhall ſubſtitute the binomial quantity $1+x$ inſtead of the binomial quantity $a+b$, which will make no change whatever in the numeral coefficients of the terms of the ſeveral quotients that will reſult from theſe diviſions: and I ſhall ſuppoſe the quantity x to be leſs than 1, to the end that the powers of x in the terms of the ſeveral quotients may be decreaſing quantities.

116. The firſt of theſe diviſions will be as follows:

Diviſor. Quotient.

$$1+x)\ (1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+\ \&c.$$

Dividend.

$$\begin{array}{l}
1\ \ *\ \ *\ \ *\ \ *\ \ *\ \ *\ \ *\ \ *\\
\underline{1+x}\\
*\ -x\\
\underline{-x-x^2}\\
*\ +x^2\\
\underline{+x^2+x^3}\\
*\ -x^3\\
\underline{-x^3-x^4}\\
*\ +x^4\\
\underline{+x^4+x^5}\\
*\ -x^5\\
\underline{-x^5-x^6}\\
*\ +x^6\\
\underline{+x^6+x^7}\\
*\ -x^7\\
\underline{-x^7-x^8}\\
*\ +x^8\\
\underline{+x^8+x^9}\\
*\ -x^9\\
\underline{-x^9-x^{10}}\\
*\ +x^{10}\\
\underline{+x^{10}+x^{11}}\\
*\ -x^{11}\\
\underline{-x^{11}-x^{12}}\\
*\ +x^{12}
\end{array}$$

By this diviſion it appears that the fraction $\frac{1}{1+x}$ is equal to the infinite ſeries $1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+$ &c, in which the ſecond, fourth, ſixth, eighth, tenth, and twelfth terms are marked with the ſign —, or are to be ſubtracted from the firſt term 1; and the third, fifth, ſeventh, ninth, and eleventh terms are marked with the ſign +, or are to be added to the firſt term 1. And it is eaſy to ſee, from the manner of making this diviſion, that, if the operation was to be continued to any greater number of terms whatſoever, the 14th, and 16th, and 18th terms, and all the following even terms in the quotient would alſo be marked with the ſign —; and that the 13th, and 15th, and 17th terms, and all the following odd terms in the quotient would be marked with the ſign +. And the numeral co-efficients of all the terms in this quotient are units, or are the terms of the firſt vertical column of terms in the foregoing table of combinations, or the figurate numbers of the firſt order; agreeably to what has been juſt now aſſerted.

117. The next diviſion will be as follows:

Diviſor. Quotient.

$1+x)\ (1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+9x^8-10x^9+11x^{10}-12x^{11}+\&c.$

Dividend.

$$\begin{array}{l}
1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+\&c.\\
\underline{1+x}\\
*-2x+x^2\\
\quad\underline{-2x-2x^2}\\
\quad\quad * +3x^2-x^3\\
\quad\quad\quad\underline{+3x^2+3x^3}\\
\quad\quad\quad\quad * -4x^3+x^4\\
\quad\quad\quad\quad\quad\underline{-4x^3-4x^4}\\
\quad\quad\quad\quad\quad\quad * +5x^4-x^5\\
\quad\quad\quad\quad\quad\quad\quad\underline{+5x^4+5x^5}\\
\quad\quad\quad\quad\quad\quad\quad\quad * -6x^5+x^6\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\underline{-6x^5-6x^6}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * +7x^6-x^7\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\underline{+7x^6+7x^7}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * -8x^7+x^8\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\underline{-8x^7-8x^8}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * +9x^8-x^9\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\underline{+9x^8+9x^9}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * -10x^9+x^{10}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\underline{-10x^9-10x^{10}}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * +11x^{10}-x^{11}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\underline{+11x^{10}+11x^{11}}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * -12x^{11}+\&c.\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\underline{-12x^{11}\ \&c.}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad *
\end{array}$$

By this diviſion it appears that the fraction $\frac{1}{\overline{1+x}|^2}$ is equal to the infinite ſeries $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + 9x^8 - 10x^9 + 11x^{10} - 12x^{11} +$ &c, in which, as in the former quotient, the ſecond, fourth, ſixth, eighth, tenth, twelfth, and other following even terms have the ſign — prefixed to them, or are to be ſubtracted from the firſt term 1; and the third, fifth, ſeventh, ninth, eleventh, and other following odd terms are marked with the ſign +, or are to be added to the ſaid firſt term. And the numeral co-efficients of the ſeveral terms of this quotient are the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, or the terms of the ſecond vertical column of terms in the foregoing table of combinations, or the figurate numbers of the ſecond order.

118. The third diviſion will be as follows:

Diviſor. **Quotient.**

$1+x)\ (1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-55x^9+66x^{10}-78x^{11}+\&c.$

Dividend.

$$\begin{array}{l}
1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+9x^8-10x^9+11x^{10}-12x^{11}+\&c.\\
\underline{1+x}\\
*-3x+3x^2\\
\underline{-3x-3x^2}\\
\quad *+6x^2-4x^3\\
\quad \underline{+6x^2+6x^3}\\
\qquad *-10x^3+5x^4\\
\qquad \underline{-10x^3-10x^4}\\
\qquad\quad *+15x^4-6x^5\\
\qquad\quad \underline{+15x^4+15x^5}\\
\qquad\qquad *-21x^5+7x^6\\
\qquad\qquad \underline{-21x^5-21x^6}\\
\qquad\qquad\quad *+28x^6-8x^7\\
\qquad\qquad\quad \underline{+28x^6+28x^7}\\
\qquad\qquad\qquad *-36x^7+9x^8\\
\qquad\qquad\qquad \underline{-36x^7-36x^8}\\
\qquad\qquad\qquad\quad *+45x^8-10x^9\\
\qquad\qquad\qquad\quad \underline{+45x^8+45x^9}\\
\qquad\qquad\qquad\qquad *-55x^9+11x^{10}\\
\qquad\qquad\qquad\qquad \underline{-55x^9-55x^{10}}\\
\qquad\qquad\qquad\qquad\quad *+66x^{10}-12x^{11}\\
\qquad\qquad\qquad\qquad\quad \underline{+66x^{10}+66x^{11}}\\
\qquad\qquad\qquad\qquad\qquad *-78x^{11}+\&c.\\
\qquad\qquad\qquad\qquad\qquad \underline{-78x^{11}-\&c.}\\
\qquad\qquad\qquad\qquad\qquad\quad *\quad +
\end{array}$$

By this diviſion it appears that the fraction $\frac{1}{\overline{1+x}|^{3}}$ is equal to the infinite ſeries $1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-55x^9+66x^{10}-78x^{11}+\&c$, in which, as in the two former quotients, the ſecond, fourth, ſixth, eighth, tenth, twelfth, and other following even terms have the ſign — prefixed to them, or are to be ſubtracted from the firſt term 1; and the third, fifth, ſeventh, ninth, eleventh, and other following odd terms have the ſign + prefixed to them, or are to be added to the ſaid firſt term. And the numeral co-efficients of the ſeveral terms of this quotient are the terms of the third vertical column in the aforeſaid table of combinations, or the figurate numbers of the third order, or the *triangular* numbers.

119. The fourth diviſion will be as follows:

Diviſor. Quotient.

$$1+x)\ (1-4x+10x^2-20x^3+35x^4-56x^5+84x^6-120x^7+165x^8-\&c.$$

Dividend.

$$\begin{array}{l}
1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-\&c.\\
\underline{1+x}\\
*-4x+6x^2\\
\quad\underline{-4x-4x^2}\\
\quad *+10x^2-10x^3\\
\qquad\underline{+10x^2+10x^3}\\
\qquad *-20x^3+15x^4\\
\qquad\quad\underline{-20x^3-20x^4}\\
\qquad\quad *+35x^4-21x^5\\
\qquad\qquad\underline{+35x^4+35x^5}\\
\qquad\qquad *-56x^5+28x^6\\
\qquad\qquad\quad\underline{-56x^5-56x^6}\\
\qquad\qquad\quad *+84x^6-36x^7\\
\qquad\qquad\qquad\underline{+84x^6+84x^7}\\
\qquad\qquad\qquad *-120x^7+45x^8\\
\qquad\qquad\qquad\quad\underline{-120x^7-120x^8}\\
\qquad\qquad\qquad\quad *+165x^8-\&c.\\
\qquad\qquad\qquad\qquad\underline{+165x^8+\&c.}\\
\qquad\qquad\qquad\qquad *-\&c.
\end{array}$$

By this diviſion it appears that the fraction $\frac{1}{\overline{1+x}|^{4}}$ is equal to the infinite ſeries $1-4x+10x^2-20x^3+35x^4-56x+84x^6-120x^7+165x^8-$ &c, *ad infinitum*, in which, as in the three former quotients, the ſecond, fourth, ſixth, and eighth, and other following even terms have the ſign — prefixed to them, or are to be ſubtracted from the firſt term 1; and the third, fifth, ſeventh, ninth, and other following odd terms have the ſign + prefixed to them, or are to be added to the ſaid firſt term. And the numeral co-efficients of the ſeveral terms of this quotient are the terms of the fourth vertical column in the aforeſaid table of combinations, or the figurate numbers of the fourth order, or the *pyramidal* numbers.

120. The fifth diviſion will be as follows:

Diviſor. Quotient.

$$1+x)\ (1-5x+15x^2-35x^3+70x^4-126x^5+210x^6-\&c.$$

Dividend.

$$\begin{array}{l}
1-4x+10x^2-20x^3+35x^4-56x^5+84x^6-120x^7+\&c.\\
\underline{1+x}\\
*-5x+10x^2\\
\underline{\quad -5x-\ 5x^2}\\
\qquad *\ +15x^2-20x^3\\
\qquad \underline{\quad +15x^2+15x^3}\\
\qquad\qquad *\ -35x^3+35x^4\\
\qquad\qquad \underline{\quad -35x^3-35x^4}\\
\qquad\qquad\qquad *\ +70x^4-56x^5\\
\qquad\qquad\qquad \underline{\quad +70x^4+70x^5}\\
\qquad\qquad\qquad\qquad *\ -126x^5+\ 84x^6\\
\qquad\qquad\qquad\qquad \underline{\quad -126x^5-126x^6}\\
\qquad\qquad\qquad\qquad\qquad *\ +210x^6-\&c.\\
\qquad\qquad\qquad\qquad\qquad \underline{\quad +210x^6+\&c.}\\
\qquad\qquad\qquad\qquad\qquad\qquad *\ -\&c.
\end{array}$$

By this diviſion it appears that the fraction $\frac{1}{\overline{1+x}|^{5}}$ is equal to the infinite ſeries $1-5x+15x^2-35x^3+70x^4-126x^5+210x^6-$ &c, *ad infinitum*, in which, as in the four preceeding quotients, the ſecond, fourth, ſixth, and other following even terms have the ſign $-$ prefixed to them, or are to be ſubtracted from the firſt term 1; and the third, fifth, ſeventh, and other following odd terms have the ſign $+$ prefixed to them, or are to be added to the ſaid firſt term. And the numeral co-efficients of the ſeveral terms of this quotient are the terms of the fifth vertical column in the foregoing table of combinations, or the figurate numbers of the fifth order.

121. The ſixth diviſion will be as follows:

Diviſor. Quotient.

$1+x)\ (1-6x+21x^2-56x^3+126x^4-252x^5+\ \&c.$

Dividend.

$$\begin{array}{l}
1-5x+15x^2-35x^3+70x^4-126x^5+\ \&c.\\
\underline{1+x}\\
*-6x+15x^2\\
\underline{\quad -6x-6x^2}\\
\quad\quad *\ +21x^2-35x^3\\
\quad\quad \underline{\quad +21x^2+21x^3}\\
\quad\quad\quad\quad *\ -56x^3+70x^4\\
\quad\quad\quad\quad \underline{\quad -56x^3-56x^4}\\
\quad\quad\quad\quad\quad\quad *\ +126x^4-126x^5\\
\quad\quad\quad\quad\quad\quad \underline{\quad +126x^4+126x^5}\\
\quad\quad\quad\quad\quad\quad\quad\quad *\ -252x^5+\ \&c.\\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{\quad -252x^5-\ \&c.}\\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad *\ \ +\ \&c.
\end{array}$$

By this diviſion it appears that the fraction $\frac{1}{\overline{1+x}|^{6}}$ is equal to the infinite ſeries $1-6x+21x^2-56x^3+126x^4-252x^5+$ &c; in which, as in the five former quotients, the ſecond, fourth, ſixth, and other following even terms have the ſign − prefixed to them, or are to be ſubtracted from the firſt term 1; and the third, and fifth, and other following odd terms have the ſign + prefixed to them, or are to be added to the ſaid firſt term. And the numeral co-efficients of the ſeveral terms of this quotient are the terms of the ſixth vertical column in the foregoing table of combinations, or the figurate numbers of the ſixth order.

Concluſions from the foregoing Operations of Diviſion.

122. From the operations of the foregoing ſix diviſions with the ſame diviſor $1 + x$, I preſume that it will be evident to the reader, that, if we were to continue each of the foregoing quotients to any number of terms, how great ſoever, the ſaid terms would continue to be marked with the ſigns + and — alternately, and that the co-efficients of the following terms after thoſe that have been above computed, would be the following numbers of the ſame order of figurate numbers to which the co-efficients of the terms above computed in the ſaid quotients, belonged, reſpectively. And I likewiſe preſume that it will be evident to him, that, if we were to divide the laſt, or ſixth, quotient by $1 + x$, and the next, or ſeventh, quotient, by the ſame quantity $1 + x$, and the ſeveral next following, or the eighth, ninth, tenth, and eleventh, &c, quotients, continued to any number whatſoever, by the ſame quantity $1 + x$ (whereby we ſhould obtain ſeveral infinite ſerieſes that would be equal to the fractions $\frac{1}{\overline{1+x}|^{7}}$, $\frac{1}{\overline{1+x}|^{8}}$, $\frac{1}{\overline{1+x}|^{9}}$, $\frac{1}{\overline{1+x}|^{10}}$, $\frac{1}{\overline{1+x}|^{11}}$, &c), and were to continue each of the ſaid diviſions till we had obtained any number of terms in the quotient, how great ſoever, the ſeveral even terms in every quotient would be marked with the ſign —, or ſubtracted from the firſt term 1; and the third, fifth, ſeventh, and other following odd terms in every quotient would be marked with the ſign +, or added to the ſaid firſt term; and alſo, that the co-efficients of the terms of the ſaid 7th quotient (that would be equal to the fraction $\frac{1}{\overline{1+x}|^{7}}$) would be the figurate numbers of the 7th order, and the co-efficients of the terms of the ſaid 8th

quotient (that would be equal to the fraction $\frac{1}{\overline{1+x}|^{8}}$.) would be the figurate numbers of the 8th order, and that the coefficients of the terms of the 9th, 10th, 11th, and other following quotients (which would be equal to the fractions $\frac{1}{\overline{1+x}|^{9}}$, $\frac{1}{\overline{1+x}|^{10}}$, $\frac{1}{\overline{1+x}|^{11}}$, &c) would be the figurate numbers of the 9th, 10th, 11th, and other following orders, reſpectively.

Obſervations on the foregoing Operations of Diviſion, tending to eſtabliſh the foregoing Concluſions.

123. The foregoing concluſions may be derived from the following obſervations, which cannot but occur to every perſon who ſhall go through the foregoing operations of algebraïck diviſion with attention, namely,

1ſt, That in every ſeparate operation of diviſion, by which a new term in the quotient is to be obtained, the dividend will always conſiſt of two terms which will have different ſigns + and — prefixed to them; ſo that, when the firſt of the two has the ſign + prefixed to it, the ſecond will be marked with the ſign —; and when the firſt has the ſign — prefixed to it, the ſecond will be marked with the ſign +.

2dly, That the ſubtrahend, or quantity which is to be ſubtracted from the ſaid dividend, will always conſiſt of two terms, which will be both marked with the ſame ſign + or —, which ſign will alſo be the ſame with that of the firſt of the two terms of the dividend from which the ſaid ſubtrahend is to be ſubtracted; and therefore the ſign which is prefixed to the ſecond term of the ſaid ſubtrahend will be contrary to that which is prefixed to the ſecond term of the ſaid dividend, from which it is to be ſubtracted; whence it follows that when, in order to ſubtract the ſaid ſecond term of the ſubtrahend from the ſecond term of the dividend, which is placed juſt above it, we ſhall (according to the rules of algebraïck ſubtraction) have changed its ſign into the contrary ſign, and have added it, with its ſign ſo changed, to the ſecond term of the dividend, the reſidue thence reſulting (which will be the firſt term of the next dividend) will have the ſame ſign prefixed to it as is prefixed to the ſecond term of the former dividend, or the contrary ſign to that which is prefixed to the firſt term of the former dividend; ſo that the firſt terms of every two contiguous dividends throughout the whole diviſion will be marked with contrary ſigns, and conſequently every two contiguous terms in the quotient (which have always the ſame ſigns with the firſt terms of the two dividends from which they are derived) will alſo be marked with contrary ſigns.

3dly, Since the two terms of the diviſor $1 + x$, to wit, 1 and x, have the ſame numeral co-efficient 1, and every new ſubtrahend is produced by multiplying the diviſor $1 + x$ into the laſt-found term of the quotient, it follows that the numeral co-efficient of the ſecond term of every new ſubtrahend muſt be the ſame with the numeral co-efficient of the firſt term of the ſame ſubtrahend. And conſequently, when the ſign of the ſecond term of the ſubtrahend is changed, and thereby become the ſame with the ſign of the ſecond term of the dividend, which is juſt above it, and it is added, with its ſign ſo changed, to the ſaid ſecond term of the dividend, the co-efficient of the quantity reſulting from this addition, or algebraïcal ſubtraction (which is evidently the ſum of the co-efficient of the ſecond term of the ſubtrahend and of the co-efficient of the ſecond term of the dividend) will alſo be the ſum of the co-efficient of the firſt term of the ſubtrahend and of the co-efficient of the ſecond term of the dividend, and conſequently (becauſe the firſt term of the ſubtrahend is always equal to, or the ſame with, the firſt term of the dividend) will alſo be the ſum of the co-efficient of the firſt term of the dividend and the co-efficient of the ſecond term of the dividend; that is, the co-efficient of the firſt term of every new dividend will be the ſum of the co-efficients of the firſt and ſecond terms of the next preceeding dividend. And conſequently the co-efficient of every new term in the quotient (which is the ſame with the co-efficient of the firſt term of the dividend from which it is derived) will be the ſum of the two co-efficients of the two terms of the next preceeding dividend. But the ſecond term of the next preceeding dividend is a term of the laſt preceeding ſeries, or quotient obtained by the diviſion by $1 + x$; and the co-efficient of the firſt term of the ſaid next preceeding dividend is equal to the ſum of the co-efficients of all the preceeding terms of the ſaid laſt preceeding ſeries, or quotient obtained by the diviſion by $1 + x$. Therefore the co-efficient of every new term in the quotient ariſing from the preſent diviſion by $1 + x$ will be equal to the ſum of all the co-efficients of the terms in the foregoing ſeries, or quotient, as far as the correſponding term, or term involving the ſame power of x, and including the ſaid term. Thus, if the former ſeries, or quotient, be called A, and the preſent quotient, now ariſing from the diviſion of the ſeries A by $1 + x$, be called B, and m be a whole number denoting the place of any term in the quotient B, the co-efficient of the mth term of the ſeries, or quotient, B, will be equal to the ſum of the co-efficients of all the terms of the preceeding ſeries, or quotient, A, as far as the mth term of the ſaid ſeries, and including the ſaid mth term.

And therefore, 4thly, that the co-efficients of the terms of the ſeveral ſerieſes, or quotients, ariſing by the continual diviſion of 1 by the binomial quantity $1 + x$ will be the ſeveral orders of figurate numbers, or the terms of the ſeveral vertical columns of terms in the foregoing table of combinations; ſince both the ſaid co-efficients of the terms of the ſaid ſerieſes, or quotients, and the ſaid figurate numbers, or terms of the ſeveral vertical columns of terms in the ſaid table of combinations, ariſe in the ſame manner from a ſeries of units, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c, to wit, by the continual addition of them to each other, and by the like continual addition of the terms of every ſeries generated from them to each other.

Application of the foregoing reaſonings to the finding of a general expreſſion of the value of the fraction $\frac{1}{\overline{a+b}|^{n}}$ in an infinite ſeries of ſimple terms.

124. It having been now proved that the terms of the ſeveral ſerieſes, or quotients, that are equal to the fractions $\frac{1}{1+x}$, $\frac{1}{\overline{1+x}|^{2}}$, $\frac{1}{\overline{1+x}|^{3}}$, $\frac{1}{\overline{1+x}|^{4}}$, $\frac{1}{\overline{1+x}|^{5}}$, $\frac{1}{\overline{1+x}|^{6}}$, &c, *ad infinitum*, (beginning with the ſecond term in each ſeries), are to be marked with the ſign — and the ſign + alternately, and that the co-efficients of the terms of the ſaid ſerieſes will be the figurate numbers of the correſponding orders; and it having been proved above in coroll. 10, that the figurate numbers of the *n*th order, or the ſignificant terms of the *n*th vertical column of terms in the foregoing table of combinations (*n* being put for any whole number whatſoever) are equal to the terms of the following ſeries, to wit, 1, $\frac{n}{1}A$, $\frac{n+1}{2}B$, $\frac{n+2}{3}C$, $\frac{n+3}{4}D$, $\frac{n+4}{5}E$, $\frac{n+5}{6}F$, $\frac{n+6}{7}G$, $\frac{n+7}{8}H$, $\frac{n+8}{9}I$, $\frac{n+9}{10}K$, $\frac{n+10}{11}L$, &c, *ad infinitum*; it follows that the fraction $\frac{1}{\overline{1+x}|^{n}}$ will be equal to the infinite ſeries $1 - \frac{n}{1}Ax$

+

$+ \frac{n+1}{2} Bx^2 - \left|\overline{\frac{n+2}{3}}\right. Cx^3 + \frac{n+3}{4} Dx^4 - \left|\overline{\frac{n+4}{5}}\right. Ex^5 + \frac{n+5}{6} Fx^6 - \left|\overline{\frac{n+6}{7}}\right. Gx^7 + \frac{n+7}{8} Hx^8 - \left|\overline{\frac{n+8}{9}}\right. Ix^9 + \frac{n+9}{10} Kx^{10} - \left|\overline{\frac{n+10}{11}}\right. Lx^{11}$ + &c, *ad infinitum.*

125. Now let $\frac{b}{a}$ be ſubſtituted inſtead of x. And the fraction $\frac{1}{1+x}$ will then be $= \frac{1}{1+\frac{b}{a}}$, and the ſeries $1 - \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 - \left|\overline{\frac{n+2}{3}}\right. Cx^3 + \frac{n+3}{4} Dx^4 - \left|\overline{\frac{n+4}{5}}\right. Ex^5 + \frac{n+5}{6} Fx^6 - \left|\overline{\frac{n+6}{7}}\right. Gx^7 + \frac{n+7}{8} Hx^8 - \left|\overline{\frac{n+8}{9}}\right. Ix^9 + \frac{n+9}{10} Kx^{10} - \left|\overline{\frac{n+10}{11}}\right. Lx^{11}$ &c, *ad infinitum,* will be $= 1 - \frac{n}{1} A \frac{b}{a} + \frac{n+1}{2} B \frac{b^2}{a^2} - \left|\overline{\frac{n+2}{3}}\right. C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} - \left|\overline{\frac{n+4}{5}}\right. E \frac{b^5}{a^5} + \frac{n+5}{6} F \frac{b^6}{a^6} - \left|\overline{\frac{n+6}{7}}\right. G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} - \left|\overline{\frac{n+8}{9}}\right. I \frac{b^9}{a^9} + \frac{n+9}{10} K \frac{b^{10}}{a^{10}} - \left|\overline{\frac{n+10}{11}}\right. L \frac{b^{11}}{a^{11}}$ + &c, *ad infinitum.* Therefore the fraction $\frac{1}{\left.\overline{1+\frac{b}{a}}\right|^{n}}$ will be equal to the ſeries $1 - \frac{n}{1} A \frac{b}{a} + \frac{n+1}{2} B \frac{b^2}{a^2} - \left|\overline{\frac{n+2}{3}}\right. C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} - \left|\overline{\frac{n+4}{5}}\right. E \frac{b^5}{a^5} + \frac{n+5}{6} F \frac{b^6}{a^6} - \left|\overline{\frac{n+6}{7}}\right. G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} - \left|\overline{\frac{n+8}{9}}\right. I \frac{b^9}{a^9} + \frac{n+9}{10} K \frac{b^{10}}{a^{10}} - \left|\overline{\frac{n+10}{11}}\right. L \frac{b^{11}}{a^{11}}$ + &c; and conſequently (dividing both ſides of this equation by a^n) we ſhall have $\left(\frac{\frac{1}{\left.\overline{1+\frac{b}{a}}\right|^{n}}}{a^n}\right.$,

or $\frac{1}{\overline{1+\frac{b}{a}}\big|^{n}\times a^{n}}$, or $\frac{1}{\overline{\overline{1+\frac{b}{a}}\big|\times a}\big|^{n}}$, or) $\frac{1}{\overline{a+b}\big|^{n}}$, $= \frac{1}{a^{n}} - \frac{n}{1}\text{A}\frac{b}{a^{n+1}} + \frac{n+1}{2}\text{B}\frac{b^{2}}{a^{n+2}} - \overline{\big|\frac{n+2}{3}}\text{C}\frac{b^{3}}{a^{n+3}} + \frac{n+3}{4}\text{D}\frac{b^{4}}{a^{n+4}} - \overline{\big|\frac{n+4}{5}}\text{E}\frac{b^{5}}{a^{n+5}} + \frac{n+5}{6}\text{F}\frac{b^{6}}{a^{n+6}} - \overline{\big|\frac{n+6}{7}}\text{G}\frac{b^{7}}{a^{n+7}} + \frac{n+7}{8}\text{H}\frac{b^{8}}{a^{n+8}} - \overline{\big|\frac{n+8}{9}}\text{I}\frac{b^{9}}{a^{n+9}} + \frac{n+9}{10}\text{K}\frac{b^{10}}{a^{n+10}} - \overline{\big|\frac{n+10}{11}}\text{L}\frac{b^{11}}{a^{n+11}}$ + &c, *ad infinitum*; or, according to Sir Iſaac Newton's notation with negative indexes of powers, we ſhall have $\overline{a+b}\big|^{-n}$ = the ſeries $a^{-n} - \frac{n}{1}\text{A}a^{-n-1}b + \frac{n+1}{2}\text{B}a^{-n-2}b^{2} - \overline{\big|\frac{n+2}{3}}\text{C}a^{-n-3}b^{3} + \frac{n+3}{4}\text{D}a^{-n-4}b^{4} - \overline{\big|\frac{n+4}{5}}\text{E}a^{-n-5}b^{5} + \frac{n+5}{6}\text{F}a^{-n-6}b^{6} - \overline{\big|\frac{n+6}{7}}\text{G}a^{-n-7}b^{7} + \frac{n+7}{8}\text{H}a^{-n-8}b^{8} - \overline{\big|\frac{n+8}{9}}\text{I}a^{-n-9}b^{9} + \frac{n+9}{10}\text{K}a^{-n-10}b^{10} - \overline{\big|\frac{n+10}{11}}\text{L}a^{-n-11}b^{11}$ + &c, *ad infinitum*. Q. E. I.

126. This laſt ſeries is the ſame with that which would reſult from Sir Iſaac Newton's original ſeries for expreſſing the value of the quantity $\overline{a+b}\big|^{n}$, or the *n*th power of the binomial quantity $a+b$, to wit, the ſeries $a^{n} + \frac{n}{1}a^{n-1}b + \frac{n}{1}\times\frac{n-1}{2}a^{n-2}b^{2} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}b^{3} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}a^{n-4}b^{4} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}a^{n-5}b^{5}$ + &c, by ſuppoſing the index n of the ſaid power to be negative, or by ſubſtituting $-n$ inſtead of n in the terms of the ſaid ſeries.

For, if this ſubſtitution be made in the terms of the ſaid ſeries, it will become equal to $a^{-n} - \frac{n}{1} a^{-n-1} b^{1} - \frac{n}{1} \times -\frac{n-1}{2} a^{-n-2} b^{2} - \frac{n}{1} \times -\frac{n-1}{2} \times -\frac{n-2}{3} a^{-n-3} b^{3} - \frac{n}{1} + -\frac{n-1}{2} \times -\frac{n-2}{3} \times -\frac{n-3}{4} \times a^{-n-4} b^{4} - \frac{n}{1} \times -\frac{n-1}{2} \times -\frac{n-2}{3} \times -\frac{n-3}{4} \times -\frac{n-4}{5} \times a^{-n-5} b^{5}$ — &c, and conſequently, (becauſe $-\frac{n}{1} \times -\frac{n-1}{2}$ is $= +\frac{n}{1} \times +\frac{n+1}{2}$, and $-\frac{n-2}{3} \times -\frac{n-3}{4}$ is $= +\frac{n+2}{3} \times +\frac{n+3}{4}$), equal to $a^{-n} - \frac{n}{1} a^{-n-1} b^{1} + \frac{n}{1} \times +\frac{n+1}{2} a^{-n-2} b^{2} + \frac{n}{1} \times +\frac{n+1}{2} \times -\frac{n-2}{3} a^{-n-3} b^{3} + \frac{n}{1} \times +\frac{n+1}{2} \times +\frac{n+2}{3} \times +\frac{n+3}{4} a^{-n-4} b^{4} + \frac{n}{1} \times +\frac{n+1}{2} \times +\frac{n+2}{3} \times +\frac{n+3}{4} \times -\frac{n-4}{5} a^{-n-5} b^{5}$ — &c, $= a^{-n} - \frac{n}{1} a^{-n-1} b^{1} + \frac{n}{1} \times \frac{n+1}{2} a^{-n-2} b^{2} + \frac{n}{1} \times \frac{n+1}{2} \times -\overline{\big|\frac{n+2}{3}} a^{-n-3} b^{3} + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} a^{-n-4} b^{4} + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times -\overline{\big|\frac{n+4}{5}} a^{-n-5} b^{5} +$ &c, $= a^{-n} - \frac{n}{1} A a^{-n-1} b^{1} + \frac{n+1}{2} B a^{-n-2} b^{2} - \overline{\big|\frac{n+2}{3}} C a^{-n-3} b^{3} + \frac{n+3}{4} D a^{-n-4} b^{4} - \overline{\big|\frac{n+4}{5}} E a^{-n-5} b^{5} +$ &c; which is the ſeries we juſt now derived from Mr. James Bernoulli's doctrine of combinations for the value of the quantity $\overline{a+b|}^{-n}$, or $\frac{1}{\overline{a+b|}^{n}}$. Q. E. D.

A general

A general expreſſion of the value of the fraction $\frac{1}{\overline{a-b}|^{n}}$, or of the reciprocal of any integral power of the reſidual quantity $a-b$, in an infinite ſeries.

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127. Coroll. 12. The fraction $\frac{1}{\overline{a-b}|^{n}}$ will be equal to the ſeries $\frac{1}{a^{n}} + \frac{n}{1} A \frac{b}{a^{n+1}} + \frac{n+1}{2} B \frac{b^{2}}{a^{n+2}} + \frac{n+2}{3} C \frac{b^{3}}{a^{n+3}} + \frac{n+3}{4} D \frac{b^{4}}{a^{n+4}} + \frac{n+4}{5} E \frac{b^{5}}{a^{n+5}} + \frac{n+5}{6} F \frac{b^{6}}{a^{n+6}} + \frac{n+6}{7} G \frac{b^{7}}{a^{n+7}} + \frac{n+7}{8} H \frac{b^{8}}{a^{n+8}} + \frac{n+8}{9} I \frac{b^{9}}{a^{n+9}} + \frac{n+9}{10} K \frac{b^{10}}{a^{n+10}} + \frac{n+10}{11} L \frac{b^{11}}{a^{n+11}} +$ &c, *ad infinitum*, which conſiſts of the very ſame terms as the ſeries obtained in the foregoing corollary for the value of the fraction $\frac{1}{\overline{a+b}|^{n}}$, but with the ſign + prefixed to all the terms after the firſt term $\frac{1}{a^{n}}$, inſtead of being prefixed only to the 3d, 5th, 7th, 9th, and other following odd terms of it, as in the former ſeries.

128. This will appear by dividing 1 two or three times following by the reſidual quantity $1-x$ inſtead of the binomial quantity $1+x$. For we ſhall eaſily perceive that all the terms, after the firſt terms, in the quotients ariſing from thoſe diviſions will be marked with the ſign +, or muſt be added to the firſt terms. The three firſt of theſe diviſions will be as follows:

The firſt Diviſion.

Diviſor. **Quotient.**

$1-x)\ (1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+$ &c.

Dividend.

$$\begin{array}{l}
1\ *\ *\ *\ *\ *\ *\ *\ *\ *\ *\ * \\
\underline{1-x} \\
+x\ * \\
\underline{+x-x^2} \\
\quad *\ +x^2\ * \\
\quad \underline{+x^2-x^3} \\
\qquad *\ +x^3\ * \\
\qquad \underline{+x^3-x^4} \\
\qquad\quad *\ +x^4\ * \\
\qquad\quad \underline{+x^4-x^5} \\
\qquad\qquad *\ +x^5\ * \\
\qquad\qquad \underline{+x^5-x^6} \\
\qquad\qquad\quad *\ +x^6\ * \\
\qquad\qquad\quad \underline{+x^6-x^7} \\
\qquad\qquad\qquad *\ +x^7\ * \\
\qquad\qquad\qquad \underline{+x^7-x^8} \\
\qquad\qquad\qquad\quad *\ +x^8\ * \\
\qquad\qquad\qquad\quad \underline{+x^8-x^9} \\
\qquad\qquad\qquad\qquad *\ +x^9\ * \\
\qquad\qquad\qquad\qquad \underline{+x^9-x^{10}} \\
\qquad\qquad\qquad\qquad\quad *\ +x^{10}
\end{array}$$

The ſecond Diviſion.

Diviſor. **Quotient.**

$$1-x)\quad(1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+10x^9+\&c.$$

Dividend.

$$\begin{array}{l}
1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+\&c.\\
\underline{1-x}\\
\quad +2x+x^2\\
\quad \underline{+2x-2x^2}\\
\qquad * \; +3x^2+x^2\\
\qquad\quad \underline{+3x^2-3x^3}\\
\qquad\qquad * \; +4x^3+x^4\\
\qquad\qquad\quad \underline{+4x^3-4x^4}\\
\qquad\qquad\qquad * \; +5x^4+x^5\\
\qquad\qquad\qquad\quad \underline{+5x^4-5x^5}\\
\qquad\qquad\qquad\qquad * \; +6x^5+x^6\\
\qquad\qquad\qquad\qquad\quad \underline{+6x^5-6x^6}\\
\qquad\qquad\qquad\qquad\qquad * \; +7x^6+x^7\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{+7x^6-7x^7}\\
\qquad\qquad\qquad\qquad\qquad\qquad * \; +8x^7+x^8\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{+8x^7-8x^8}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad * \; +9x^8\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{+9x^8}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad *
\end{array}$$

The third Diviſion.

Diviſor. Quotient.

$1-x)\quad(1+3x+6x^2+10x^3+15x^4 21x^5+36x^6+$ &c.

Dividend.

$$\begin{array}{l}
1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+\text{\&c.}\\
\underline{1-x}\\
\quad +3x+3x^2\\
\quad \underline{+3x-3x^2}\\
\qquad * \quad +6x^2+4x^3\\
\qquad\qquad \underline{+6x^2-6x^3}\\
\qquad\qquad\quad * \quad +10x^3+5x^4\\
\qquad\qquad\qquad\quad \underline{+10x^3-10x^4}\\
\qquad\qquad\qquad\qquad * \quad +15x^4+6x^5\\
\qquad\qquad\qquad\qquad\qquad \underline{+15x^4-15x^5}\\
\qquad\qquad\qquad\qquad\qquad\quad * \quad +21x^5+7x^6\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{+21x^5-21x^6}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad * \quad +28x^6+8x^7\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{+28x^6-28x^7}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad * \quad +36x^7
\end{array}$$

129. It is eaſy to ſee that, both in theſe three diviſions, and in all the following diviſions that may be made of the laſt quotient hereby obtained, by the ſame diviſor $1-x$, all the terms of the ſeveral quotients, after the firſt terms, will be marked with the ſign +, or be added to the firſt terms, and that the co-efficients of the ſeveral terms will be the very ſame numbers as the co-efficients of the correſponding terms in the former quotients which reſulted from the diviſions by the binomial quantity $1+x$. It follows therefore that the fraction $\frac{1}{\overline{1-x}\rvert^{n}}$ will be equal to the infinite ſeries $1+\frac{n}{1}Ax+\frac{n+1}{2}Bx^2+\frac{n+2}{3}Cx^3+\frac{n+3}{4}Dx^4+\frac{n+4}{5}Ex^5+\frac{n+5}{6}Fx^6+\frac{n+6}{7}Gx^7+\frac{n+7}{8}Hx^8+\frac{n+8}{9}Ix^9+\frac{n+9}{10}Kx^{10}+\frac{n+10}{11}Lx^{11}+$ &c, *ad infinitum*, and conſequently

quently (ſubſtituting $\frac{b}{a}$ inſtead of x in this equation) that the fraction $\frac{1}{\overline{1-\frac{b}{a}}\big|^{n}}$ will be equal to the infinite ſeries $1+\frac{n}{1}A\frac{b}{a}+\frac{n+1}{2}B\frac{b^2}{a^2}+\frac{n+2}{3}C\frac{b^3}{a^3}+\frac{n+3}{4}D\frac{b^4}{a^4}+\frac{n+4}{5}E\frac{b^5}{a^5}+\frac{n+5}{6}F\frac{b^6}{a^6}+\frac{n+6}{7}G\frac{b^7}{a^7}+\frac{n+7}{8}H\frac{b^8}{a^8}+\frac{n+8}{9}I\frac{b^9}{a^9}+\frac{n+9}{10}K\frac{b^{10}}{a^{10}}+\frac{n+10}{11}L\frac{b^{11}}{a^{11}}+$ &c, *ad infinitum*, and conſequently (dividing both ſides of the laſt equation by a^n) that the fraction ($\frac{\frac{1}{\overline{1-\frac{b}{a}}\big|^{n}}}{a^n}$, or $\frac{1}{a^n\times\overline{1-\frac{b}{a}}\big|^{n}}$, or $\frac{1}{\overline{a\times 1-\frac{b}{a}}\big|^{n}}$, or) $\frac{1}{\overline{a-b}\big|^{n}}$ will be equal to the infinite ſeries $\frac{1}{a^n}+\frac{n}{1}A\frac{b}{a^{n+1}}+\frac{n+1}{2}B\frac{b^2}{a^{n+2}}+\frac{n+2}{3}C\frac{b^3}{a^{n+3}}+\frac{n+3}{4}D\frac{b^4}{a^{n+4}}+\frac{n+4}{5}E\frac{b^5}{a^{n+5}}+\frac{n+5}{6}F\frac{b^6}{a^{n+6}}+\frac{n+6}{7}G\frac{b^7}{a^{n+7}}+\frac{n+7}{8}H\frac{b^8}{a^{n+8}}+\frac{n+8}{9}I\frac{b^9}{a^{n+9}}+\frac{n+9}{10}K\frac{b^{10}}{a^{n+10}}+\frac{n+10}{11}L\frac{b^{11}}{a^{n+11}}+$ &c, *ad infinitum.* Q. E. D.

130. If we make uſe of Sir Iſaac Newton's notation with negative indexes of powers, the laſt equation will be as follows, to wit, $\overline{a-b}\big|^{-n}=$ the infinite ſeries $a^{-n}+\frac{n}{1}Aa^{-n-1}b+\frac{n+1}{2}Ba^{-n-2}b^2+\frac{n+2}{3}Ca^{-n-3}b^3+\frac{n+3}{4}Da^{-n-4}b^4+\frac{n+4}{5}Ea^{-n-5}b^5+\frac{n+5}{6}Fa^{-n-6}b^6+\frac{n+6}{7}$

$\mathrm{G}a^{-n-7}b^7 + \frac{n+7}{8}\mathrm{H}a^{-n-8}b^8 + \frac{n+8}{9}\mathrm{I}a^{-n-9}b^9 + \frac{n+9}{10}\mathrm{K}a^{-n-10}b^{10} + \frac{n+10}{11}\mathrm{L}a^{-n-11}b^{11}$ + &c, *ad infinitum.* But the other way of expreſſing this equation ſeems to be clearer and more natural than this way, and, for ordinary purpoſes, preferable to it.

131. This laſt ſeries is the ſame with that which would reſult from Sir Iſaac Newton's original ſeries for expreſſing the value of the quantity $\overline{a-b}|^n$, or the *n*th power of the reſidual quantity $a-b$, to wit, the ſeries $a^n - \frac{n}{1}a^{n-1}b^1 + \frac{n}{1}\times\frac{n-1}{2}a^{n-2}b^2 - \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}a^{n-4}b^4 - \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}a^{n-5}b^5$ + &c, by ſuppoſing the index n of the ſaid power to be negative, or by ſubſtituting $-n$ inſtead of n in the terms of the ſaid ſeries.

For, if this ſubſtitution be made in the terms of the ſaid ſeries, it will become equal to $a^{-n} + \frac{n}{1}a^{-n-1}b^1 - \frac{n}{1}\times\frac{-n-1}{2}a^{-n-2}b^2 + \frac{n}{1}\times\frac{-n-1}{2}\times\frac{-n-2}{3}a^{-n-3}b^3 - \frac{n}{1}\times\frac{-n-1}{2}\times\frac{-n-2}{3}\times\frac{-n-3}{4}a^{-n-4}b^4 + \frac{n}{1}\times\frac{-n-1}{2}\times\frac{-n-2}{3}\times\frac{-n-3}{4}\times\frac{-n-4}{5}a^{-n-5}b^5$ + &c, $= a^{-n} + \frac{n}{1}a^{-n-1}b^1 + \frac{n}{1}\times\frac{+n+1}{2}a^{-n-2}b^2 + \frac{n}{1}\times\frac{+n+1}{2}\times\frac{+n+2}{3}a^{-n-3}b^3 + \frac{n}{1}\times\frac{+n+1}{2}\times\frac{+n+2}{3}\times\frac{+n+3}{4}a^{-n-4}b^4 + \frac{n}{1}\times\frac{+n+1}{2}\times\frac{+n+2}{3}\times\frac{+n+3}{4}\times\frac{+n+4}{5}a^{-n-5}b^5$ + &c, $= a^{-n} + \frac{n}{1}$

$a^{-n-1}b^{1} + \frac{n}{1} \times \frac{n+1}{2} a^{-n-2} b^{2} + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$
$a^{-n-3} b^{3} + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} a^{-n-4} b^{4} + \frac{n}{1}$
$\times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} a^{-n-5} b^{5} +$ &c, $= a^{-n}$
$+ \frac{n}{1} \mathrm{A} a^{-n-1} b^{1} + \frac{n+1}{2} \mathrm{B} a^{-n-2} b^{2} + \frac{n+2}{3} \mathrm{C} a^{-n-3} b^{3}$
$+ \frac{n+3}{4} \mathrm{D} a^{-n-4} b^{4} + \frac{n+4}{5} \mathrm{E} a^{-n-5} b^{5} +$ &c; which is the ſeries we juſt now derived in art. 130, from Mr. James Bernoulli's doctrine of combinations for the value of the quantity $\overline{a-b}|^{-n}$, or $\frac{1}{\overline{a-b}|^{n}}$. Q. E. D.

132. We have now ſeen how from Mr. James Bernoulli's doctrine of combinations, and his explanation of the properties of the figurate numbers derived from it, may be deduced juſt and regular demonſtrations of Sir Iſaac Newton's famous binomial and reſidual theorems in the caſe of the integral and negative powers of a binomial and a reſidual quantity, or of the reciprocals of their integral and affirmative powers, as well as in the caſe of their integral and affirmative powers themſelves, in which Mr. Bernoulli himſelf has demonſtrated them above in coroll. 5. And I doubt whether any other method of demonſtrating theſe two famous theorems in the caſe of the integral and negative powers of a binomial and a reſidual quantity has yet been found out, that is equally clear and ſatisfactory.

A Difficulty that may occur concerning the foregoing Theorem relating to the integral and negative Powers of a reſidual Quantity, as a — b, *or* 1 — x.

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133. Before we conclude this ſubject of the binomial and reſidual theorems in the caſe of integral and negative powers, I will endeavour to clear up a difficulty which may, perhaps, occur to the reader's mind concerning the latter of the ſaid theorems.

It has been ſhewn in art. 129, that $\overline{1-x}|^{-n}$, or $\frac{1}{\overline{1-x}|^{n}}$, is equal to the infinite ſeries $1 + \frac{n}{1}Ax + \frac{n+1}{2}Bx^2 + \frac{n+2}{3}Cx^3 + \frac{n+3}{4}Dx^4 + \frac{n+4}{5}Ex^5 + \frac{n+5}{6}Fx^6 + \frac{n+6}{7}Gx^7 +$ &c, *ad infinitum*, in which all the terms following the firſt term 1 are marked with the ſign +, or are to be added to the ſaid firſt term. And the co-efficients of the terms in this ſeries continually increaſe, when n is of any magnitude greater than 1. Thus, if n is $= 2$, we ſhall have $\frac{n}{1}$ $(= \frac{2}{1})$ $= 2$, and $\frac{n+1}{2}$ $(= \frac{2+1}{2})$ $= \frac{3}{2}$, and $\frac{n+2}{3}$ $(= \frac{2+2}{3})$ $= \frac{4}{3}$, and $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, equal to $\frac{5}{4}$, $\frac{6}{5}$, $\frac{7}{6}$, $\frac{8}{7}$, &c, reſpectively; in all which fractions the numerators exceed the denominators by an unit; and, if n is $= 3$, we ſhall have $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, equal to $\frac{3}{1}$, $\frac{4}{2}$, $\frac{5}{3}$, $\frac{6}{4}$, $\frac{7}{5}$, $\frac{8}{6}$, $\frac{9}{7}$, &c, reſpectively; in all which fractions the numerators exceed the denominators by 2. And the like exceſs of the numerators above the denominators will take place in a ſtill higher degree in the ſaid generating fractions $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, when the index n is equal to 4, or 5, or 6, or any higher number. And conſequently the co-efficients B, C, D, E, F, G, &c, which are derived from the firſt term 1, or A, by the continual multiplication of the ſaid generating fractions $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, muſt continually increaſe in all theſe ſerieſes. And accordingly we find that the figurate numbers of every order, or the ſeveral ſignificant terms in every vertical column of terms in the foregoing table of combinations, page 74, (which are equal to the co-efficients of the terms of the foregoing ſeries $1 + \frac{n}{1}Ax + \frac{n+1}{2}Bx^2 + \frac{n+2}{3}Cx^3 + \frac{n+3}{4}Dx^4 + \frac{n+4}{5}Ex^5 + \frac{n+5}{6}Fx^6$ + &c, *ad infinitum*), increaſe continually. And hence it may happen that, if x is but little leſs than 1, the whole terms at the beginning of the ſeries $1 + \frac{n}{1}Ax + \frac{n+1}{2}Bx^2 + \frac{n+2}{3}Cx^3 + \frac{n+3}{4}Dx^4 + \frac{n+4}{5}Ex^5 + \frac{n+5}{6}Fx^6$ + &c, *ad infinitum*, may (by means of this increaſe of their co-efficients) be increaſing quantities.

Now, from this circumſtance it may, perhaps, be apprehended, that all the terms of this ſeries will in ſome caſes diverge, or increaſe, continually, to what number of terms ſoever the ſaid ſeries may be continued, and conſequently that the ſaid ſeries (conſiſting of an infinite number of terms that are every one greater than that next before it) will be infinite in magnitude as well as in the number of its terms, and therefore cannot be equal to the finite quantity $\frac{1}{\overline{1-x}|^n}$. This is a difficulty that ſeems naturally to ariſe upon this ſubject. But it may be removed by the following conſiderations.

An Explanation of the foregoing Difficulty.

134. The proportion of the numerators of the ſeveral generating fractions $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, to their denominators (though it is always a proportion of majority, when n is greater than 1) approaches continually nearer and nearer to a ratio of equality, as its limit; ſo that, if n be ever ſo great a number, we may, by continuing the ſeries of theſe generating fractions to a great number of terms, come to one in which the ratio of the numerator to the denominator ſhall be leſs than any propoſed ratio of majority. Thus, for example, if n is = 1000, and the ratio of majority that is propoſed, or given, and with which the ratios of the numerators of theſe generating fractions to their denominators is to be compared, is that of 1 to 0.99999, or of 100,000 to 99,999, it will be poſſible, by continuing the ſeries of the ſaid generating fractions, to aſſign one in which the ratio of the numerator to the denominator ſhall be leſs than the ratio of 1 to 0.99999, or of 100,000 to 99,999. This may be ſhewn in the manner following. Let z be the general repreſentative of the ſeveral numbers added to the index n in the numerators of theſe ſucceſſive generating fractions; ſo that the ſaid numerators ſhall be equal to the ſeveral ſucceſſive values of $n+z$, or, on the preſent ſuppoſition that n is = 1000, to the ſeveral ſucceſſive values of $1000 + z$. Then will the denominators of the ſaid ſucceſſive generating fractions be denoted by the ſucceſſive values of $z + 1$, and the ſaid generating fractions themſelves will be equal to the ſeveral ſucceſſive values of the fraction $\frac{1000+z}{z+1}$, or $\frac{z+1000}{z+1}$. Now it is evident that, by continually increaſing the number z, the proportion of the numerator $z + 1000$ to the denominator $z + 1$ may be made to ap-

proach as near as we pleaſe to the proportion of z to z, or the proportion of equality. The number z may therefore be increaſed till the ſaid proportion of $z + 1000$ to $z + 1$ ſhall be nearer to a proportion of equality, or ſhall be a leſs ratio of majority, than the propoſed ratio of 1 to 0.99999, or of 100,000 to 99,999. And the ſame thing might be done if the propoſed ratio, inſtead of being that of 1 to 0.99999, or of 100,000 to 99,999, had been that of 1 to 0.999,999, or of 1000,000 to 999,999, or that of 1 to 0.999,999,9, or of 10,000,000 to 9,999,999, or any other ratio of majority, how ſmall ſoever. Therefore, however nearly the quantity x may approach to an equality with 1 (than which it is always ſuppoſed to be ſomewhat leſs) it will always be poſſible to increaſe the number z till the proportion of $z + n$ to $z + 1$, or of $n + z$ to $z + 1$, becomes leſs than the proportion of 1 to x, or till the fraction $\frac{n+z}{z+1}$ becomes leſs than the fraction $\frac{1}{x}$. And, as the number z increaſes with the number of terms of the ſeries $1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 + \frac{n+6}{7} Gx^7$ + &c (being always leſs by 2 than the number of the terms from the beginning of the ſeries to the term in which it occurs, including the ſaid term), it is evident, that, by continuing the terms of the ſeries, we muſt always come to a term in which the generating fraction $\frac{n+z}{z+1}$ ſhall be leſs than the fraction $\frac{1}{x}$. And when we are arrived at this term, the next term of the ſeries will be leſs increaſed by being multiplied into the next generating fraction (which will be leſs than the fraction $\frac{1}{x}$) than it will be diminiſhed by being multiplied into the fraction $\frac{x}{1}$, or the reciprocal of the fraction $\frac{1}{x}$; and conſequently it will be leſs than the laſt preceeding term of the ſeries from which it is derived. And therefore, when we are come to this term, all the following terms of the ſaid ſeries (which have hitherto been increaſing quantities) will decreaſe continually, and in a greater and greater proportion continually, as the ſeries advances. And conſequently the ſaid ſeries will in all caſes be of a finite magnitude, however nearly the quantity x may approach to an equality with 1. Q. E. D.

End of the Additional Corollaries not contained in the original text of Mr. James Bernoulli, *which began in page* 123.

A SCHOLIUM.

135. We may here take occaſion to obſerve, that, though many writers on mathematical ſubjects (as for example, Faulhaber and Remmelin of the city of Ulm in Germany, and Dr. John Wallis of Oxford, Mr. Nicholas Mercator in his Logarithmotechnia *, and Monſieur Preſtet, a learned French mathematician) have made the properties of the figurate numbers the ſubject of their conſideration, yet no one has hitherto given the publick a general and ſcientifick demonſtration of the foregoing important 12th property of them. At leaſt I may ſay, that no ſuch demonſtration has ever come to my knowledge. Dr. Wallis, indeed, in that part of his learned treatiſe on the arithmetick of infinites, in which he eſtabliſhes the foundations of his method, has inveſtigated by arguments of induction the proportions which a ſeries of the ſquares of a given number of the natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, and a ſeries of their cubes, and a ſeries of their fourth powers, and ſerieſes

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* See Vol. I. of the Collection of Tracts, in quarto, intitled, *Scriptores Logarithmici*, pages 192, 193.

of their following higher powers, would bear to a ſeries conſiſting of the ſame number of terms all equal to the laſt, or greateſt, term of the former ſeries; and, after performing theſe inveſtigations, has, in his 176th propoſition, made a tranſition to the contemplation of the trigonal, or triangular, and the pyramidal, and trigono-pyramidal, or triangulo-pyramidal, numbers, and other following orders of the figurate numbers. But I apprehend he would have acted more judiciouſly and more agreeably to the nature of the ſubject he was conſidering, if he had taken the contrary courſe, and begun with the inveſtigation of the properties of the figurate numbers, and then, after having diſcovered thoſe properties, and given a juſt and general demonſtration of them, had proceeded to inveſtigate the ſums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. For, beſides the objections that may be juſtly made to his method of making theſe inveſtigations by inductions from particular examples, as being by no means ſcientifick or ſatisfactory to a mind accuſtomed to more accurate modes of reaſoning, and likewiſe as being more prolix and tedious than need be, on account of the neceſſity of having a ſeparate inveſtigation for every new ſeries of powers;—I ſay, beſides theſe objections to his method of treating this ſubject, it may be conſidered as inelegant and unnatural on another account, namely, becauſe it treats of the more abſtruſe parts of the ſubject, to wit, the inveſtigation of the ſums of powers, before the more ſimple and eaſy parts of it, or the doctrine of the figurate numbers. For theſe numbers may be juſtly eſteemed to be more ſimple and eaſy to be underſtood than the powers of the natural numbers, partly, becauſe the ſeveral orders of them are generated one from another by the eaſy operation of addition; whereas, the powers of numbers are produced by the more complicated operation of multiplication; and partly and eſpecially, becauſe the ſums of the ſeveral orders of figurate numbers (reckoning from the beginning of the foregoing table of them, or including the ſeveral cyphers prefixed to the ſignificant terms of the ſeveral vertical columns of the ſaid table) are (as we have ſeen) exact aliquot parts of the ſerieſes that conſiſt of the ſame numbers of terms all equal to their laſt, or greateſt, terms, reſpectively; whereas the ſums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, never are exact aliquot parts of the ſums of equal numbers of terms equal to the laſt, or greateſt, of them, reſpectively, but always exceed, or fall ſhort of, ſuch aliquot parts by ſome ſmall finite quantity, how great ſoever the number of the terms of ſuch ſerieſes may be ſuppoſed to be, and what number of cyphers ſoever we may prefix to the ſaid ſerieſes conſiſting of the powers of the natural numbers. Nor can it be alledged, that it was neceſſary for Dr. Wallis to begin by inveſtigating the ſums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, on account of the difficulty of deducing the values of thoſe ſums from the doctrine of the ſums of the figurate numbers. For, it is full as eaſy to deduce the ſums of the ſaid powers from the ſums of the ſeveral orders of figurate numbers, as to deduce the latter from the ſums of the powers of numbers in the manner adopted by Dr. Wallis: as I ſhall now proceed to ſhew by deducing the ſums of the ſaid powers from the ſums of the ſeveral orders of figurate numbers, which we have already inveſtigated.

An inveſtigation of the ſum of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to any given number of terms, and of the ſums of their ſquares, and of their cubes, and of their fourth powers, and other higher powers, continued to the ſame number of terms.

136. If the letter x be made to denote the ſeveral ſucceſſive terms of the ſeries 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to the nth term, which, it is evident, will be n,

the ſucceſſive values of the reſidual quantities $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, &c, continued to n terms, will be $1-1$, $2-1$, $3-1$, $4-1$, $5-1$, $6-1$, $7-1$, $8-1$, $9-1$, $10-1$, $11-1$, &c, continued to n terms, or 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c, continued to n terms. But, by coroll. 2, of the foregoing propoſition, art. 81, the ſeries 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c, (which are the terms contained in the ſecond vertical column of the foregoing table of combinations), continued to n terms, is $=\frac{n\times\overline{n-1}}{2}=\frac{nn-n}{2}=\frac{nn}{2}-\frac{n}{2}$. Therefore the ſum of all the n ſucceſſive values of $x-1$ will be equal to $\frac{nn}{2}-\frac{n}{2}$; and conſequently, if we denote the ſaid ſum by $S.\overline{x-1}$, we ſhall have $S.\overline{x-1}=\frac{nn}{2}-\frac{n}{2}$. But the ſum of the n ſucceſſive values of $x-1$ is equal to the exceſs of the n ſucceſſive values of x above the n ſucceſſive values of 1, or (making uſe of the ſame kind of notation) to $S.x-S.1$. Therefore $S.x-S.1$ will be $=\frac{nn}{2}-\frac{n}{2}$, and conſequently (adding $S.1$ to both ſides) $S.x$ will be $=\frac{nn}{2}-\frac{n}{2}+S.1$. But the ſum of the n ſucceſſive values of 1 is evidently the number n. Therefore $S.x$ will be $=\frac{nn}{2}-\frac{n}{2}+n=\frac{nn}{2}+\frac{n}{2}$, or the ſum of all the n ſucceſſive values of x, to wit, $1+2+3+4+5+6+7+8+9+10+11+$ &c $+\,n$ will be $=\frac{nn}{2}+\frac{n}{2}$. Q. E. D.

Thus, for example, if n is $=12$, the ſum of the twelve terms of the ſeries $1+2+3+4+5+6+7+8+9+10+11+12$ will be $=\frac{12\times12}{2}+\frac{12}{2}=12\times6+6$

$=72$

$= 72 + 6 = 78$. And ſo we ſhall find it to be by actually adding up the terms.

$$\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ \hline 78 \end{array}$$

Of the ſum of the ſquares of the natural numbers, 1, 2, 3, 4, 5, *&c, continued to any given number* n.

137. Let it now be required to find the ſum of the ſquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, continued to n, or the ſum of the numbers 1, 4, 9, 16, 25, 36, 49, &c, continued to the nth term, which will be nn. This may be done in the manner following.

Let x be put, as before, for the ſeveral ſucceſſive terms of the ſeries 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n. Then, ſince by coroll. 4, of the foregoing propoſition, art. 83, the nth term of the third vertical column of the foregoing table of combinations is $= \frac{\overline{n-1} \times \overline{n-2}}{2} = \frac{nn-3n+2}{2}$, it follows,

lows, that every xth term of the ſame vertical column will be $= \frac{xx-3x+2}{2}$, or that, if x be made ſucceſſively equal to 1, 2, 3, 4, 5, 6, &c, the ſucceſſive values of the fraction $\frac{xx-3x+2}{2}$, will produce the firſt, ſecond, third, fourth, fifth, ſixth, and other following terms of the ſame vertical column, which are 0, 0, 1, 3, 6, 10, &c. Thus, for example, if x is $= 1$, we ſhall have $\frac{xx-3x+2}{2} = \frac{1-3+2}{2} = \frac{0}{2} = 0$; and, if x is $= 2$, we ſhall have $\frac{xx-3x+2}{2} = \frac{4-6+2}{2} = \frac{0}{2} = 0$; and, if x is $= 3$, we ſhall have $\frac{xx-3x+2}{2} = \frac{9-9+2}{2} = \frac{2}{2} = 1$; and, if x is $= 4$, we ſhall have $\frac{xx-3x+2}{2} = \frac{16-12+2}{2} = \frac{6}{2} = 3$; and, if x is $= 5$, we ſhall have $\frac{xx-3x+2}{2} = \frac{25-15+2}{2} = \frac{12}{2} = 6$; and, if x is $= 6$, we ſhall have $\frac{xx-3x+2}{2} = \frac{36-18+2}{2} = \frac{20}{2} = 10$; which numbers 0, 0, 1, 3, 6, and 10, are the firſt ſix terms of the ſaid third vertical column. And the ſame thing will be found to be true in any greater number of its terms. But, by the ſecond corollary of the foregoing propoſition, art. 81, the ſum of all the $n - 2$ ſignificant terms, or, including the two cyphers at the beginning of it, the ſum of all the n terms, of the ſaid third vertical column is $= \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3} = \frac{n \times \overline{nn-3n+2}}{2 \times 3} = \frac{n^3-3nn+2n}{6}$. Therefore the ſum of all the n ſucceſſive values of the fraction $\frac{xx-3x+2}{2}$ will be $= \frac{n^3-3nn+2n}{6}$. But the ſum of all the n ſucceſſive values of $\frac{xx-3x+2}{2}$ is evidently equal to the ſum of all the n ſucceſſive values of $\frac{xx}{2}$, together with the ſum of all the n ſucceſſive values of $\frac{2}{2}$, or 1, diminiſhed by the ſum of all the n ſucceſſive values of $\frac{3x}{2}$, or (according to our former notation) $\mathrm{S}.\overline{\frac{xx-3x+2}{2}}$ is $= \mathrm{S}.\frac{xx}{2} + \mathrm{S}.\frac{2}{2} - \mathrm{S}.\frac{3x}{2} = \mathrm{S}.\frac{xx}{2} + \mathrm{S}.1 - \mathrm{S}.\frac{3x}{2} = \mathrm{S}.\frac{xx}{2} + n - \mathrm{S}.\frac{3x}{2}$, Therefore $\mathrm{S}.\frac{xx}{2} + n - \mathrm{S}.\frac{3x}{2}$ will be $= \frac{n^3-3nn+2n}{6}$. But $\mathrm{S}.\frac{3x}{2}$ is evidently $= \frac{3}{2} \times \mathrm{S}.x$. Therefore $\mathrm{S}.\frac{xx}{2} + n - \frac{3}{2} \times \mathrm{S}.x$ will be $= \mathrm{S}.\frac{xx}{2} + n - \mathrm{S}.\frac{3x}{2}$, and conſequently will be $= \frac{n^3-3nn+2n}{6}$. But it has been ſhewn in art, 136, that $\mathrm{S}.x$ is $= \frac{nn}{2} + \frac{n}{2}$. Therefore $\frac{3}{2} \times \mathrm{S}.x$ will be $= \frac{3}{2} \times \overline{\frac{nn}{2} + \frac{n}{2}} = \frac{3nn}{4} + \frac{3n}{4}$. Therefore $\mathrm{S}.\frac{xx}{2} + n - \frac{3nn}{4} - \frac{3n}{4}$ will be $= \frac{n^3-3nn+2n}{6}$, or $\mathrm{S}.\frac{xx}{2} - \frac{3nn}{4} + \frac{n}{4}$ will be $= \frac{n^3-3nn+2n}{6}$, or $\mathrm{S}.\frac{xx}{2} - \frac{9nn}{12} + \frac{3n}{12}$ will be $= \frac{n^3}{6} - \frac{6nn}{12} + \frac{4n}{12}$. Therefore (adding $\frac{9nn}{12}$ to both ſides) we ſhall have $\mathrm{S}.\frac{xx}{2} + \frac{3n}{12} = \frac{n^3}{6} + \frac{3nn}{12} + \frac{4n}{12}$; and (ſubtracting $\frac{3n}{12}$ from both ſides) we ſhall have $\mathrm{S}.\frac{xx}{2} = \frac{n^3}{6} + \frac{3nn}{12} + \frac{n}{12} = \frac{n^3}{6} + \frac{nn}{4} + \frac{n}{12}$, and conſequently $\mathrm{S}.xx$ ($= 2 \times \mathrm{S}.\frac{xx}{2}$) $= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$; that is, the ſum of all the n ſquares 1, 4, 9, 16, 25, 36, 49, &c, of the n firſt natural numbers 1, 2, 3, 4, 5, 6, 7, &c . . . . n will be $= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$, or one-third of the cube of the greateſt number n, together with half the ſquare of the ſaid number, and a ſixth part of the ſaid number itſelf. Q. E. I.

Thus, for example, if n is $= 12$, we ſhall have $nn = 144$, and $n^3 = 1728$, and $\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$ $(= \frac{1728}{3} + \frac{144}{2} + \frac{12}{6} = 576 + 72 + 2) = 650$. Therefore the ſum of the following twelve numbers, to wit, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, and 144, (which are the ſquares of the twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12) will be equal to 650. And ſo, upon adding them up together, we ſhall find them to be

$$\begin{array}{r}1\\4\\9\\16\\25\\36\\49\\64\\81\\100\\121\\144\\\hline 650\end{array}$$

Of the ſum of the cubes of the natural numbers 1, 2, 3, 4, 5, *&c, continued to any given number* n.

138. Let x be put, as before, for the ſeveral ſucceſſive terms of the ſeries 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n. Then, ſince by coroll. 4, of the foregoing propoſition, art. 83, the nth term of the fourth vertical column of the foregoing table of combinations is $= \frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times 3} = \frac{nn-3n+2}{2}\times\frac{n-3}{3} = \frac{n^3-6nn+11n-6}{6}$; it follows, that every xth term of the ſame fourth column will be $= \frac{x^3-6xx+11x-6}{6}$, or that, if x be made ſucceſſively equal to 1, 2, 3, 4, 5, 6, &c, the ſucceſſive values of the fraction $\frac{x^3-6xx+11x-6}{6}$ will produce the firſt, ſecond, third, fourth, fifth, ſixth, and other following terms of the ſame vertical column, which are 0, 0, 0, 1, 4, 10, 20, 35, &c. But, by the 2d corollary of the foregoing propoſition, art. 81, the ſum of all the terms of the ſaid fourth vertical column (the number of which, including the three cyphers at the beginning of it, is n) is $= \frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times 3\times 4} = \frac{n^4-6n^3+11nn-6n}{24}$. Therefore the ſum of the n ſucceſſive values of the fraction $\frac{x^3-6xx+11x-6}{6}$ will be $= \frac{n^4-6n^3+11nn-6n}{24}$, or, according to our former notation, $S.\overline{\Big|\frac{x^3-6xx+11x-6}{6}}$ will be $= \frac{n^4-6n^3+11nn-6n}{24}$. But $S.\overline{\Big|\frac{x^3-6xx+11x-6}{6}}$ is $= S.\frac{x^3}{6} - S.\frac{6xx}{6} + S.\frac{11x}{6} - S.\frac{6}{6} = S.$

$\frac{x^3}{6} - \mathrm{S}.\,xx + \frac{11}{6} \times \mathrm{S}.\,x - \mathrm{S}.\,1 =$ (by the two foregoing articles 136 and 137) $\mathrm{S}.\frac{x^3}{6} - \frac{n^3}{3} - \frac{nn}{2} - \frac{n}{6} + \frac{11}{6} \times \overline{\frac{nn}{2} + \frac{n}{2}} - n = \mathrm{S}.\frac{x^3}{6} - \frac{n^3}{3} - \frac{nn}{2} - \frac{n}{6} + \frac{11nn}{12} + \frac{11n}{12} - n = \mathrm{S}.\frac{x^3}{6} - \frac{n^3}{3} - \frac{6nn}{12} - \frac{2n}{12} + \frac{11nn}{12} + \frac{11n}{12} - \frac{12n}{12} = \mathrm{S}.\frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12} - \frac{3n}{12} = \mathrm{S}.\frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12} - \frac{n}{4}$. Therefore $\mathrm{S}.\frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12} - \frac{n}{4}$ will be $= \frac{n^4 - 6n^3 + 11nn - 6n}{24}$; and conſequently (adding $\frac{n^3}{3} + \frac{n}{4}$ to both ſides) $\mathrm{S}.\frac{x^3}{6} + \frac{5nn}{12}$ will be $= \frac{n^4 - 6n^3 + 11nn - 6n}{24} + \frac{n^3}{3} + \frac{n}{4} = \frac{n^4 - 6n^3 + 11nn - 6n}{24} + \frac{8n^3}{24} + \frac{6n}{24} = \frac{n^4 + 2n^3 + 11nn}{24}$; and (ſubtracting $\frac{5nn}{12}$ from both ſides) $\mathrm{S}.\frac{x^3}{6}$ will be $= \frac{n^4 + 2n^3 + 11nn}{24} - \frac{5nn}{12} = \frac{n^4 + 2n^3 + 11nn}{24} - \frac{10nn}{24} = \frac{n^4 + 2n^3 + nn}{24}$. Therefore $\mathrm{S}.\,x^3$ will be $= 6 \times \frac{n^4 + 2n^3 + nn}{24} = \frac{n^4 + 2n^3 + nn}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$, or the ſum of the n ſucceſſive values of x^3, or of the ſeveral cube numbers 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, &c, continued to n^3, will be $= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$. Q. E. I.

Thus, for example, if n is $= 12$, we ſhall have $nn = 144$, and $n^3 = 1728$, and $n^4 = 20{,}736$, and conſequently $\frac{n^4}{4}$ $(= \frac{20736}{4}) = 5184$, and $\frac{n^3}{2}$ $(= \frac{1728}{2}) = 864$, and $\frac{nn}{4}$ $(= \frac{144}{4} = 36$, and $\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$ $(= 5184 + 864 + 36) = 6084$. Therefore the ſum of the twelve numbers 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, and 1728, (which are the cubes of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12), will be = 6084; And ſo we ſhall find the ſaid ſum to be, if we actually add up together the ſaid twelve cube numbers.

$$\begin{array}{r} 1 \\ 8 \\ 27 \\ 64 \\ 125 \\ 216 \\ 343 \\ 512 \\ 729 \\ 1000 \\ 1331 \\ 1728 \\ \hline 6084 \end{array}$$

Of the ſum of the fourth powers of the natural numbers 1, 2, 3, 4, 5, *&c; continued to any given number* n.

139. Let n be put, as before, for the ſeveral ſucceſſive terms of the ſeries 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n.

Then, ſince by coroll. 4, of the foregoing propoſition, art. 83, the nth term of the fifth vertical column of the foregoing table of combinations is $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4}$ $= \frac{n^3 - 6nn + 11n - 6}{6} \times \frac{n-4}{4} = \frac{n^4 - 10n^3 + 35nn - 50n + 24}{24}$, it follows that every xth term of the ſame fifth column will be $=$ $\frac{x^4-10x^3+35xx-50x+24}{24}$, or that, if x be made ſucceſſively equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, the ſucceſſive values of the fraction $\frac{x^4-10x^3+35xx-50x+24}{24}$ will produce the firſt, ſecond, third, fourth, fifth, ſixth, and other following terms of the ſame vertical column, which are 0, 0, 0, 0, 1, 5, 15, 35, 70, &c. But, by the ſecond corollary of the foregoing propoſition, art. 81, the ſum of all the terms of the ſaid fifth vertical column (the number of which, including the four cyphers at the beginning of it, is n) is $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5} = \frac{n^5-10n^4+35n^3-50nn+24n}{120}$. Therefore the ſum of all the n ſucceſſive values of the fraction $\frac{x^4-10x^3+35xx-50x+24}{24}$ will be $= \frac{n^5-10n^4+35n^3-50nn+24n}{120}$; or, according to our former notation,

$$S.\overline{\left|\frac{x^4-10x^3+35xx-50x+24}{24}\right.} \text{ will be } = \frac{n^5-10n^4+35n^3-50nn+24n}{120}.$$

But $S.\overline{\left|\frac{x^4-10x^3+35xx-50x+24}{24}\right.}$ is $= S.\frac{x^4}{24} - S.\frac{10x^3}{24} + S.\frac{35xx}{24} - S.\frac{50x}{24} + S.\frac{24}{24} = S.\frac{x^4}{24} - \frac{10}{24} \times S.x^3 + \frac{35}{24} \times S.xx - \frac{50}{24} \times S\,x + S.1 = S.\frac{x^4}{24} - \frac{10}{24} \times \overline{\left|\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}\right.} + \frac{35}{24} \times \overline{\left|\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}\right.} - \frac{50}{24} \times \overline{\left|\frac{nn}{2} + \frac{n}{2}\right.} + n = S.\frac{x^4}{24} - \frac{10n^4}{96} - \frac{10n^3}{48} - \frac{10nn}{96} + \frac{35n^3}{72} + \frac{35nn}{48} + \frac{35n}{144} - \frac{50nn}{48} - \frac{50n}{48} + n = S.\frac{x^4}{24} - \frac{5n^4}{48} - \frac{30n^3}{144} - \frac{10nn}{96} + \frac{70n^3}{144} + \frac{70nn}{96} + \frac{35n}{144} - \frac{100nn}{96} - \frac{150n}{144} + \frac{144n}{144} = S.\frac{x^4}{24} - \frac{5n^4}{48} + \frac{40n^3}{144} - \frac{40nn}{96} + \frac{29n}{144}$.

Therefore S . $\frac{x^4}{24} - \frac{5n^4}{48} + \frac{40n^3}{144} - \frac{40nn}{96} + \frac{29n}{144}$ will be $= \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120}$; and conſequently (adding $\frac{5n^4}{48} + \frac{40nn}{96}$ to both ſides) we ſhall have S . $\frac{x^4}{24} + \frac{40n^3}{144} + \frac{29n}{144}$ $(= \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120} + \frac{5n^4}{48} + \frac{40nn}{96} = \frac{n^5}{120} - \frac{n^4}{12} + \frac{7n^3}{24} - \frac{5nn}{12} + \frac{n}{5} + \frac{5n^4}{48} + \frac{5nn}{12} = \frac{n^5}{120} - \frac{4n^4}{48} + \frac{7n^3}{24} - \frac{5nn}{12} + \frac{n}{5} + \frac{5n^4}{48} + \frac{5nn}{12}) = \frac{n^5}{120} + \frac{n^4}{48} + \frac{7n^3}{24} + \frac{n}{5}$, and ſubtracting $\frac{40n^3}{144} + \frac{29n}{144}$ from both ſides) S . $\frac{x^4}{24} (= \frac{n^5}{120} + \frac{n^4}{48} + \frac{7n^3}{24} + \frac{n}{5} - \frac{40n^3}{144} - \frac{29n}{144} = \frac{n^5}{120} + \frac{n^4}{48} + \frac{42n^3}{144} + \frac{144n}{720} - \frac{40n^3}{144} - \frac{145n}{720} = \frac{n^5}{120} + \frac{n^4}{48} + \frac{2n^3}{144} - \frac{n}{720}) = \frac{n^5}{120} + \frac{n^4}{48} + \frac{n^3}{72} - \frac{n}{720}$; and conſequently S . x^4 will be $(= 24 \times \frac{n^5}{120} + 24 \times \frac{n^4}{48} + 24 \times \frac{n^3}{72} - 24 \times \frac{n}{720}) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$; or the ſum of the n ſucceſſive values of x^4, or of the ſeveral fourth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n, will be $= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Q. E. I.

Thus, for example, if n is $= 12$, we ſhall have $n^3 = 1728$, and $n^4 = 20,736$, and n^5 $(= n^4 \times n = 20,736 \times 12) = 248,832$, and conſequently $\frac{n^5}{5} = \frac{248,832}{5}$, and $\frac{n^4}{2}$ $(= \frac{20,736}{2} = 10,368$, and $\frac{n^3}{3}$ $(= \frac{1728}{3})$ 576, and $\frac{n}{30}$ $(= \frac{12}{30}) = \frac{2}{5}$, and $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ $(= \frac{248,832}{5} + 10,368 + 576 - \frac{2}{5} = \frac{248,830}{5} + 10,368 + 576 = 49,766 + 10,368 + 576) = 60,710$. Therefore the ſum of the twelve numbers

bers 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, 14641, and 20,736, (which are the fourth powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12), will be = 60,710. And ſo we ſhall find the ſaid ſum to be, if we actually add up together the ſaid twelve numbers, or fourth powers of the firſt twelve natural numbers; which may be done as follows.

$$\begin{array}{r} 1 \\ 16 \\ 81 \\ 256 \\ 625 \\ 1{,}296 \\ 2{,}401 \\ 4{,}096 \\ 6{,}561 \\ 10{,}000 \\ 14{,}641 \\ 20{,}736 \\ \hline 60{,}710. \end{array}$$

140. The foregoing examples are, I preſume, ſufficient to ſhew how the ſums of the ſeveral powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to any number *n*, may be deduced from the ſums of the ſeveral correſponding orders of figurate numbers contained in the foregoing table of combinations. I ſhall not therefore add the inveſtigations of the ſums of any higher powers of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, than the foregoing ones, but ſhall only ſet down the reſults of the like inveſtigations which I have made for my own ſatisfaction with reſpect to the ſums of the ſix next higher powers of thoſe numbers, to wit, the fifth powers, the ſixth powers, the ſeventh powers, the eighth powers, the ninth powers, and the tenth powers of them. Theſe reſults are as follows.

The

And for the ſum of the ſixth powers of the firſt twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 64; 729; 4,096; 15,625; 46,656; 117,649; 262,144; 531,441; 1,000,000; 1,771,561; and 2,985,984), we ſhall have $\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$, or $\frac{\overline{12}|^7}{7} + \frac{\overline{12}|^6}{2} + \frac{\overline{12}|^5}{2} - \frac{\overline{12}|^3}{6} + \frac{12}{42}$, $= \frac{\overline{12}|^6 \times 12}{7} + \frac{\overline{12}|^6}{2} + \frac{\overline{12}|^5}{2} - \frac{\overline{12}|^3}{6} + \frac{2}{7} = \frac{2{,}985{,}984 \times 12}{7} + \frac{2{,}985{,}984}{2} + \frac{248{,}832}{2} - \frac{1728}{6} + \frac{2}{7} = \frac{35{,}831{,}808}{7} + 1{,}492{,}992 + 124{,}416 - 288 + \frac{2}{7} = \frac{35{,}831{,}810}{7} + 1{,}492{,}992 + 124{,}416 - 288 = 5{,}118{,}830 + 1{,}492{,}992 + 124{,}416 - 288 = 5{,}118{,}830 + 1{,}492{,}992 + 124{,}128) = 6{,}735{,}950$. And ſo we ſhall find the ſaid ſum to be, if we add up together the ſaid twelve numbers, or ſixth powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

1
64
729
4,096
15,625
46,656
117,649
262,144
531,441
1,000,000
1,771,561
2,985,984

6,735,950.

And

And for the ſum of the ſeventh powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 128; 2,187; 16,384; 78,125; 279,936; 823,543; 2,097,152; 4,782,969; 10,000,000; 19,487,171; and 35,831,808), we ſhall have $\frac{n^8}{8}+\frac{n^7}{2}+\frac{7n^6}{12}-\frac{7n^4}{24}+\frac{nn}{12}$, or $\frac{\overline{12}|^8}{8}+\frac{\overline{12}|^7}{2}+\frac{7\times\overline{12}|^6}{12}-\frac{7\times\overline{12}|^4}{24}+\frac{\overline{12}|^2}{12}$ ($=\frac{12\times\overline{12}|^7}{8}+\frac{\overline{12}|^7}{2}+7\times\overline{12}|^5-\frac{7\times\overline{12}|^3}{2}+12$ $=\frac{3\times\overline{12}|^7}{2}+\frac{\overline{12}|^7}{2}+7\times\overline{12}|^5-\frac{7\times\overline{12}|^3}{2}+12=\frac{3\times 35,831,808}{2}$ $+\frac{35,831,808}{2}+7\times 248,832-\frac{7\times 1728}{2}+12=3\times$ $17,915,904+17,915,904+1,741,824-7\times 864+12$ $=4\times 17,915,904+1,741,824-6048+12=$ $71,663,616+1,741,824-6048+12=73,405,452-$ 6048) $=73,399,404$. And ſo we ſhall find the ſaid ſum to be, if we add up together the ſaid twelve numbers, or ſeventh powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

$$\begin{array}{r}
1\\
128\\
2,187\\
16,384\\
78,125\\
279,936\\
823,543\\
2,097,152\\
4,782,969\\
10,000,000\\
19,487,171\\
35,831,808\\
\hline
73,399,404.
\end{array}$$

And for the ſum of the eighth powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 256; 6,561; 65,536; 390,625; 1,679,616; 5,764,801; 16,777,216; 43,046,721; 100,000,000; 214,358,881; and 429,981,696) we ſhall have $\frac{n^9}{9}+\frac{n^8}{2}+\frac{2n^7}{3}-\frac{7n^5}{15}+\frac{2n^3}{9}-\frac{n}{30}$, or

$$\frac{\overline{12}|^{9}}{9}+\frac{\overline{12}|^{8}}{2}+\frac{2\times\overline{12}|^{7}}{3}-\frac{7\times\overline{12}|^{5}}{15}+\frac{2\times\overline{12}|^{3}}{9}-\frac{12}{30}\ (=\frac{12\times\overline{12}|^{8}}{9}+\frac{\overline{12}|^{8}}{2}+\frac{2\times\overline{12}|^{7}}{3}-\frac{7\times\overline{12}|^{5}}{15}+\frac{2\times\overline{12}|^{3}}{9}-\frac{2}{5}=\frac{4\times\overline{12}|^{8}}{3}+\frac{\overline{12}|^{8}}{2}+\frac{2\times\overline{12}|^{7}}{3}-\frac{7\times\overline{12}|^{5}}{15}+\frac{2\times1728}{9}-\frac{2}{5}=\frac{4\times429{,}981{,}696}{3}+\frac{429{,}981{,}696}{2}+\frac{2\times35{,}831{,}808}{3}-\frac{7\times248{,}832}{15}+\frac{3456}{9}-\frac{2}{5}=\frac{1{,}719{,}926{,}784}{3}+214{,}990{,}848+\frac{71{,}663{,}616}{3}-\frac{1{,}741{,}824}{15}+384-\frac{2}{5}=573{,}308{,}928+214{,}990{,}848+23{,}887{,}872-\frac{580{,}608}{5}+384-\frac{2}{5}=812{,}188{,}032-\frac{580{,}610}{5}=812{,}188{,}032-116{,}12\ )=812{,}071{,}910.$$

And ſo we ſhall find the ſaid ſum to be, if we add up together the ſaid twelve numbers, or eighth powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

$$\begin{array}{r}
1\\
256\\
6,561\\
65,536\\
390,625\\
1,679,616\\
5,764,801\\
16,777,216\\
43,046,721\\
100,000,000\\
214,358,881\\
429,981,696\\
\hline
812,071,910
\end{array}$$

And for the ſum of the ninth powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, (which powers are 1; 512; 19,683; 262,144; 1,953,125; 10,077,696; 40,353,607; 134,217,728; 387,420,489; 1,000,000,000; 2,357,947,691; and 5,159,780,352;) we ſhall have $\frac{n^{10}}{10}+\frac{n^{9}}{2}+\frac{3n^{8}}{4}-\frac{7n^{6}}{10}+\frac{n^{4}}{2}-\frac{3nn}{20}$, or $\frac{\overline{12}|^{10}}{10}+\frac{\overline{12}|^{9}}{2}+\frac{3\times\overline{12}|^{8}}{4}-\frac{7\times\overline{12}|^{6}}{10}+\frac{\overline{12}|^{4}}{2}-\frac{3\times\overline{12}|^{2}}{20}$ $\left(=\frac{12\times\overline{12}|^{9}}{10}+\frac{\overline{12}|^{9}}{2}+\frac{3\times\overline{12}|^{8}}{4}-\frac{7\times\overline{12}|^{6}}{10}+\frac{\overline{12}|^{4}}{2}-\frac{3\times\overline{12}|^{2}}{20}\right.$ $=\frac{12\times 5,159,780,352}{10}+\frac{5,159,780,352}{2}+\frac{3\times 429,981,696}{4}-\frac{7\times 2,985,984}{10}+\frac{20,736}{2}-\frac{3\times 144}{20}=\frac{61,917,364,224}{10}+2,579,890,176+\frac{1,289,945,088}{4}-\frac{20,901,888}{10}+10,368-\frac{3\times 72}{10}=\frac{61,917,364,224}{10}+2,579,890,176+322,486,272-\frac{20,901,888}{10}+10,368-\frac{216}{10}=\frac{61,917,364,224}{10}-\frac{20,902,104}{10}+2,579,890,$

176 + 322,486,272 + 10,368 $= \frac{61,896.462,120}{10}$ + 2,902, 386,816 = 6,189,646,212 + 2,902,386,816) = 9,092, 033,028. And ſo we ſhall find the ſaid ſum to be, if we add up together the ſaid twelve numbers, or ninth powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows:

1
512
19,683
262,144
1,953,125
10,077,696
40,353,607
134,217,728
387,420,489
1,000,000,000
2,357,947,691
5,159,780,352
9,092,033,028.

And for the ſum of the tenth powers of the firſt twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, (which powers are 1; 1,024; 59,049; 1,048,576; 9,765,625; 60,466,176; 282,475, 249; 1,073,741,824; 3,486,784,401; 10,000,000,000; 25,937,424,601; and 61,917,364,224), we ſhall have $\frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$, or $\frac{\overline{12}|^{11}}{11} + \frac{\overline{12}|^{10}}{2} + \frac{5 \times \overline{12}|^9}{6} - \overline{12}|^7 + \overline{12}|^5 - \frac{\overline{12}|^3}{2} + \frac{5 \times 12}{66}$ $(= \frac{12 \times \overline{12}|^{10}}{11} + \frac{\overline{12}|^{10}}{2} + \frac{5 \times \overline{12}|^9}{6} - \overline{12}|^7 + \overline{12}|^5 - \frac{\overline{12}|^3}{2} + \frac{10}{11} = \frac{12 \times 61,917,364,224}{11} + \frac{61,917,364,224}{2}$

$\frac{61{,}917{,}364{,}224}{2} + \frac{5 \times 5{,}159{,}780{,}352}{6} - 35{,}831{,}808 + 248{,}832 - \frac{1728}{2} + \frac{10}{11} = \frac{743{,}008{,}370{,}688}{11} + 30{,}958{,}682{,}112 + 5 \times 859{,}963{,}392 - 35{,}831{,}808 + 248{,}832 - 864 + \frac{10}{11} = \frac{743{,}008{,}370{,}698}{11} + 30{,}958{,}682{,}112 + 4{,}299{,}816{,}960 - 35{,}831{,}808 + 248{,}832 - 864 = 67{,}546{,}215{,}518 + 30{,}958{,}682{,}112 + 4{,}299{,}816{,}960 - 35{,}831{,}808 + 248{,}832 - 864 = 102{,}804{,}963{,}422 - 35{,}832{,}672) = 102{,}769{,}130{,}750$. And ſo we ſhall find the ſaid ſum to be, if we add up together the ſaid twelve numbers, or tenth powers of the firſt twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows:

$$\begin{array}{r} 1 \\ 1{,}024 \\ 59{,}049 \\ 1{,}048{,}576 \\ 9{,}765{,}625 \\ 60{,}466{,}176 \\ 282{,}475{,}249 \\ 1{,}073{,}741{,}824 \\ 3{,}486{,}784{,}401 \\ 10{,}000{,}000{,}000 \\ 25{,}937{,}424{,}601 \\ 61{,}917{,}364{,}224 \\ \hline 102{,}769{,}130{,}750. \end{array}$$

141. If the foregoing expreſſions of the values of the ſums of theſe ſeveral ſets of powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to *n*, be ſet down one under the other in a regular table, the ſaid table will be as follows:

A Table of the values of the ſums of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, *&c, continued to* n *terms, and of the ſquares, and the cubes, and the fourth powers, and other following powers of the ſaid numbers, as far as the tenth powers, incluſively.*

The ſum of the firſt n terms of the ſaid ſeries of natural numbers is equal to

$$\frac{nn}{2} + \frac{n}{2}.$$

The ſum of the ſquares of the ſaid n terms is

$$= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}.$$

The ſum of the cubes of the ſaid n terms is

$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}.$$

The ſum of the fourth powers of the ſaid n terms is

$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} \ast - \frac{n}{30}.$$

The ſum of the fifth powers of the ſaid n terms is

$$= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} \ast - \frac{nn}{12}.$$

The ſum of the ſixth powers of the ſaid n terms is

$$= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} \ast - \frac{n^3}{6} \ast + \frac{n}{42}.$$

The ſum of the ſeventh powers of the ſaid n terms is

$$= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} \ast - \frac{7n^4}{24} \ast + \frac{nn}{12}.$$

The ſum of the eighth powers of the ſaid n terms is

$$= \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} \ast - \frac{7n^5}{15} \ast + \frac{2n^3}{9} \ast - \frac{n}{30}.$$

The ſum of the ninth powers of the ſaid n terms is

$$= \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} \ast - \frac{7n^6}{10} \ast + \frac{n^4}{2} \ast - \frac{3nn}{20}.$$

And the ſum of the tenth powers of the ſaid n terms is

$$= \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} \ast - n^7 \ast + n^5 \ast - \frac{n^3}{2} \ast + \frac{5n}{66}.$$

The law of the generation, or derivation, of the terms of the ſeveral expreſſions ſet down in the foregoing table, one from the other.

142. By an attentive conſideration of the foregoing table we may diſcover the law by which the terms of the ſeveral expreſſions of which it conſiſts, may be derived one from the other; after which we ſhall be able to continue the ſaid table to the ſums of the eleventh and twelfth and other higher powers of the numbers 1, 2, 3, 4, 5, 6, &c, without entering into the long trains of reaſoning and making the laborious ſubſtitutions of the ſums already known in the expreſſion of the value of the new ſum, which have been uſed in obtaining the foregoing ſums. This law will be found to be as follows.

Let the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, be ſuppoſed to be continued to any number n; and let it be required to find the ſum of the cth powers of the ſaid n terms, or the value of the ſeries $\overline{1}|^{c} + \overline{2}|^{c} + \overline{3}|^{c} + \overline{4}|^{c} + \overline{5}|^{c} + \overline{6}|^{c} + \overline{7}|^{c} + \overline{8}|^{c} + \overline{9}|^{c} + \overline{10}|^{c} + \overline{11}|^{c}$ + &c, continued to $\overline{n}|^{c}$. Let the capital letters A, B, C, D, &c, be put for the co-efficients of the laſt terms of the ſums of the ſquares, and the fourth powers, and the ſixth powers, and the eighth powers, and the other following even powers of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, already computed, with the ſame ſigns + or — prefixed to them, as are prefixed to the ſaid laſt terms, of which they are the co-efficients. Thus, becauſe the laſt term of the ſum of the ſquares of the numbers 1, 2, 3, 4, 5, 6, 7, &c, is $+ \frac{n}{6}$, or $+ \frac{1}{6} \times n$, A will be $= + \frac{1}{6}$; and, becauſe the laſt term of the ſum of the fourth powers of the ſaid numbers is $-\frac{n}{30}$, or $-\frac{1}{30}\times n$, B will be $=-\frac{1}{30}$; and, becauſe the laſt term of the ſum of the ſixth powers of the ſaid numbers is $+\frac{n}{42}$, or $+\frac{1}{42}\times n$, C will be $=+\frac{1}{42}$; and, becauſe the laſt term of the ſum of the eighth powers of the ſaid numbers is $-\frac{n}{30}$, or $-\frac{1}{30}\times n$, D will be $=-\frac{1}{30}$; and, becauſe the laſt term of the ſum of the tenth powers of the ſaid numbers is $+\frac{5n}{66}$, or $+\frac{5}{66}\times n$, E will be $=+\frac{5}{66}$. Theſe being the ſeveral values of the capital letters A, B, C, D, E, &c, the ſeries $\overline{1|}^{c}+\overline{2|}^{c}+\overline{3|}^{c}+\overline{4|}^{c}+\overline{5|}^{c}+\overline{6|}^{c}+\overline{7|}^{c}+\overline{8|}^{c}+\overline{9|}^{c}+\overline{10|}^{c}+\overline{11|}^{c}+$ &c, $+\overline{n|}^{c}$ will be equal to the ſeries $\frac{1}{c+1}\times n^{c+1}+\frac{1}{2}$

$$\times n^{c}+\frac{c}{2}\times An^{c-1}+\frac{c\times\overline{c-1}\times\overline{c-2}}{2\times3\times4}\times Bn^{c-3}$$

$$+\frac{c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\times\overline{c-4}}{2\times3\times4\times5\times6}\times Cn^{c-5}$$

$$+\frac{c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\times\overline{c-4}\times\overline{c-5}\times\overline{c-6}}{2\times3\times4\times5\times6\times7\times8}\times Dn^{c-7}$$

$$+\frac{c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\times\overline{c-4}\times\overline{c-5}\times\overline{c-6}\times\overline{c-7}\times\overline{c-8}}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}\times En^{c-9}$$

+ &c; in which the indexes of the powers of n, after the third term $\frac{c}{2}\times An^{c-1}$, decreaſe continually by 2, till we come at laſt to n or nn, and the co-efficients of An^{c-1}, Bn^{c-3}, Cn^{c-5}, Dn^{c-7}, En^{c-9}, &c, are formed by the continual multiplication of 1 into the fractions $\frac{c}{2}$, $\frac{c-1}{3} \times \frac{c-2}{4}$, $\frac{c-3}{5} \times \frac{c-4}{6}$, $\frac{c-5}{7} \times \frac{c-6}{8}$, $\frac{c-7}{9} \times \frac{c-8}{10}$, $\frac{c-9}{11} \times \frac{c-10}{12}$, &c, till we come to $\frac{c-\overline{c-1}}{c+1} \times \frac{c-c}{c+2}$, or $\frac{c-c+1}{c+1} \times \frac{c-c}{c+2}$, or $\frac{1}{c+1} \times \frac{c-c}{c+2}$, or $\frac{1}{c+1} \times \frac{0}{c+2}$, which will be $= 0$. And thus we ſhall determine the powers of n in all the terms of the ſaid ſeries, and alſo the co-efficients of the ſaid powers of n in all the terms of the ſaid ſeries, except the laſt term. And this laſt co-efficient may be derived from the co-efficients of the preceeding terms, by an eaſy addition or ſubtraction, being always the quantity which is neceſſary to be added to, or ſubtracted from, the reſult of all the preceeding co-efficients, in order to make ſuch reſult become equal to 1. Thus, in the firſt ſum in the foregoing table, to wit, $\frac{nn}{2}+\frac{n}{2}$, the co-efficient of the laſt term $\frac{n}{2}$ is $\frac{1}{2}$, which is the quantity which muſt be added to $\frac{1}{2}$, the co-efficient of the firſt term $\frac{nn}{2}$, in order to make it equal to 1; and, in the ſecond ſum, $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$, the co-efficient of the laſt term $\frac{n}{6}$ is $\frac{1}{6}$, which is the quantity that muſt be added to $\frac{1}{3} + \frac{1}{2}$, or the ſum of the co-efficients of the firſt and ſecond terms, $\frac{n^3}{3} + \frac{n^2}{2}$, in order to make it equal to 1; and in the third ſum, $\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$, the co-efficient of the laſt term is $\frac{1}{4}$, which is the quantity that muſt be added to $\frac{1}{4} + \frac{1}{2}$, or the ſum of the co-efficients of the firſt and ſecond terms, $\frac{n^4}{4} + \frac{n^3}{2}$, in order to make it equal to 1; and in the fourth ſum, $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$, the co-efficient of the laſt term is $\frac{1}{30}$, which is the quantity which muſt be ſubtracted from $\frac{1}{5} + \frac{1}{2} + \frac{1}{3}$, or the ſum of the co-efficients of the three firſt terms $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3}$, in order to make it equal to 1. And the ſame thing is true with reſpect to the co-efficients of the laſt terms of all the following ſums*; and conſequently the ſaid co-efficients, and the ſigns + or —, to be prefixed to them, may always be determined, by means of the co-efficients of the preceeding terms of the ſums to which they belong. And thus all the terms of the ſeries $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times An^{c-1} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times Bn^{c-3} + \frac{c}{2} \times \frac{c-1}{3}$

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* And hence it will appear that the co-efficient of the laſt term of the expreſſion that is equal to the ſum of the ninth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n, muſt be $-\frac{3}{20}$, and not $-\frac{1}{12}$; agreeably to what was obſerved above in the note at the bottom of page 180. For, as the ſeveral terms of that expreſſion preceeding the laſt term are $\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2}$, of which the co-efficients are $\frac{1}{10} + \frac{1}{2} + \frac{3}{4} - \frac{7}{10} + \frac{1}{2}$, which are $= \frac{2}{20} + \frac{10}{20} + \frac{15}{20} - \frac{14}{20} + \frac{10}{20}$ or $\frac{22}{20} + \frac{1}{20}$, or $\frac{23}{20}$, or $\frac{20}{20} + \frac{3}{20}$, or $1 + \frac{3}{20}$ (from which it is neceſſary to ſubtract the fraction $\frac{3}{20}$ in order to make it become equal to 1), it follows, according to the rule here laid down by the author, that the co-efficient of the laſt term of the ſaid expreſſion muſt be $-\frac{3}{20}$, and conſequently that the laſt term of the ſaid expreſſion muſt be $-\frac{3nn}{20}$. Q. E. D.

$$\times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times \mathrm{C}n^{c-5} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5}$$
$$\times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times \mathrm{D}n^{c-7} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times$$
$$\frac{c-3}{5} \times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times \frac{c-7}{9} \times \frac{c-8}{10} \times \mathrm{E}n^{c-9} + \&c,$$

may be determined, and conſequently the value of the ſeries $\overline{1|}^{c} + \overline{2|}^{c} + \overline{3|}^{c} + \overline{4|}^{c} + \overline{5|}^{c} + \overline{6|}^{c} + \overline{7|}^{c} + \overline{8|}^{c} + \overline{9|}^{c} + \overline{10|}^{c} + \overline{11|}^{c} + \&c + \overline{n|}^{c}$ (to which the ſaid ſeries is equal) may be aſſigned, without the help of the long reaſonings and laborious ſubſtitutions that were employed for this purpoſe in the foregoing articles. The method of doing this will appear more clearly from the following example.

An example of the computation of the expreſſion that is equal to the ſum of certain powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, *&c, continued to* n, *by means of the foregoing ſeries.*

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143. As an example to the foregoing ſeries, let it be required to find the value of the ſeries $\overline{1|}^{10} + \overline{2|}^{10} + \overline{3|}^{10} + \overline{4|}^{10} + \overline{5|}^{10} + \overline{6|}^{10} + \overline{7|}^{10} + \overline{8|}^{10} + \overline{9|}^{10} + \overline{10|}^{10} + \overline{11|}^{10} + \&c + \overline{n|}^{10}$, or the ſum of the 10th powers of the ſeveral natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n terms; which ſum, as ſet down in the foregoing table, is $= \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^{9}}{6} - n^{7} + n^{5} - \frac{n^{3}}{2} + \frac{5n}{66}$.

In this caſe the index c is $= 10$, and conſequently $c + 1$ is $= 11$, and $c - 1$ is $= 9$, and $c-2$, $c-3$, $c-4$, $c-5$,

$c-6$, $c-7$, $c-8$, $c-9$, and $c-10$ are, reſpectively, equal to 8, 7, 6, 5, 4, 3, 2, 1, and 0. We ſhall therefore have

$\frac{c}{2}\left(=\frac{10}{2}\right)=5$,

and $\frac{c-1}{3}\times\frac{c-2}{4}\left(=\frac{9}{3}\times\frac{8}{4}=3\times2\right)=6$,

and $\frac{c-3}{5}\times\frac{c-4}{6}\left(=\frac{7}{5}\times\frac{6}{6}\right)=\frac{7}{5}$,

and $\frac{c-5}{7}\times\frac{c-6}{8}\left(=\frac{5}{7}\times\frac{4}{8}=\frac{5}{7}\times\frac{1}{2}\right)=\frac{5}{14}$,

and $\frac{c-7}{9}\times\frac{c-8}{10}\left(=\frac{3}{9}\times\frac{2}{10}=\frac{1}{3}\times\frac{1}{5}\right)=\frac{1}{15}$,

and $\frac{c-9}{11}\times\frac{c-10}{12}\left(=\frac{1}{11}\times\frac{0}{12}\right)=0$.

Therefore the ſeries $\frac{1}{c+1}\times n^{c+1}+\frac{1}{2}\times n^{c}+\frac{c}{2}\times An^{c-1}+\frac{c}{2}\times\frac{c-1}{3}\times\frac{c-2}{4}\times Bn^{c-3}+\frac{c}{2}\times\frac{c-1}{3}\times\frac{c-2}{4}\times\frac{c-3}{5}\times\frac{c-4}{6}\times Cn^{c-5}+\frac{c}{2}\times\frac{c-1}{3}\times\frac{c-2}{4}\times\frac{c-3}{5}\times\frac{c-4}{6}\times\frac{c-5}{7}\times\frac{c-6}{8}\times Dn^{c-7}+\frac{c}{2}\times\frac{c-1}{3}\times\frac{c-2}{4}\times\frac{c-3}{5}\times\frac{c-4}{6}\times\frac{c-5}{7}\times\frac{c-6}{8}\times\frac{c-7}{9}\times\frac{c-8}{10}\times En^{c-9}$ will be $=\frac{1}{11}\times n^{11}+\frac{1}{2}\times n^{10}+5\times An^{9}+5\times6\times Bn^{7}+5\times6\times\frac{7}{5}\times Cn^{5}+5\times6\times\frac{7}{5}\times\frac{5}{14}\times Dn^{3}+5\times6\times\frac{7}{5}\times\frac{5}{14}\times\frac{1}{15}\times En=\frac{n^{11}}{11}+\frac{n^{10}}{2}+5An^{9}+30Bn^{7}+42Cn^{5}+42\times\frac{5}{14}\times Dn^{3}+42\times\frac{5}{14}\times\frac{1}{15}\times En=\frac{n^{11}}{11}+\frac{n^{10}}{2}+5An^{9}+30Bn^{7}+42Cn^{5}+15Dn^{3}+En=\frac{n^{11}}{11}+\frac{n^{10}}{2}+5\times+\frac{1}{6}\times n^{9}+30\times-\frac{1}{30}\times n^{7}+42\times+\frac{1}{42}\times n^{5}+15$

$\times - \frac{1}{30} \times n^3 - En = \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + En$; of which expreſſion all the terms, except the laſt term En, are known quantities. And this laſt term En may be found in the following manner. The co-efficients of all the preceeding terms are $\frac{1}{11} + \frac{1}{2} + \frac{5}{6} - 1 + 1 - \frac{1}{2}$, which are equal to $\frac{1}{11} + \frac{5}{6} = \frac{6}{66} + \frac{55}{66} = \frac{61}{66}$; to which it is neceſſary to add $\frac{5}{66}$, in order to make the reſult equal to 1. Therefore E, or the co-efficient of the laſt term En, will be $= + \frac{5}{6}$; and conſequently the compleat value of the foregoing ſeries in this caſe of the 10th powers of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n terms, will be $\frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$; which is the value ſet down for the ſum of the ſaid 10th powers in the foregoing table.

A numerical example of the computation of the ſum of the tenth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, *&c, continued to* 1000, *by means of the foregoing expreſſion.*

144. If n is $= 1000$, we ſhall have

$n^3 \ (= \overline{1000}|^{3})$
$= 1000{,}000{,}000,$
and $n^5 \ (= \overline{1000}|^{5})$
$= 1000{,}000{,}000{,}000{,}000;$
and $n^7 \ (= \overline{1000}|^{7})$
$= 1000{,}000{,}000{,}000{,}000{,}000{,}000,$
and $n^9 \ (= \overline{1000}|^{9})$
$= 1000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000,$
and $n^{10} \ (= \overline{1000}|^{10})$
$= 1000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000;$
and $n^{11} \ (= \overline{1000}|^{11})$
$= 1000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000,$
and conſequently

$\frac{{}^{11}}{11} \left(= \frac{1000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}{11}\right) =$
$90{,}909{,}090{,}909{,}090{,}909{,}090{,}909{,}090{,}909{,}090,$
and $\frac{n^{10}}{2} \left(= \frac{1000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}{2}\right) =$
$500{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000,$
and $\frac{5n^9}{6} \left(= \frac{5}{6} \times 1000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000,\right.$
$\left. = \frac{5000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}{6}\right) =$
$833{,}333{,}333{,}333{,}333{,}333{,}333{,}333{,}333 \cdot 333{,}333,$ &c.
and $\frac{n^3}{2} \left(= \frac{1000{,}000{,}000}{2} = 500{,}000{,}000;\right.$

and

$$\text{and } \frac{5n}{6} \left(= \frac{5 \times 1000}{66} = \frac{5000}{66}\right) = 75\cdot757,575, \text{ \&c};$$

$$\text{and } \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$$

$$= \left\{ \begin{array}{rl} 90,909,090,909,090,909,090,909,090,909,090.909,090, & \text{\&c} \\ +\ 500,000,000,000.000,000,000,000,000,000.000,000, & \\ +\ 833,333,333,333,333,333,333,333,333\cdot333,333, & \text{\&c} \\ -\ 1,000,000,000,000,000,000,000.000,000, & \\ +\ 1,000,000,000,000,000.000,000, & \\ -\ 500,000,000.000,000, & \\ +\ 75\cdot757,575, & \text{\&c} \end{array} \right.$$

$$= \left\{ \begin{array}{rl} 91,409,924,242,424,243,424,242,424,242,499\cdot999,998, & \text{\&c} \\ -\ 1,000,000,000,000,500,000,000.000,000, & \end{array} \right.$$

$$= 91,409,924,241,424,243,424,241,924,242,499\cdot999,998,$$

$$\text{or } 91,409,924,241,424,243,424,241,924,242,500.$$

Therefore the ſum of all the tenth powers of the firſt thouſand natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, ... 1000, is 91,409,924,241,424,243,424,241,924,242, 500, or more than 91 quintillions, or 91 times the fifth power of a million.

145. I cannot but obſerve on this occaſion, that the learned *Iſmael Bullialdus*, or *Bouillaud*, has been rather unfortunate in his manner of treating this ſubject, in his Treatiſe on the Arithmetick of Infinites [*]; ſince the whole of the folio volume which he has written upon it does nothing more than enable us to find the ſums of the firſt ſix powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, continued to any given number n; which is only a part of what we have here accompliſhed in the compaſs of a dozen pages.

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[*] See an account of this book of Monſieur Bouillaud in Dr. Wallis's Algebra, chapter lxxx. pages 310, 311.

A computation of all the other expreſſions given above in the Table ſet down in art. 141, *page* 188, *for the values of the ſums of the powers of the natural numbers* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *&c, continued to the number* n, *by means of the foregoing general ſeries* $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} +$

$$\frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{c-7}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6} \times \overline{c-7} \times \overline{c-8}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10} \times En^{c-9}$$

$$+ \&c.$$

146. If the foregoing ſeries be applied in the ſame manner to the computation of the ſums of the preceeding powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; 11, 12, &c, continued to the number *n*, as it has here been applied to the computation of the ſum of their tenth powers, it will be found to produce the ſeveral expreſſions ſet down above in the table in art. 141, for the values of the ſums of thoſe powers; to wit, for the value of the ſum of the firſt, or ſimple, powers of the ſaid natural numbers, continued to the number *n*, or for the ſum of the ſaid natural numbers themſelves, continued to the number *n*, the expreſſion

$$\frac{nn}{2} + \frac{n}{2};$$

And for the ſum of the ſquares of the ſaid *n* terms the expreſſion

$$\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6};$$

And for the ſum of the cubes of the ſaid n terms the expreſſion

$$\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4};$$

And for the ſum of the fourth powers of the ſaid n terms the expreſſion

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} * - \frac{n}{30};$$

And for the ſum of the fifth powers of the ſaid n terms the expreſſion

$$\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{nn}{12};$$

And for the ſum of the ſixth powers of the ſaid n terms the expreſſion

$$\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} * - \frac{n^3}{6} * + \frac{n}{42};$$

And for the ſum of the ſeventh powers of the ſaid n terms the expreſſion

$$\frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12};$$

And for the ſum of the eighth powers of the ſaid n terms the expreſſion

$$\frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} * - \frac{7n^5}{15} * + \frac{2n^3}{9} * - \frac{n}{30};$$

And for the ſum of the ninth powers of the ſaid n terms the expreſſion

$$\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3nn}{20}.$$

This may be done in the manner following.

In applying this ſeries to the firſt caſe, or the ſum of the firſt, or ſimple, powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n, or to the ſum of the ſaid natural numbers themſelves, it is evident that we muſt compute only the two firſt terms of the ſaid ſeries,

ſo

to wit, the terms $\frac{1}{c+1} \times n^{c+1}$ and $\frac{1}{2} \times n^{c}$; becauſe the following terms involve in them the numbers A, B, C, D, E, &c, which are derived from the values of the ſums of the ſquares, and the fourth powers, and the ſixth powers, and the eighth powers, and the tenth powers, and the other following even powers of the ſaid natural numbers, with which ſeveral ſums we have as yet nothing to do.

Now, becauſe c is in this caſe $= 1$, the two firſt terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c}$ of the foregoing ſeries will be $= \frac{1}{1+1} \times n^{1+1} + \frac{1}{2} \times n^{1} (= \frac{1}{2} \times n^{2} + \frac{1}{2} \times n) = \frac{nn}{2} + \frac{n}{2}$. Therefore the ſum of the firſt, or ſimple, powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n, or the ſum of the ſaid natural numbers themſelves, will be $= \frac{nn}{2} + \frac{n}{2}$. Q. E. I.

Secondly, When c is $= 2$, and the ſum of the ſquares of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt compute only the three firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times A n^{c-1}$; becauſe the following terms involve in them the numbers B, C, D, E, &c, which relate to the ſums of the fourth, and the ſixth, and the eighth, and the tenth, and the other following even powers of the ſaid natural numbers, with which ſums we have as yet nothing to do.

Now, when c is $= 2$, the three terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times A n^{c-1}$ will be $(= \frac{1}{2+1} \times n^{2+1} + \frac{1}{2} \times n^{2} + \frac{2}{2} + A n^{2-1} = \frac{1}{3} \times n^{3} + \frac{1}{2} \times n^{2} + 1 \times An^{1})$

$=$

$= \frac{1}{3} \times n^3 + \frac{1}{2} \times n^2 + A n$; of which expreſſion the two firſt terms are known quantities, and only the third, or laſt, term $A n$ remains to be inveſtigated. Now this laſt term $A n$ is to be found in the following manner. The co-efficients of the two preceeding terms are $\frac{1}{3} + \frac{1}{2}$, which are equal to $\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$; to which it is neceſſary to add $\frac{1}{6}$, in order to make the reſult equal to 1. Therefore A, or the co-efficient of the laſt term $A n$, will be $= + \frac{1}{6}$; and conſequently the compleat value of the three firſt terms of the foregoing ſeries in this caſe of the ſquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, continued to the number n, will be $\frac{1}{3} \times n^3 + \frac{1}{2} \times n^2 + \frac{1}{6} \times n$, or $\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$. Q. E. I.

3dly, When c is $= 3$, and the ſum of the cubes of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt (as in the laſt caſe) compute only the three firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1}$; becauſe the following terms involve in them the numbers B, C, D, F, &c, which relate to the ſums of the fourth, and the ſixth, and the eighth, and the tenth, and the other following even powers of the ſaid natural numbers, with which ſums we have as yet nothing to do.

Now, when c is $= 3$, the three terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1}$ will be $= \frac{1}{3+1} \times n^{3+1} + \frac{1}{2} \times n^3 + \frac{3}{2} \times A n^{3-1}$ ($= \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{3}{2} \times A n^2$

$= \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{3}{2} \times + \frac{1}{6} \times n^2 = \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{1}{4} \times n^2) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$. Therefore the ſum of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$. Q. E. I.

4thly, When c is $= 4$, and the ſum of the fourth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt only compute the four firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$; becauſe the following terms involve in them the numbers C, D, E, &c, which relate to the ſums of the ſixth, and the eighth, and the tenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 4$, the four terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ will be $= \frac{1}{4+1} \times n^{4+1} + \frac{1}{2} \times n^4 + \frac{4}{2} \times An^{4-1} + \frac{4 \times \overline{4-1} \times \overline{4-2}}{2 \times 3 \times 4} \times Bn^{4-3}$ $(= \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times An^3 + \frac{4 \times 3 \times 2}{2 \times 3 \times 4} \times Bn^1 = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times An^3 + Bn = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times + \frac{1}{6} \times n^3 + Bn) = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3 + Bn$; of which expreſſion the laſt term Bn is to be determined in the manner following.

ing. The co-efficients of the three firſt terms $\frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3$ are $\frac{1}{5} + \frac{1}{2} + \frac{1}{3}$ $(= \frac{6}{30} + \frac{15}{30} + \frac{10}{30} = \frac{31}{30}) = 1 + \frac{1}{30}$; from which it is neceſſary to ſubtract $\frac{1}{30}$, in order to make the reſult equal to 1. Therefore B, or the co-efficient of the laſt term Bn, will be $= -\frac{1}{30}$, and conſequently the compleat value of the four firſt terms of the ſaid ſeries in this caſe will be $\frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3 - \frac{1}{30} \times n$, or $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Therefore the ſum of the fourth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Q. E. I.

5thly, When c is $= 5$, and the ſum of the fifth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt, as in the laſt caſe, compute only the four firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$; becauſe the following terms involve in them the numbers C, D, E, &c, which relate to the ſums of the ſixth, and the eighth, and the tenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 5$, the four terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ will be $= \frac{1}{5+1} \times n^{5+1} + \frac{1}{2} \times n^5 + \frac{5}{2} \times An^{5-1}$

$+ \frac{5 \times \overline{5-1} \times \overline{5-2}}{2 \times 3 \times 4} \times Bn^{5-3}$ $(= \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times An^4 + \frac{5 \times 4 \times 3}{2 \times 3 \times 4} \times Bn^2 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times An^4 + \frac{5}{2} \times Bn^2 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times + \frac{1}{6} \times n^4 + \frac{5}{2} \times - \frac{1}{30} \times n^2 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{12} \times n^4 - \frac{5}{12} \times n^2) = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{5nn}{12}$. Therefore the ſum of the fifth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n, will be $= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{5nn}{12}$. Q. E. I.

6thly, When c is $= 6$, and the ſum of the ſixth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt compute only the five firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$; becauſe the following terms involve in them the numbers D, E, F, G, &c, which relate to the ſums of the eighth, and the tenth, and the twelfth, and the fourteenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 6$, the five terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ will be $= \frac{1}{6+1}$

$\times n^{6+1} + \frac{1}{2} \times n^{6} + \frac{6}{2} \times An^{6-1} + \frac{6 \times \overline{6-1} \times \overline{6-2}}{2 \times 3 \times 4} \times Bn^{6-3} + \frac{6 \times \overline{6-1} \times \overline{6-2} \times \overline{6-3} \times \overline{6-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{6-5}$ ($= \frac{1}{7} \times n^{7} + \frac{1}{2} \times n^{6} + 3 \times An^{5} + \frac{6 \times 5 \times 4}{2 \times 3 \times 4} \times Bn^{3} + \frac{6 \times 5 \times 4 \times 3 \times 2}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{1} = \frac{1}{7} \times n^{7} + \frac{1}{2} \times n^{6} + 3 \times An^{5} + 5 \times Bn^{3} + Cn = \frac{1}{7} \times n^{7} + \frac{1}{2} \times n^{6} + 3 \times + \frac{1}{6} \times n^{5} + 5 \times -\frac{1}{30} \times n^{3} + Cn$) $= \frac{1}{7} \times n^{7} + \frac{1}{2} \times n^{6} + \frac{1}{2} \times n^{5} - \frac{1}{6} \times n^{3} + Cn$; of which expreſſion the laſt term Cn is to be determined in the manner following. The co-efficients of the four firſt terms of the expreſſion $\frac{1}{7} \times n^{7} + \frac{1}{2} \times n^{6} + \frac{1}{2} \times n^{5} - \frac{1}{6} \times n^{3} + Cn$ are $\frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6}$ ($= \frac{6}{42} + \frac{21}{42} + \frac{21}{42} - \frac{7}{42} = \frac{6}{42} + \frac{42}{42} - \frac{7}{42} = \frac{6}{42} + 1 - \frac{7}{42}$) $= 1 - \frac{1}{42}$; to which it is neceſſary to add $\frac{1}{42}$, in order to make the reſult be equal to 1. Therefore C, or the co-efficient of the laſt term Cn, will be $+ \frac{1}{42}$, and conſequently the whole expreſſion $\frac{1}{7} \times n^{7} + \frac{1}{2} \times n^{6} + \frac{1}{2} \times n^{5} - \frac{1}{6} \times n^{3} + Cn$ will be $= \frac{1}{7} \times n^{7} + \frac{1}{2} \times n^{6} + \frac{1}{2} \times n^{5} - \frac{1}{6} \times n^{3} + \frac{1}{42} \times n$, or $\frac{n^{7}}{7} + \frac{n^{6}}{2} + \frac{n^{5}}{2} \ast - \frac{n^{3}}{6} \ast + \frac{n}{42}$. Therefore the ſum of the ſixth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^{7}}{7} + \frac{n^{6}}{2} + \frac{n^{5}}{2} \ast - \frac{n^{3}}{6} \ast + \frac{n}{42}$.

Q. E. I.

7thly, When c is $= 7$, and the ſum of the ſeventh powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt, as in the laſt caſe, compute only the five firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$; becauſe the following terms involve the numbers D, E, F, G, &c, which relate to the ſums of the eighth, and the tenth, and the twelfth, and the fourteenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 7$, the ſaid five terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ will be $= \frac{1}{7+1} \times n^{7+1} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times An^{7-1} + \frac{7 \times \overline{7-1} \times \overline{7-2}}{2 \times 3 \times 4} \times Bn^{7-3} + \frac{7 \times \overline{7-1} \times \overline{7-2} \times \overline{7-3} \times \overline{7-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{7-5}$ $(= \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times An^{6} + \frac{7 \times 6 \times 5}{2 \times 3 \times 4} \times Bn^{4} + \frac{7 \times 6 \times 5 \times 4 \times 3}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{2} = \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times An^{6} + \frac{7 \times 5}{4} \times Bn^{4} + \frac{7}{2} \times Cn^{2} = \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times + \frac{1}{6} \times n^{6} + \frac{7 \times 5}{4} \times - \frac{1}{30} \times n^{4} + \frac{7}{2} \times + \frac{1}{42} \times n^{2} = \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{12} \times n^{6} - \frac{7}{24} \times n^{4} + \frac{1}{12} \times n^{2}) = \frac{n^{8}}{8} + \frac{n^{7}}{2} + \frac{7n^{6}}{12} - \frac{7n^{4}}{24} + \frac{nn}{12}$. Therefore the ſum of the ſeventh powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12}$. Q. E. I.

8thly, When c is $= 8$, and the ſum of the eighth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt compute only the ſix firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1}$ $+ \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{c-7}$; becauſe the following terms involve in them the numbers E, F, G, &c, which relate to the ſums of the tenth, and the twelfth, and the fourteenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 8$, the ſix terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2}$ $\times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{c-7}$ will be $=$ $\frac{1}{8+1} \times n^{8+1} + \frac{1}{2} \times n^8 + \frac{8}{2} \times An^{8-1} + \frac{8 \times \overline{8-1} \times \overline{8-2}}{2 \times 3 \times 4}$ $\times Bn^{8-3} + \frac{8 \times \overline{8-1} \times \overline{8-2} \times \overline{8-3} \times \overline{8-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{8-5}$ $+ \frac{8 \times \overline{8-1} \times \overline{8-2} \times \overline{8-3} \times \overline{8-4} \times \overline{8-5} \times \overline{8-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{8-7}$

$(= \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + 4 \times An^7 + \frac{8 \times 7 \times 6}{2 \times 3 \times 4} \times Bn^5 + \frac{8 \times 7 \times 6 \times 5 \times 4}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^3 + \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^1 = \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + 4 \times An^7 + 7 \times 2 \times Bn^5 + \frac{7 \times 4}{3} \times Cn^3 + Dn = \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + 4 \times + \frac{1}{6} \times n^7 + 7 \times 2 \times - \frac{1}{30} \times n^5 + \frac{7 \times 4}{3} \times + \frac{1}{42} \times n^3 + Dn) = \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + \frac{2}{3} \times n^7 - \frac{7}{15} \times n^5 + \frac{2}{9} \times n^3 + Dn$; of which expreſſion the laſt term Dn is to be determined in the manner following. The co-efficients of the five firſt terms of the expreſſion $\frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + \frac{2}{3} \times n^7 - \frac{7}{15} \times n^5 + \frac{2}{9} \times n^3 + Dn$ are $\frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9}$, which are $(= \frac{10}{90} + \frac{45}{90} + \frac{60}{90} - \frac{42}{90} + \frac{20}{90} = \frac{135}{90} - \frac{42}{90} = \frac{93}{90} = \frac{90}{90} + \frac{3}{90} = 1 + \frac{3}{90}) = 1 + \frac{1}{30}$; from which it is neceſſary to ſubtract $\frac{1}{30}$, in order to make the reſult be equal to 1. Therefore D, or the co-efficient of the laſt term Dn, will be $= -\frac{1}{30}$, and conſequently the whole expreſſion $\frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + \frac{2}{3} \times n^7 - \frac{7}{15} \times n^5 + \frac{2}{9} \times n^3 + Dn$ will be $= \frac{1}{9} \times n^9 + \frac{1}{2} \times n^8 + \frac{2}{3} \times n^7 - \frac{7}{15} \times n^5 + \frac{2}{9} \times n^3 - \frac{1}{30} \times n = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} * - \frac{7n^5}{15} * + \frac{2n^3}{9} * - \frac{n}{30}$. Therefore the ſum of the eighth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} * - \frac{7n^5}{15} * + \frac{2n^3}{9} - \frac{n}{30}$. Q. E. I.

And, 9thly, When c is $=9$, and the ſum of the ninth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt, as in the laſt caſe, compute only the ſix firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1}\times n^{c+1}+\frac{1}{2}\times n^{c}+\frac{c}{2}\times An^{c-1}$

$$+\frac{c\times\overline{c-1}\times\overline{c-2}}{2\times3\times4}\times Bn^{c-3}$$

$$+\frac{c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\times\overline{c-4}}{2\times3\times4\times5}\times Cn^{c-5}$$

$+\frac{c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\times\overline{c-4}\times\overline{c-5}\times\overline{c-6}}{2\times3\times4\times5\times6}\times Dn^{c-7}$; becauſe the following terms involve the numbers E, F, G, &c, which relate to the ſums of the tenth, and the twelfth, and the fourteenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $=9$, the ſaid ſix terms $\frac{1}{c+1}\times n^{c+1}+\frac{1}{2}\times n^{c}+\frac{c}{2}\times An^{c-1}+\frac{c\times\overline{c-1}\times\overline{c-2}}{2\times3\times4}\times Bn^{c-3}$ $+\frac{c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\times\overline{c-4}}{2\times3\times4\times5\times6}\times Cn^{c-5}$ $+\frac{c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\times\overline{c-4}\times\overline{c-5}\times\overline{c-6}}{2\times3\times4\times5\times6\times7\times8}\times Dn^{c-7}$ will be $=$ $\frac{1}{9+1}\times n^{9+1}+\frac{1}{2}\times n^{9}+\frac{9}{2}\times An^{9-1}+\frac{9\times\overline{9-1}\times\overline{9-2}}{2\times3\times4}\times Bn^{9-3}+\frac{9\times\overline{9-1}\times\overline{9-2}\times\overline{9-3}\times\overline{9-4}}{2\times3\times4\times5\times6}\times Cn^{9-5}$ $+\frac{9\times\overline{9-1}\times\overline{9-2}\times\overline{9-3}\times\overline{9-4}\times\overline{9-5}\times\overline{9-6}}{2\times3\times4\times5\times6\times7\times8}\times Dn^{9-7}$ $(=\frac{1}{10}\times n^{10}+\frac{1}{2}\times n^{9}+\frac{9}{2}\times An^{8}+\frac{9\times8\times7}{2\times3\times4}\times Bn^{6}+\frac{9\times8\times7\times6\times5}{2\times3\times4\times5\times6}\times Cn^{4}+\frac{9\times8\times7\times6\times5\times4\times3}{2\times3\times4\times5\times6\times7\times8}\times Dn^{2}=\frac{1}{10}\times n^{10}$

$+ \frac{1}{2} \times n^9 + \frac{9}{2} \times An^8 + 7 \times 3 \times Bn^6 + 7 \times 3 \times Cn^4 + \frac{9}{2} \times Dn^2 = \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{9}{2} \times + \frac{1}{6} \times n^8 + 7 \times 3 \times - \frac{1}{30} \times n^6 + 7 \times 3 \times + \frac{1}{42} \times n^4 + \frac{9}{2} \times - \frac{1}{30} \times n^2 (= \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{3}{4} \times n^8 - \frac{7}{10} \times n^6 + \frac{1}{2} \times n^4 - \frac{3}{20} \times n^2) = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3n^2}{20}$. Therefore the ſum of the ninth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3nn}{20}$. Q. E. I.

Theſe ſeveral expreſſions of the values of the ſums of the firſt nine powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, are the ſame with thoſe ſet down above in the table in art. 141, page 188. And it is evident that this way of obtaining them, by means of the foregoing general ſeries, is much leſs laborious than the former method of obtaining them, ſet forth above in art. 136, 137, 138, and 139.

End of the applications of the foregoing general ſeries $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times n^{c-3}$ *+ &c, to the inveſtigation of the expreſſions ſet down above in art.* 141, *page* 188.

Of

Of certain ſerieſes conſiſting of numbers analogous to the figurate numbers.

147. Before we conclude this chapter, it will not be amiſs to ſhew how certain other ſerieſes, which bear a great reſemblance to the ſerieſes formed by the figurate numbers, may be reduced to, or compared with, the correſponding ſerieſes of thoſe numbers, and how their values, or the ſums of their terms, and likewiſe the values of their laſt terms, may be thereby determined. The ſerieſes I here ſpeak of, and which I call *analogous to the figurate numbers*, are ſuch as have the differences of their terms, or the differences of thoſe differences, or the differences of thoſe ſecond differences, or the differences of the differences of ſome remoter order, equal to each other, and which therefore are generated by the continual addition of a ſet of equal quantities. Let d, d, d, d, d, d, &c, be a ſet of equal quantities, by the continual addition of which to another quantity c we obtain the quantities c, $c+d$, $c+2d$, $c+3d$, $c+4d$, $c+5d$, &c. And let the terms of this ſecond ſeries c, $c+d$, $c+2d$, $c+3d$, $c+4d$, &c, be continually added to each other, and to a third quantity b, whereby we ſhall obtain a third ſeries of terms, which will be b, $b+c$, $b+2c+d$, $b+3c+3d$, $b+4c+6d$, $b+5c+10d$, &c; and let the terms of this third ſeries be continually added to each other, and to a fourth quantity a, whereby we ſhall obtain a fourth ſeries of terms, which will be a, $a+b$, $a+2b+c$, $a+3b+3c+d$, $a+4b+6c+4d$, $a+5b+10c+10d$, &c. And let the firſt ſeries d, d, d, d, d, d, &c, be called D; the ſecond ſeries c, $c+d$, $c+2d$, $c+3d$, $c+4d$, $c+5d$, &c, be called C; the third ſeries b, $b+c$, $b+2c+d$, $b+3c+3d$, $b+4c+6d$, $b+5c+10d$, &c, be called B; and the fourth, or laſt, ſeries a, $a+b$, $a+2b+c$, $a+3b+3c+d$, $a+4b+6c+4d$, and $a+5b+10c+10d$, &c, be called A. This laſt ſeries A (the firſt differences of

the terms of which conſtitute the terms of the third ſeries B, and the ſecond differences of the terms of which conſtitute the terms of the ſecond ſeries C, and the third differences of the terms of which conſtitute the terms of the firſt ſeries D, which conſiſts of the equal quantities d, d, d, d, d, d, &c), may, I think, with propriety be called a ſeries *analogous to the figurate numbers.* The generation of the terms of this ſeries will, perhaps, appear more clearly from the following table.

D	C	B	A
d	c	b	a
d	$c+d$	$b+c$	$a+b$
d	$c+2d$	$b+2c+d$	$a+2b+c$
d	$c+3d$	$b+3c+3d$	$a+3b+3c+d$
d	$c+4d$	$b+4c+6d$	$a+4b+6c+4d$
d	$c+5d$	$b+5c+10d$	$a+5b+10c+10d$

148. Now in the laſt ſeries A it is obvious that the co-efficients of the letters a, which are the firſt members of the ſeveral terms a, $a+b$, $a+2b+c$, $a+3b+3c+d$, $a+4b+6c+4d$, and $a+5b+10c+10d$, are a ſet of units, or the firſt order of the figurate numbers; and that the co-efficients of the ſeveral letters b in the ſecond members of the ſaid terms are the lateral, or natural, numbers 1, 2, 3, 4, 5, &c, or the ſecond order of the figurate numbers; and that the co-efficients of the ſeveral letters c in the third members of the ſaid terms are the trigonal, or triangular, numbers 1, 3, 6, 10, &c, or the third order of the figurate numbers; and that the co-efficients of the ſeveral letters d in the fourth members of the ſaid terms are the pyramidal numbers 1, 4, 10, &c, or the fourth order of the figurate numbers. And therefore, as we have above ſhewn how the ſums of the figurate numbers of the ſeveral ſucceſſive orders, and likewiſe the values of the laſt terms in them, may be determined, when the number of terms contained in them is known; it will be eaſy to find both the ſum of all the terms of the ſeries A, by multiplying the ſums of the ſucceſſive columns of figurate numbers, into the letters

a, *b*, *c*, and *d*, reſpectively, and adding the products ſo obtained into one ſum, and likewiſe to find the value of the laſt term of the ſaid ſeries, by multiplying the laſt terms of the ſeveral columns of co-efficients, or figurate numbers, into the letters *a*, *b*, *c*, and *d*, reſpectively, and adding the ſaid products into one ſum. For, if the number of terms in the ſeries A be denoted by the letter *n*, it follows from coroll. 2, art. 81, pages 109, 110, 111, that the ſum of the co-efficients of the letter *a* will be *n*; and the ſum of the co-efficients of the letter *b* will be $n \times \frac{n-1}{2}$; and the ſum of the co-efficients of *c* will be $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$; and the ſum of the co-efficients of *d* will be $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$; and conſequently the ſum of all the *n* terms of the ſaid ſeries A will be $= n \times a + \frac{n \times \overline{n-1}}{2} \times b + n \times \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3} \times c + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4} \times d$. And it follows from coroll. 4, art. 83, papes 112, 113, that the co-efficients of the letters *a*, *b*, *c*, and *d*, in the laſt, or *n*th, term of the ſeries A will be 1, $n-1$, $\frac{\overline{n-1} \times \overline{n-2}}{2}$, and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3}$, reſpectively; and conſequently that the ſaid laſt, or *n*th, term will be $= a + \overline{n-1} \times b + \frac{\overline{n-1} \times \overline{n-2}}{2} \times c + \frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3} \times d$.

Q. E. I *.

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* See upon this ſubject Mr. Thomas Simpſon's *Eſſays on ſeveral curious and uſeful ſubjects in ſpeculative and mixed mathematics*, publiſhed in the year 1740, pages 98, 99, 100, 101, 102, 103, 104, and 105; and likewiſe his Algebra, 6th Edition, publiſhed in the year 1790, Sections XIV and XV, pages 201, 202, &c. — — — 228.

End of the Tranſlation of the foregoing Extract from Mr. James Bernoulli's Treatiſe De Arte Conjectandi.

A
NEW AND GENERAL METHOD
OF FINDING THE
SUM OF ANY SERIES OF POWERS
OF A SET OF
QUANTITIES THAT ARE IN ARITHMETICAL PROGRESSION;

BEING THE TENTH OF THE LATE LEARNED

MR. THOMAS SIMPSON'S MATHEMATICAL ESSAYS,

PUBLISHED IN THE YEAR 1740.

Mr. SIMPSON's *Tenth Mathematical Eſſay*[*].

PROPOSITION.

To find the ſum of any Series of Powers whoſe roots are in arithmetical progreſſion, as $\overline{m+d}|^n + \overline{m+2d}|^n + \overline{m+3d}|^n \ldots\ldots\ldots x^n$, *m, d, and n, being any numbers whatſoever.*

LET $Ax^{n+1} + Bx^n + Cx^{n-1} + Dx^{n-2} + Ex^{n-3} + Fx^{n-4}$, &c. — K, if poſſible, be always equal to $\overline{m+d}|^n + \overline{m+2d}|^n \ldots\ldots x^n$, and A, B, C, &c, determinate quantities. Then, if any other number in the progreſſion $m+d$, $m+2d$, $m+3d \ldots\ldots x+d$, $x+2d$, $x+3d$, &c, as $x+d$, be ſubſtituted inſtead of x, the equality will ſtill continue; and we ſhall have

$A \times \overline{x+d}|^{n+1} + B \times \overline{x+d}|^n + C \times \overline{x+d}|^{n-1} + D \times \overline{x+d}|^{n-2}$ &c. — K equal $\overline{m+d}|^n + \overline{m+2d}|^n \ldots\ldots \overline{x+d}|^n$; from which if we take the former equation, there will remain $A \times \overline{\overline{x+a}|^{n+1} - x^{n+1}} + B \times \overline{\overline{x+d}|^n - x^n} + C \times \overline{\overline{x+d}|^{n-1} - x^{n-1}}$, &c. $= \overline{x+d}|^n$, ſhewing how much each ſide is increaſed by augmenting the number of terms in the given ſeries by unity; where, by tranſpoſing $\overline{x+d}|^n$, and throwing the ſeveral powers of $x+d$ into ſerieſes, we ſhall have

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[*] This Eſſay of Mr. Simpſon's is the part of his Eſſays alluded to in the Note at the bottom of page 213. As it is ſo nearly connected with the ſubject of the latter part of the foregoing Extract from Mr. James Bernoulli's Treatiſe *De Arte Conjectandi*, relating to the ſums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, and is not very long, I thought it would be agreeable to the Reader to ſee it here immediately after the ſaid Extract, and therefore I have cauſed it to be re-printed. F. M.

$$\left.\begin{array}{lllll}
-x^n & -nx^{n-1}d & -n\times\frac{n-1}{2}\times x^{n-2}d^2 & -n\times\frac{n-1}{2}\times\frac{n-2}{3}\,x^{n-3}d^3, & \&c. \\
\overline{n+1}\,Ax^nd + & n+1\times\frac{n}{2}\,Ax^{n-1}d^2 + & n+1\times\frac{n}{2}\times\frac{n-1}{3}\,Ax^{n-2}d^3 + & n+1\times\frac{n}{2}\times\frac{n-1}{3}\times\frac{n-2}{4}\,Ax^{n-3}d^4, & \&c. \\
* & +\,nBx^{n-1}d^2 + & n\times\frac{n-1}{2}\,Bx^{n-2}d^3 + & n\times\frac{n-1}{2}\times\frac{n-2}{3}\,Bx^{n-3}d^4, & \&c. \\
* & * & +\,n-1\times Cx^{n-2}d^3 + & n-1\times\frac{n-2}{2}\,Cx^{n-3}d^4, & \&c. \\
* & * & * & n-2\,Dx^{n-3}d^4, & \&c. \\
 & & & \&c. &
\end{array}\right\} = 0.$$

From which, by equating the homologous terms, A will come out $= \frac{1}{\overline{n+1} \times d}$, $B = \frac{1}{2}$, $C = \frac{nd}{3.4}$, $D = 0$, $E = -\frac{n \times \overline{n-1} \times \overline{n-2} \times d^3}{2.3.4.5.6}$, $F = 0$, $G = \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times d^5}{2.3.4.5.6.7.6}$ $H = 0$, &c. wherefore the values of A, B, C, &c, being ſo aſſigned, the whole expreſſion, or its equal $- \overline{x+d}|^{n}$ $+ A \times \overline{\overline{x+d}|^{n+1} - x^{n+1}} + B \times \overline{\overline{x+d}|^{n} - x^{n}}$, &c, muſt be equal 0, and conſequently $A \times \overline{\overline{x+d}|^{n+1} - x^{n+1}} + B \times \overline{\overline{x+d}|^{n} - x^{n}}$, &c, $= \overline{x+d}|$; that is, let x and n be what they will, the foreſaid increments of $A x^{n-1} + B x^{n} + C x^{n-1}$, &c, $-$ K and $\overline{m+d}|^{n} + \overline{m+2d}|^{n}$, &c. will, under the above aſſigned values of A, B, &c, be equal to one another: Therefore, if K be taken equal $A m^{n+1} + B m^{n} + C m^{n-1}$, &c, ſo that when x equal m, or the propoſed ſeries is equal to nothing, $A x^{n+1} + B x^{n}$, &c, $-$ K may be alſo $= 0$, it is manifeſt, that theſe two expreſſions, as they are increaſed alike, will, in all other circumſtances, be equal; that is, let x be what it will, $A x^{n+1} + B x^{n} + C x^{n-1} + D x^{n-2}$, &c, $- A m^{n+1} - B m^{n} - C m^{n-1} - D m^{n-2}$, &c, under the ſaid values of A, B, C, &c, will be always equal to $\overline{m+d}|^{n} + \overline{m+2d}|^{n} + \overline{m+3d}|^{n} \ldots x^{n}$; which values being therefore ſubſtituted, there will be $\frac{x^{n+1}}{\overline{n+1} \times d} + \frac{x^{n}}{2} + \frac{dnx^{n-1}}{3.4} - \frac{n \times \overline{n-1} \times \overline{n-2}\, d^3 x^{n-3}}{2.3.4.5.6}$ $+ \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2.3.4.5.6.7.6} d^5 x^{n-5}$

$$- \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5} \times \overline{n-6}}{2.3.4.5.6.7.8.5.6} d^7 x^{n-7}$$

$$+ \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5} \times \overline{n-6} \times \overline{n-7} \times \overline{n-8}}{2.3.4.5.6.7.8.9.11.12} d^9$$

$$x^{n-9}, \text{ \&c}, - \frac{m^{n+1}}{\overline{n+1} \times d} - \frac{m^n}{2} - \frac{ndm^{n-1}}{3.4} + \frac{n \times \overline{n-1} \times \overline{n-2}}{2.3.4.5.6} \times d^3$$

$$m^{n-3}, \text{ \&c}, = \overline{m+d}|^n + \overline{m+2d}|^n + \overline{m+3d}| \ldots\ldots x^n.$$

Q. E. I.

COROL. I.

Hence, if n be a whole poſitive number, and m be taken equal o; then all the terms in the ſecond ſeries $- \frac{m^{n+1}}{\overline{n+1} \times d} - \frac{m^n}{2} + \frac{ndm^{n-1}}{3.4}$, &c, vaniſhing when n is even, and all but that where the exponent of m is nothing, when odd, we ſhall, in this caſe, have $d^n + \overline{2d}|^n + \overline{3d}|^n + \overline{4d}|^n \ldots\ldots\ldots x^n$ barely equal to $\frac{x^{n+1}}{\overline{n+1} \times d} + \frac{x^n}{2} + \frac{ndx^{n-1}}{3.4} - \frac{n \times \overline{n-1} \times \overline{n-2}\, d^3 x^{n-3}}{2.3.4.5.6}$, &c, the firſt ſeries continued 'till it terminates, provided that the laſt term, when n is an odd number, be rejected,

COROL. II.

Wherefore, by taking d equal to 1, and n equal to 2, 3, 4, 5, &c, ſucceſſively, we have

$$
\begin{aligned}
&1+2+3+4+5\ldots+x &&= \frac{x^2}{2}+\frac{x}{2}\\
&1^2+2^2+3^2+4^2+5^2\ldots+x^2 &&= \frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{6}\\
&1^3+2^3+3^3+4^3+5^3\ldots+x^3 &&= \frac{x^4}{4}+\frac{x^3}{2}+\frac{x^2}{4}\\
&1^4+2^4+3^4+4^4+5^4\ldots+x^4 &&= \frac{x^5}{5}+\frac{x^4}{2}+\frac{x^3}{3}-\frac{x}{30}\\
&1^5+2^5+3^5+4^5+5^5\ldots+x^5 &&= \frac{x^6}{6}+\frac{x^5}{2}+\frac{5x^4}{12}-\frac{x^2}{12}\\
&1^6+2^6+3^6+4^6+5^6\ldots+x^6 &&= \frac{x^7}{7}+\frac{x^6}{2}+\frac{x^5}{2}-\frac{x^3}{6}+\frac{x}{42}\\
&\quad\&c. &&\quad\&c,
\end{aligned}
$$

C O R O L. III.

Moreover, if d be taken equal to 1, and m equal to 1, our general equation will become $2^n+3^n+4^n\ldots\ldots+x^n=\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+\frac{nx^{n-1}}{3.4}$, &c, $-\frac{1}{n+1}-\frac{1}{2}-\frac{n}{3.4}+\frac{n\times n-1\times n-2}{2.3.4.5.6}$, &c, each ſide of which being increaſed by unity, and the whole multiplied by d^n, gives $d^n+\overline{2d}|^n+\overline{3d}|^n+\overline{4d}|^n\ldots\ldots+\overline{dx}|^n=d^n$ into $\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+\frac{nx^{n-1}}{3.4}-\frac{n\times n-1\times n-2\,x^{n-3}}{2.3.4.5.6}$, &c, $-\frac{1}{n+1}+\frac{1}{2}-\frac{n}{3.4}+\frac{n\times n-1\times n-2}{2.3.4.5.6}$, &c.

EXAMPLE

EXAMPLE I.

Let it be required to find the ſum of a ſeries, conſiſting of 100 cube numbers, whoſe roots are, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, &c.

Here d, the common difference of the roots, being equal $\frac{1}{2}$, $n = 3$, and $x = 0$, let theſe values be ſubſtituted in the equation in Cor. II. and it will become ($\left.\overline{\frac{1}{2}}\right|^3$ *in*, $\frac{\left.\overline{100}\right|^4}{4} + \frac{\left.\overline{100}\right|^3}{2} + \frac{\left.\overline{100}\right|^2}{4} =$) 3187812.5, the number that was to be found.

EXAMPLE II.

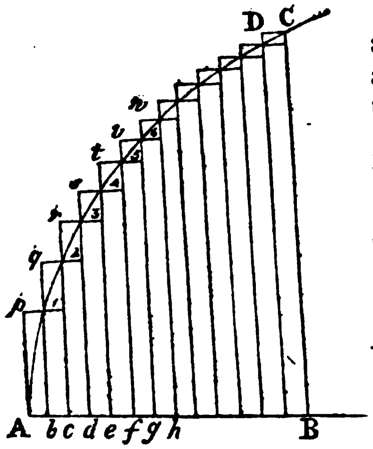
Let $n = \frac{1}{2}$, $d = \frac{1}{4}$. Then the equation in the laſt Corollary will become $\left.\overline{\frac{1}{4}}\right|^{\frac{1}{2}} + \left.\overline{\frac{2}{4}}\right|^{\frac{1}{2}} + \left.\overline{\frac{3}{4}}\right|^{\frac{1}{2}} \ldots + \left.\overline{\frac{x}{4}}\right|^{\frac{1}{2}} = \overline{\left.\overline{\frac{1}{4}}\right|^{\frac{1}{2}} \times \frac{2 \times x^{\frac{3}{2}}}{3} + \frac{x^{\frac{1}{2}}}{2} + \frac{1}{24 \times x^{\frac{1}{2}}}}$, &c, $- \frac{339}{1920}$ very nearly; ſo that, taking x equal 4, it will be $\left.\overline{\frac{1}{4}}\right|^{\frac{1}{2}} + \left.\overline{\frac{2}{4}}\right|^{\frac{1}{2}} + \left.\overline{\frac{3}{4}}\right|^{\frac{1}{2}} + 1 = 3.0731$; which differs from the true value by leſs than $\frac{1}{10000}$; and if more terms had been uſed, the anſwer would ſtill have been more exact; but never can come *accurately* true, when n is negative or a fraction, becauſe then both ſerieſes run on *ad infinitum*.

SCHOLIUM.

SCHOLIUM.

The theorems, above found, are not only uſeful in finding the ſum of a Series of Powers, but may be of ſervice alſo in the Quadrature of Curves, &c, eſpecially as the concluſions will be accurately true, and the reaſoning thereupon ſcientific.

This I ſhall endeavour to ſhew by the following inſtance; wherein A C, being ſuppoſed a curve, whoſe equation is $y = z^n$ (A B being equal z, and C B equal y) the area A B C is required.



Let A B be divided into any number, x, of equal parts, as Ab, bc, cd, &c, and from the points of diviſion let perpendiculars be raiſed, cutting the curve in the points, 1, 2, 3, &c, and having made $p1$, $q2$, $r3$, $s4$, &c, parallel to A B, let the baſe Ab, bc, cd, &c, of each of the rectangles pb, qc, rd, &c, be repreſented by d: Then $b1$, $c2$, $d3$, &c, the heights of thoſe rectangles, being ordinates to the curve, will be d^n, $\overline{2d}|^n$, $\overline{3d}|^n$, &c, reſpectively, each of which ∵ being multiplied by d, the common baſe, and the ſum of all the products taken, will give d into $d^n + \overline{2d}|^n + \overline{3d}|^n \ldots \overline{xa}|^n$, (= A$p1q2r$, &c, C B A) for the area of the whole circumſcribing polygon; and this ſeries, according to the above ſaid Theorem (Cor. III.) is equal to d^{n+1} in, $\frac{nx^{n+1}}{n+1} + \frac{x^n}{2}$, &c, $= \frac{\overline{dx}|^{n+1}}{n+1}$

+

$+ \frac{d\times\overline{dx}|^{n}}{2}$, &c, or, becauſe $dx = z$, it will be $= \frac{z^{n+1}}{n+1} + \frac{dz^{n}}{2}$, &c. Now, if from this the difference of the inſcribed and circumſcribed polygons, or the rectangle BD $= dz^{n}$ be taken, there will remain $\frac{z^{n+1}}{n+1} - \frac{dz^{n}}{2}$, for the area of the inſcribed polygon. Hence, it is manifeſt, that, let d be what it will, the inſcribed polygon can never be ſo great, nor the circumſcribed ſo ſmall, as $\frac{z^{n+1}}{n+1}$ $(= \frac{AB \times BC}{n+1})$: And therefore this expreſſion muſt be *accurately* equal to the required curvilinear area ACB.

A CON-

AN
INVESTIGATION
AND
DEMONSTRATION
OF
SIR ISAAC NEWTON'S BINOMIAL THEOREM,
IN THE CASE OF
INTEGRAL AND AFFIRMATIVE POWERS;
IN WHICH

The Law of the generation of the numeral co-efficients of the Series which is equal to the quantity $\overline{a+b}|^{m}$, is diſcovered by a conjecture grounded on the obſervation of ſome particular inſtances; but, when ſo diſcovered, is ſhewn to be true univerſally in all other Integral and Affirmative Powers whatſoever, by a ſtrict and accurate Demonſtration.

A General Statement, or Expreſſion, of the Binomial Theorem.

Art. 1. SIR ISAAC NEWTON's Binomial Theorem is a Propoſition affirming that, if m be any number whatſoever, either integral or fractional, affirmative or negative, the quantity $\overline{a+b}|^{m}$, or the mth power of the binomial quantity $a+b$, will be equal to the ſeries $a^{m} + \frac{m}{1} a^{m-1}b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2}b^{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m-3}{4} a^{m-4}b^{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5}b^{5}$ + &c, or (if we put A for 1, or the co-efficient of the firſt term a^{m}, and B for $\frac{m}{1}$, or the co-efficient of the ſecond term $\frac{m}{1} \times a^{m-1}b$, and C for $\frac{m}{1} \times \frac{m-1}{2}$, or the co-efficient of the third term $\frac{m}{1} \times \frac{m-1}{2} \times a^{m-2}b^{2}$, and D, E, F, G, H, I, K, &c, for the co-efficients of the fourth, fifth, ſixth, ſeventh, eighth, ninth, tenth, and other following terms, reſpectively), to the ſeries $a^{m} + \frac{m}{1} Aa^{m-1}b + \frac{m-1}{2} Ba^{m-2}b^{2} + \frac{m-2}{3} Ca^{m-3}b^{3} + \frac{m-3}{4} Da^{m-4}b^{4} + \frac{m-4}{5} Ea^{m-5}b^{5} + \frac{m-5}{6} Fa^{m-6}b^{6}$

$+ \frac{m-6}{7} Ga^{m-7}b^{7} + \frac{m-7}{8} Ha^{m-8}b^{8} + \frac{m-8}{9} Ia^{m-9}b^{9} +$ &c, in which ſeries the powers of a, to wit, a^{m}, a^{m-1}, a^{m-2}, a^{m-3}, a^{m-4}, a^{m-5}, a^{m-6}, a^{m-7}, a^{m-8}, a^{m-9}, &c, are produced from each other by a continual diviſion by a, and the powers of b, to wit, b, b^{2}, b^{3}, b^{4}, b^{5}, b^{6}, b^{7}, b^{8}, b^{9}, &c, are produced from each other by a continual multiplication by b, and the numeral co-efficients B, C, D, E, F, G, H, I, K, &c, of the ſecond, third, fourth, fifth, ſixth, ſeventh, eighth, ninth, tenth, and other following terms are derived, or generated, from 1, or A, the numeral co-efficient of the firſt term a^{m}, by a continual multiplication of it into the fractions $\frac{m}{1}$, or $\frac{m-0}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, $\frac{m-8}{9}$, &c, continued *ad infinitum*, or to the end of the ſeries when the number of its terms is finite.

Of the Invention of the ſaid Theorem.

2. Sir Iſaac Newton was the firſt perſon that expreſſed this important Theorem in the foregoing ſhort and convenient Algebräick notation, and likewiſe the firſt perſon that diſcovered that it would be true, not only when the index m of the power to which the binomial quantity is to be raiſed is a whole number, as 2, 3, 4, 5, 6, &c, but alſo when it is a fraction of any kind, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{17}$, or $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{17}$, or

$\frac{3}{2}$, $\frac{5}{3}$, $\frac{17}{4}$, &c, or even a negative quantity, as -2, -3, -5, -17, or $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{17}$, or $-\frac{2}{3}$, $-\frac{3}{5}$, $-\frac{4}{17}$, or $-\frac{3}{2}$, $-\frac{5}{3}$, $-\frac{17}{4}$, &c. But he was not the firſt perſon that diſcovered it to be true in the firſt, or ſimpleſt, caſe, or when the index m is equal to an integral and affirmative number. For in that caſe it was known to Mr. Henry Briggs, the celebrated improver and computer of Logarithms, above 40 years before it was diſcovered by Sir Iſaac Newton; and it was publiſhed by Mr. Briggs, in his learned Treatiſe on Logarithms, intitled, *Arithmetica Logarithmica*, in the year 1624; as has been clearly ſhewn by the learned Dr. Hutton, of Woolwich Academy, in his very curious, hiſtorical, Introduction to the new edition of Sherwin's Mathematical Tables, publiſhed in the year 1784.

3. But, though Mr. Briggs had publiſhed this famous Theorem, in this firſt caſe of it, in his *Arithmetica Logarithmica*, in the year 1624, yet it ſeems to have been but little known to Mathematicians till about 60 years after. For even the famous Dr. John Wallis, of Oxford, (who was a very extenſive reader of Mathematical Works, as well as a great improver of the Science,) appears to have been ignorant of it till a little before the year 1685, in which he publiſhed his learned, hiſtorical, Treatiſe of Algebra, at which time he was about 69 years old. For he there tells us, in page 319, that he had formerly ſought to diſcover the law by which the numeral co-efficients of the terms of the ſeries which is equal to $\overline{a+b}|^m$ are generated from each other, but had not been able to find it; and that he had lately been made acquainted with it by the peruſal of a very learned letter of Mr. Iſaac Newton, the Profeſſor of Mathematicks in the Univerſity of Cambridge, to Mr. Oldenburgh, Secretary to the Royal Society, written in the year 1676. His words are theſe, after ſpeaking of ſome other excellent inventions in the Mathematicks contained in the ſaid letter—

“ *He* [Mr. Newton] *then obſerves (what I had formerly* “ *ſought after, but unſucceſsfully), that the following numbers* “ *are, from the two firſt, to be found by continual multiplication* “ *of this ſeries* $1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ “ *&c.*” From this paſſage of Dr. Wallis's Algebra, I am inclined to think that this famous theorem was never generally known to Mathematicians till this publication of it in that work. And from its having thus been communicated to the learned world as a diſcovery of Mr. Newton (who was afterwards better known by the title of Sir Iſaac Newton), it has uſually been called *his* Theorem.

4. This Theorem had been diſcovered by Sir Iſaac Newton about the year 1665, as appears from his letters to Mr. Oldenburgh in the year 1676, copies of which were ſent to Mr. Leibnitz by Mr. Newton's direction. But theſe letters do not appear to have been known to the Mathematical world in general, till the year 1712, when they were printed in the *Commercium Epiſtolicum* by the order of the Royal Society. And no part of them ſeems to have been publiſhed before the year 1685, when the foregoing account of the generation of the numeral co-efficients of the terms of the ſeries that is equal to the *m*th power of a binomial quantity, and a few more curious diſcoveries contained in them, were inſerted by Dr. Wallis, in his Treatiſe on Algebra.

5. It has been obſerved above, that Mr. Briggs, and not Sir Iſaac Newton, was *the firſt inventor* of this Theorem in the firſt and ſimpleſt caſe of it, or when the index *m* is an affirmative whole number. Yet I am inclined to think that Sir Iſaac Newton was likewiſe *an inventor* of it even in that caſe, though *not the firſt inventor*. For it is well known that he was not an extenſive reader of Mathematical Works; and he appears to have applied himſelf principally in his younger years to the ſtudy of Des Cartes's Geometry, with Schooten's Commentary on it, and the other Tracts publiſhed by Schooten with it, and of Dr. Wallis's *Arithmetica Infinitorum*, and his other works on mathematical ſubjects

then publiſhed; in none of which books is any mention made of this uſeful Theorem that had been diſcovered ſo many years before by Mr. Briggs. And, as theſe were the books to which Mr. Newton is known to have given the greateſt part of his attention at that time, he may very well be ſuppoſed not to have ſeen Mr. Briggs's *Arithmetica Logarithmica*, in which this Theorem is contained, at the time of his diſcovering it himſelf, which was about the year 1665, or when he was only 23 years old. And, if he had ſeen that book, and obſerved this diſcovery to be contained in it, I can hardly conceive that, when he was ſpeaking of this Theorem, and ſetting forth its great utility in mathematical inveſtigations, he would have omitted to make mention of the name of Mr. Briggs, and to acknowledge that what he had delivered upon the ſubject in his *Arithmetica Logarithmica* contained the ſubſtance of the ſaid Binomial Theorem in the caſe of Integral Powers, though not expreſſed in Algebräick Symbols. For theſe reaſons I am inclined to think that Sir Iſaac Newton had not ſeen Mr. Briggs's *Arithmetica Logarithmica* when he invented the Binomial Theorem, and conſequently that he was truly *an inventor* of it even in the caſe of Integral Powers, though *not the firſt inventor.*

6. But it ſeems more ſurpriſing that Dr. Wallis, who was a much more copious reader of Mathematical Works than Sir Iſaac Newton, and who actually had ſeen and read Mr. Briggs's *Arithmetica Logarithmica*, and makes mention of it in his Algebra, chapter XII, page 60, ſhould not have attended to the contents of that ingenious Treatiſe enough to have obſerved that it contained this moſt uſeful Theorem. Yet this appears to have been the fact, from what the Doctor tells us in the 85th chapter of his Algebra, page 319, in the paſſage that has been already cited in art. 3, where he mentions the law of the generation of the co-efficients of the terms of the ſeries that is equal to the *m*th power of a binomial quantity, as a diſcovery that had recently come to his knowledge by the peruſal of Mr. Newton's letter to Mr. Oldenburgh. For, ſurely, it muſt be concluded from this declaration, that, though he had ſeen Mr. Briggs's *Arithmetica*

metica Logarithmica, he had not read it with ſufficient attention to diſcover that this method of generating the co-efficients of the terms of the ſeries that is equal to the *m*th power of a binomial quantity, when *m* was equal to any whole number whatſoever, was contained in it: though it ſeems indeed unaccountably ſtrange that he ſhould not have taken notice of it.

7. We may therefore, upon the whole matter, conſider the Binomial Theorem, in the caſe of Integral and Affirmative Powers, as having been firſt invented by Mr. Briggs before the year 1624, and publiſhed by him in that year in his *Arithmetica Logarithmica*, but in ſuch a manner, and in ſuch expreſſions, as did not much engage the attention of Mathematicians towards it; ſince it does not ſeem to have been generally known amongſt Mathematicians till it was afterwards publiſhed in the year 1685, as an invention of Mr. Iſaac Newton, by Dr. Wallis in his Algebra. And we may conſider it as having been invented a ſecond time by Mr. Newton about the year 1665, and extended by him at the ſame time to the other caſes of Fractional and Negative Powers, and alſo expreſſed in the very ſhort and convenient Algebräick notation, in which it is ſet forth above in art. 1, and which has contributed ſo much to give it currency amongſt Mathematicians. And, laſtly, we may conſider it as having been communicated by Mr. Newton to Mr. Oldenburgh and Mr Leibnitz, and probably alſo to his friend and patron Dr. Iſaac Barrow, the Maſter of Trinity College, Cambridge, and a few more of his Mathematical friends, in the year 1676, in the letter above-mentioned; and as having afterwards been communicated to the world at large in the aforeſaid extracts from the ſaid letter to Mr. Oldenburgh, which Dr. Wallis publiſhed in his Algebra in the year 1685.

Of Mr. James Bernoulli's demonſtration of the ſaid Theorem, in his Treatiſe on the Doctrine of Chances, intitled, De Arte Conjectandi.

8, But, by what ſteps, or what train of reaſoning, Sir Iſaac Newton diſcovered this law of the ſaid co-efficients to be ſuch as he deſcribed it, is not known; nor is any demonſtration of it, even in the eaſieſt caſe of it (or when the index *m* of the power to which the binomial quantity is to be raiſed, is a whole number), any where to be found in all his works. Nor has Dr. Wallis attempted to ſupply this defect, nor, as I believe, any other mathematical author whatſoever in the laſt century, from the year 1685 (when the Theorem was firſt publiſhed by Dr. Wallis) to the end of it; nor do I know of any demonſtration of it given in the beginning of the preſent century before the year 1713, when the learned and ſagacious Mr. James Bernoulli's excellent Treatiſe on the Doctrine of Chances, intitled, *De Arte Conjectandi*, was publiſhed at Baſil, or Baſle, in Switzerland. But there we find an excellent demonſtration of it, in the caſe of Integral Powers, derived from the doctrine of Permutations and Combinations, and the properties of the Figurate numbers, which are the true principles to which it ought to be referred. This demonſtration is contained in the 3d chapter of the ſecond part of that valuable Treatiſe, and may be perfectly underſtood by a careful peruſal of the three firſt chapters of that ſecond part, without the help of the firſt part of the Treatiſe. For the doctrine of Permutations and Combinations is explained from its firſt principles in the two firſt chapters of that ſecond part of the Treatiſe, without any reference to the firſt part; and the properties of the Figurate numbers are derived from that doctrine in a moſt ſtrict and ſcientific manner, in the third chapter of the ſame ſecond part; and amongſt theſe properties of the Figurate numbers, ſo derived, is the Binomial Theorem, in the caſe of Integral and Affirmative Powers, or the law of the generation of the numeral co-efficients of the terms of a ſeries that is equal to any integral and affirmative power of the binomial quantity $a + b$. This demonſtration therefore deſerves to be generally known and ſtudied by Mathematicians, as the true foundation of this celebrated and moſt uſeful Theorem. And upon that account I have re-printed the ſaid three firſt chapters of the ſecond Part of that excellent Treatiſe *De Arte Conjectandi*, in the foregoing part of this volume, in the author's original Latin text, with ſome explanatory notes on a few of the moſt difficult paſſages of it, and have afterwards added a very full Tranſlation of the ſame three chapters, with ſome examples and additions of my own, which I thought might be uſeful to my readers, and which I have taken care to diſtinguiſh from the other parts which are tranſlated from the Author's text. And I hope that, by thus exhibiting this part of that excellent work in an Engliſh dreſs, and removing the difficulties that occur in the original, in conſequence of the Author's extreme conciſeneſs, I ſhall induce the young Students of the Mathematicks in England, to make themſelves acquainted with this maſterly and ſcientific demonſtration of this moſt important Theorem, which ſeems hitherto to have been adopted by too many Mathematicians, upon the mere ground of induction, and the experience of its truth in the ſeveral trials they have made of it, without endeavouring to find a demonſtration of it.

Another demonſtration of it, in the ſame caſe of Integral Powers, will be given in the enſuing part of this Diſcourſe.

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9. But, though the demonſtration of this propoſition given by Mr. James Bernoulli in this excellent Treatiſe, *De Arte Conjectandi*, Part 2d, Chapter 3d, (and which may be ſeen above in this volume in the original Latin text of Mr. James Bernoulli in page 28, and in my Tranſlation of it in pages 115 and 116), is the firſt, and, in my opinion, the beſt that has yet been given of it, yet I doubt not that the Mathematical Reader will be pleaſed to ſee another demonſtration of it, that is ſomewhat ſhorter than Mr. Bernoulli's (inaſmuch as it does not require the previous knowledge of the Doctrine of Permutations and Combinations, and the properties of the Figurate Numbers), and yet is equally accurate and concluſive. Such a demonſtration I ſhall therefore now endeavour to lay before him in the remaining part of this Diſcourſe.

10. Now in order to diſcover the general relation of the terms of the ſeries that is equal to $\overline{a+b}|^{m}$ to each other, when m denotes any whole number whatſoever, it will be proper in the firſt place to examine their relation to each other when m is equal to ſome particular whole numbers, and thoſe not large ones, that they may be more eaſily managed and their properties more readily ſeen into. And, if, when we have examined theſe particular ſerieſes that are equal to certain particular values of $\overline{a+b}|^{m}$, when m is equal to certain ſmall whole numbers, we can find any common properties that belong equally to all of them, and can alſo perceive that the ſame properties muſt likewiſe belong to all the ſerieſes that ſhall be equal to any other values of $\overline{a+b}|^{m}$, as well as to thoſe which we have conſidered; or, if we cannot immediately perceive this to be the caſe, but can find ſome method of demonſtrating that it is ſo; we ſhall then arrive at the

knowledge of the general relation of the terms of the ſeries that is equal to $\overline{a+b}|^{m}$, to each other, which is the object of our purſuit. We will therefore raiſe the binomial quantity $a+b$ to its ſquare, and cube, and fourth power, and fifth power; and ſixth power, by multiplication; which may be done in the manner following.

$$
\begin{array}{l}
1 = \overline{a+b}|^{0}. \\
a+b \\
\hline
a+b = \overline{a+b}|^{1}. \\
a+b \\
\hline
aa+ab \\
\quad +ab+bb \\
\hline
aa+2ab+bb = \overline{a+b}|^{2}. \\
\qquad a+b \\
\hline
a^{3}+2a^{2}b+ab^{2} \\
\quad +a^{2}b+2ab^{2}+b^{3} \\
\hline
a^{3}+3a^{2}b+3ab^{2}+b^{3} = \overline{a+b}|^{3}. \\
\qquad\qquad a+b \\
\hline
a^{4}+3a^{3}b+3a^{2}b^{2}+ab^{3} \\
\quad +a^{3}b+3a^{2}b^{2}+3ab^{3}+b^{4} \\
\hline
a^{4}+4a^{3}b+6a^{2}b^{2}+4ab^{3}+b^{4} = \overline{a+b}|^{4}. \\
\qquad\qquad\qquad a+b \\
\hline
a^{5}+4a^{4}b+6a^{3}b^{2}+4a^{2}b^{3}+ab^{4} \\
\quad +a^{4}b+4a^{3}b^{2}+6a^{2}b^{3}+4ab^{4}+b^{5}. \\
\hline
a^{5}+5a^{4}b+10a^{3}b^{2}+10a^{2}b^{3}+5ab^{4}+b^{5} = \overline{a+b}|^{5}. \\
\qquad\qquad\qquad\qquad a+b \\
\hline
a^{6}+5a^{5}b+10a^{4}b^{2}+10a^{3}b^{3}+5a^{2}b^{4}+ab^{5} \\
\quad +a^{5}b+5a^{4}b^{2}+10a^{3}b^{3}+10a^{2}b^{4}+5ab^{5}+b^{6} \\
\hline
a^{6}+6a^{5}b+15a^{4}b^{2}+20a^{3}b^{3}+15a^{2}b^{4}+6ab^{5}+b^{6} = \overline{a+b}|^{6}.
\end{array}
$$

Obſerve.

Obſervations on the terms of the foregoing Serieſes that are equal to $\overline{a+b}|^{1}$, $\overline{a+b}|^{2}$, $\overline{a+b}|^{3}$, $\overline{a+b}|^{4}$, $\overline{a+b}|^{5}$, and $\overline{a+b}|^{6}$, explaining the Compoſition of the Literal parts of the ſaid terms.

11. If we examine the compoſition of the foregoing products, or ſerieſes, which are obtained by continual multiplications by the binomial quantity $a+b$, the firſt obſervation that will occur to us will be, that the firſt term of the ſeries $aa+2ab+bb$, which is equal to the ſquare of $a+b$, is aa or a^2; and that the firſt term of the ſeries $a^3+3a^2b+3ab^2+b^3$, which is equal to the cube of $a+b$, is a^3; and that the firſt term of the ſeries $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, which is equal to the fourth power of $a+b$, is a^4; and that the firſt term of the ſeries $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, which is equal to the fifth power of $a+b$, is a^5; and that the firſt term of the ſeries $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$, is a^6; or that the firſt term of the ſeries that is equal to any one of the ſaid five powers of the binomial quantity $a+b$, is the ſame power of the ſingle quantity a.

And it is eaſy to ſee that, if we were to continue theſe multiplications by the binomial quantity $a+b$ ever ſo far, the ſame thing would take place with reſpect to the firſt terms of the following products, or ſerieſes, which are equal to any higher powers of the binomial quantity $a+b$, let their number be ever ſo great; or that, if the letter m be any number, how great ſoever, the firſt term of the product, or ſeries, that is equal to $\overline{a+b}|^{m}$, or the mth power of the binomial quantity $a+b$, will be a^m, or the ſame power of the ſingle quantity a.

For, as the firſt term of every new product, or ſeries, is produced by the multiplication of the firſt term of the next preceeding product, or ſeries, by a, or $1 \times a$, the co-efficient of the firſt term of the new ſeries, which is the product of the ſaid multiplication, muſt be the ſame with the co-efficient of the firſt term of the next preceeding ſeries, which is the multiplicand of the ſaid multiplication. And conſequently, as the co-efficient of the ſaid multiplicand, or firſt term of the preceeding ſeries is originally 1, (namely, when $a+b$ is multiplied into $a+b$, in order to produce the ſeries $aa+2ab+bb$, which is equal to its ſquare), the co-efficients of the firſt terms of all the following products, or ſerieſes, which are equal to $\overline{a+b}|^{2}$, $\overline{a+b}|^{3}$, $\overline{a+b}|^{4}$, $\overline{a+b}|^{5}$, $\overline{a+b}|^{6}$, $\overline{a+b}|^{7}$, $\overline{a+b}|^{8}$, $\overline{a+b}|^{9}$, $\overline{a+b}|^{10}$, &c, *ad infinitum*, muſt likewiſe all be equal to 1. Q. E. D.

12. The ſecond obſervation that will occur to us, is, that the indexes of the ſeveral ſucceſſive powers of a in the terms of every product, or ſeries, that is equal to any power of the binomial quantity $a+b$, decreaſe continually by an unit, and that the indexes of the powers of b in the terms of the ſaid products, or ſerieſes, increaſe by an unit at the ſame time. Thus, in the ſeries $a^2+2ab+b^2$, or (as it is ſometimes expreſſed in Sir Iſaac Newton's Notation of Indexes, becauſe a^0 is $= 1$, and b^0 is likewiſe $= 1$), $a^2b^0 + 2a^1b^1+a^0b^2$, which is equal to the ſquare of the binomial quantity $a+b$, the indexes of the powers of a are 2, 1, and 0, or 2, $2-1$, and $2-2$, and the indexes of the powers of b are 0, 1, and 2, or 0, $0+1$, and $0+2$; and in the ſeries $a^3+3a^2b+3ab^2+b^3$, or $a^3b^0+3a^2b^1+3a^1b^2+a^0b^3$, which is equal to the cube of the binomial quantity $a+b$, the indexes of the powers of a are 3, 2, 1, and 0, or 3, $3-1$, $3-2$, and $3-3$, and the indexes of the powers of b are 0, 1, 2, and 3. And the ſame thing takes place in the following products, or ſerieſes, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, and $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, and $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, which are equal to the fourth, fifth, and ſixth powers of $a+b$. And it is eaſy to ſee that the ſame thing will likewiſe take place in the terms of the products, or ſerieſes, that are equal

to any higher powers of $a+b$ whatſoever, if the ſaid multiplications by $a+b$ were to be continued till the ſerieſes equal to ſuch higher powers were produced. Therefore the literal parts of the ſecond term, and of all the following terms of each of the ſaid products, or ſerieſes, may always be generated, or derived, from the firſt term of it, by the continual multiplication of it by the fraction $\frac{b}{a}$.

13. But, by the firſt obſervation, the firſt term of the ſeries which is equal to $\overline{a+b}|^{m}$, or the mth power of the binomial quantity $a+b$, when m is any whole number whatſoever, is am.

It follows therefore, in the 3d place, that the literal parts of the terms of the ſeries that is equal to $\overline{a+b}|^{m}$, will be a^{m}, $a^{m-1}b$, $a^{m-2}b^{2}$, $a^{m-3}b^{3}$, $a^{m-4}b^{4}$, $a^{m-5}b^{5}$, $a^{m-6}b^{6}$, $a^{m-7}b^{7}$, $a^{m-8}b^{8}$, $a^{m-9}b^{9}$, &c, till we come to the quantity $a^{m-m}\,b^{m}$, (or $a^{0}\,b^{m}$, or $1\times b^{m}$), or b^{m}.

And thus we have diſcovered the compoſition of the literal parts of the terms of the ſeries which is equal to $\overline{a+b}|^{m}$, as fully as can be deſired. And we have likewiſe diſcovered that the co-efficient of the firſt term, of which the literal part is a^{m}, is always 1, or that the firſt term of the ſaid ſeries is a^{m} itſelf, and not any multiple of a^{m}. Q. E. I.

14. In the 4th place it is evident that all the terms of every product, or ſeries, ariſing from the multiplication of the binomial quantity $a+b$ into itſelf, muſt be connected together by the ſign +, or added to each other. And conſequently the literal parts of the ſeries that is equal to $\overline{a+b}|^{m}$,

will be $a^m + a^{m-1}b + a^{m-2}b^2 + a^{m-3}b^3 + a^{m-4}b^4 + a^{m-5}b^5 + a^{m-6}b^6 + a^{m-7}b^7 + a^{m-8}b^8 + a^{m-9}b^9$ + &c. $+ b^m$.

Of the numeral co-efficients of the ſecond and other following terms of the product, or ſeries, that is equal to $\overline{a+b}|^m$.

15. We come now to inquire into the numeral co-efficients of the ſecond and other following terms of the product, or ſeries, which is equal to $\overline{a+b}|^m$, or the mth power of the binomial quantity $a+b$.

Now the numeral co-efficient of the ſecond term of this ſeries will always be equal to m, or the index of the power to which the binomial quantity $a+b$ is to be raiſed. This may be demonſtrated in the manner following.

In raiſing the ſeveral powers of the binomial quantity $a+b$ by the continual multiplication of that quantity into itſelf, in the manner above exemplified in art. 10, it is evident that the ſaid ſecond term of every new product, or ſeries, that is equal to a new power of $a+b$, is always produced by adding the product of the multiplication of the firſt term of the ſeries that is equal to the next lower power of $a+b$ (of which firſt term we have ſeen that 1 is always the co-efficient) by b to the product of the multiplication of the ſecond term of the ſaid foregoing ſeries by a; the effect of which addition is, to increaſe the co-efficient of the ſecond term of the new ſeries by an unit, or ſo as to make it exceed the co-efficient of the ſecond term of the foregoing ſeries by an unit. Thus, the ſecond term, $2ab$, of the ſeries

a^2+

$2ab + b^2$, which is equal to the ſquare of $a+b$, is produced by the addition of the product ba, or ab, (which ariſes from the multiplication of a, the firſt term of the former ſeries $a+b$, by b,) to the product ab, which ariſes from the multiplication of the ſecond term b of the former ſeries $a+b$ by a; the effect of which addition is, to make the co efficient, 2, of the ſecond term $2ab$ in the new ſeries, exceed the co-efficient, 1, of the ſecond term b of the former ſeries, by an unit. And, in like manner, the ſecond term, $3a^2b$, of the ſeries $a^3 + 3a^2b + 3ab^2 + b^3$, which is equal to the cube of the binomial quantity $a + b$, is produced by the addition of the product a^2b (which ariſes from the multiplication of a^2, the firſt term of the former ſeries $a^2 + 2ab + b^2$, by b), to the product $2a^2b$, which ariſes from the multiplication of the ſecond term $2ab$ of the ſaid former ſeries $a^2 + 2ab + b^2$ by a; the effect of which addition is, to make the co-efficient, 3, of the ſecond term $3a^2b$ of the new ſeries, exceed the co-efficient, 2, of the ſecond term $2ab$ of the former ſeries, by an unit. And, in like manner, $4a^3b$, the ſecond term of the next ſeries, is $= a^3 \times b + 3a^2b \times a$, or $a^3b + 3a^3b = \overline{1 + 3|} \times a^3b$; and $5a^4b$, the ſecond term of the next ſeries, is $= a^4 \times b + 4a^3b \times a = a^4b + 4a^4b = \overline{1 + 4|} \times a^4b$; and $6a^5b$, the ſecond term of the next ſeries, is $= a^5 \times b + 5a^4b \times a = a^5b + 5a^5b = \overline{1+5|} \times a^5b$. And this, it is eaſy to ſee, muſt be the caſe in any higher powers whatſoever of $a+b$, if we were to continue the multiplications by $a+b$ till the ſerieſes that were equal to ſuch higher powers of $a+b$ were produced. And conſequently, ſince in the firſt power of the binomial quantity $a+b$, to wit, in the ſaid quantity itſelf, the co-efficient of the ſecond term b, to wit, 1, is equal to the index of the ſaid firſt power, which is alſo 1, and in the ſecond, and third, and fourth, and fifth, and ſixth powers of the ſaid binomial quantity, the co-efficient of the ſecond term of the ſeries that is equal to each of the ſaid powers of $a+b$ is alſo equal to the index of the ſaid powers; it follows that in all higher powers whatſoever of the ſaid binomial quantity $a+b$, the co-efficient of the ſecond term of the ſeries which is equal to every ſuch power will be equal to the index of the ſaid power; or, in other words, the co-efficient of the ſecond term of the ſeries which is equal to $\overline{a+b}\rvert^{m}$ will always be equal to the index m.

Q. E. D.

16. From what has been ſhewn in the foregoing articles we may conclude with certainty, that the two firſt terms of the ſeries that is equal to $\overline{a+b}\rvert^{m}$, when m is equal to any whole number whatſoever, will be $a^{m} + m \times a^{m-1}b$, and that the literal parts of the following terms of the ſaid ſeries will be $a^{m-2}b^{2} + a^{m-3}b^{3} + a^{m-4}b^{4} + a^{m-5}b^{5} + a^{m-6}b^{6} + a^{m-7}b^{7} + a^{m-8}b^{8} + a^{m-9}b^{9} +$ &c. $+ a^{m-m}b^{m}$, (or $a^{0}b^{m}$, or $1\,b^{m}$, or) $+ b^{m}$. It remains that we inquire what will be the numeral co-efficients of the ſaid third and other following terms of the ſaid ſeries, or by what law, or rule, they may be generated, or derived, from the two firſt co-efficients, 1 and m. This is a matter of conſiderable difficulty; and I am not acquainted with any direct and ſcientific method of inveſtigating this law of the generation of the ſaid co-efficients, beſides that of Mr. James Bernoulli above-mentioned, which is grounded on the Doctrine of Permutations and Combinations, and the properties of the Figurate numbers. But I can point out a manner of conſidering the ſubject and attempting to find this law of generation, which ſeems likely enough to have occurred to a Mathematician who was in purſuit of this inquiry, and which, if it had occurred to him, would have led him directly to form a juſt conjecture concerning this law by which theſe co-efficients are to be generated; after which he would have been induced to try the law, ſo diſcovered by conjecture, in ſome eaſy particular inſtances, and, having found it to be true in all of them, he would naturally conclude that it was true in all other caſes whatſoever. This conjectural method of inveſtigation, I conceive, may have been as follows.

A Conjectural Inveſtigation of the Law by which the co-efficients of the third, and fourth, and fifth, and other following terms of the ſeries which is equal to $\overline{a+b}|^m$, *or the* m*th power of the binomial quantity* a + b, *may be generated, or derived, from* 1 *and* m, *the co-efficients of the two firſt terms of the ſaid ſeries.*

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17. Now, in order to diſcover the manner in which theſe co-efficients may be derived from the two firſt co-efficients 1 and m, I ſhould think it would be natural to examine the co-efficients of the terms of the ſeries that is equal to $\overline{a+b}|^m$ in ſome of the lower powers of $a+b$ which we have actually raiſed by multiplication, as, for example, in the ſeries which is equal to $\overline{a+b}|^6$, and which we have found above in art. 10, to be $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. In this ſeries the co-efficients of the terms are 1, 6, 15, 20, 15, 6, and 1; and our object is to diſcover, 1ſt, by what number, integral or fractional, the ſecond of theſe co-efficients, to wit, 6, ought to be multiplied in order to produce the third co-efficient, to wit, 15; and, 2dly, by what number, integral or fractional, the third co-efficient, to wit, 15, ought to be multiplied in order to produce the fourth co-efficient 20; and, 3dly, by what number the fourth co-efficient, 20, ought to be multiplied in order to produce the fifth co-efficient 15; and, 4thly, by what number the fifth co-efficient, 15, ought to be multiplied in order to produce the ſixth co-efficient 6; and, laſtly, by what number the ſixth co-efficient, 6, ought to be multiplied in order to produce the ſeventh and laſt co-efficient 1. Now theſe multiplying numbers are evidently $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$. For 6 $\times \frac{15}{6}$ is $= 15$, and $15 \times \frac{20}{15}$ is $= 20$, and $20 \times \frac{15}{20}$ is $= 15$,

and 15 × $\frac{6}{15}$ is = 6, and 6 × $\frac{1}{6}$ is = 1. We muſt therefore now examine theſe five generating fractions $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$, together with the preceeding generating fraction $\frac{6}{1}$, by the multiplication of which into the firſt co-efficient 1 the ſecond co-efficient 6 is produced; and muſt endeavour to find out ſome remarkable, or regular, property in them, which we may reaſonably ſuppoſe to belong alſo to the co-efficients of the terms of other powers of $a+b$, as well as to thoſe of the terms of this, its ſixth power. And, in order to examine theſe fractions with the greater eaſe, it ſeems natural, in the firſt place, to reduce them to their loweſt denominations, by dividing both their numerators and their denominators by the factors which are common to them both. Now, if this be done, the ſaid generating fractions $\frac{6}{1}$, $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$, will be found to be equal to ($\frac{6}{1}$, $\frac{5\times3}{2\times3}$, $\frac{4\times5}{3\times5}$, $\frac{3\times5}{4\times5}$, $\frac{2\times3}{5\times3}$, and $\frac{1}{6}$, or) $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$; in which laſt fractions it is impoſſible not to obſerve that the numerators regularly decreaſe by an unit from 6, which is the index of the power to which the binomial quantity $a+b$ has been raiſed, to 1, and the denominators regularly increaſe at the ſame time by the ſame quantity of an unit from 1 to the ſaid index 6. This regularity is very ſtriking, and naturally raiſes a ſuſpicion that the ſame thing may take place in the generating fractions of the co-efficients of the terms of the ſerieſes that are equal to other powers of the binomial quantity $a+b$, and is an inducement to try it in the other ſerieſes that have been produced above in art. 10, by multiplication, and which are equal to $\overline{a+b}|^{5}$, $\overline{a+b}|^{4}$, $\overline{a+b}|^{3}$, and $\overline{a+b}|^{2}$. We will therefore now proceed to try it in thoſe inſtances.

18. Now we have ſeen in art. 10, that $\overline{a+b}|^{5}$ is $=$ the ſeries $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, in which the numeral co-efficients of the terms are 1, 5, 10, 10, 5, and 1. Therefore the generating fractions, by the multiplication of which the ſecond of theſe co-efficients, to wit, 5, is derived from the firſt, or 1, and every following co-efficient from that which is next before it, will be $\frac{5}{1}$, $\frac{10}{5}$, $\frac{10}{10}$, $\frac{5}{10}$, and $\frac{1}{5}$; which are reſpectively equal to the fractions $\frac{5}{1}$, $\frac{4}{2}$, $\frac{3}{3}$, $\frac{2}{4}$, and $\frac{1}{5}$. And in theſe laſt fractions we cannot but obſerve that the numerators 5, 4, 3, 2, and 1, regularly decreaſe by an unit from 5, or the index of the power to which the binomial quantity $a+b$ is raiſed, to 1, and the denominators 1, 2, 3, 4, and 5, regularly increaſe at the ſame time, by the ſame quantity of an unit from 1 to the ſaid index 5. It appears therefore that the ſame rule takes place amongſt theſe generating fractions $\frac{5}{1}$, $\frac{4}{2}$, $\frac{3}{3}$, $\frac{2}{4}$, and $\frac{1}{5}$, as took place amongſt the generating fractions $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$, of the co-efficients of the terms of the former ſeries which was equal to $\overline{a+b}|^{6}$.

19. We will now try whether the ſame rule will take place in the ſeries which is equal to the fourth power of $a+b$.

This ſeries is $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, in which the numeral co-efficients of the terms are 1, 4, 6, 4, and 1. Now the generating fractions by the multiplication of which the ſecond of theſe co-efficients, to wit, 4, is generated from the firſt, or 1, and every following co-efficient is generated from that which is next before it, are evidently $\frac{4}{1}$, $\frac{6}{4}$, $\frac{4}{6}$, and $\frac{1}{4}$; which are reſpectively equal to $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{1}{4}$. And in theſe laſt fractions the numerators 4, 3, 2, and 1, regularly decreaſe by an unit from 4, which is the index of the power to which the binomial quantity $a+b$ has been raiſed, to 1, and the denominators 1, 2, 3, and 4, regularly increaſe at the ſame time by an unit from 1 to the ſaid index 4.

20. We come next to the ſeries which is equal to the cube of $a+b$.

This ſeries is $a^3 + 3a^2b + 3ab^2 + b^3$, in which the co-efficients of the terms are 1, 3, 3, and 1. Now the generating fractions, by the multiplication of which the ſecond of theſe co-efficients is derived from the firſt, and the third from the ſecond, and the fourth from the third, are evidently $\frac{3}{1}$, $\frac{3}{3}$, and $\frac{1}{3}$, which are reſpectively equal to $\frac{3}{1}$, $\frac{2}{2}$, and $\frac{1}{3}$. And in theſe laſt fractions the numerators 3, 2, and 1, decreaſe regularly by an unit from 3, which is the index of the power to which the binomial quantity $a+b$ has been raiſed, to 1, and the denominators 1, 2, and 3, increaſe regularly at the ſame time by an unit from 1 to the ſaid index 3.

21. And the ſame thing takes place in the ſeries which is equal to the ſquare of $a+b$. For this ſeries is $a^2+2ab+b^2$, in which the co-efficients of the terms are 1, 2, and 1. Now the generating fractions, by the multiplication of which the ſecond co-efficient 2 is derived from the firſt co-efficient 1, and the third co-efficient 1 is derived from the ſecond co-efficient 2, are evidently $\frac{2}{1}$, and $\frac{1}{2}$; which admit of no reduction to lower denominations. And in theſe fractions $\frac{2}{1}$ and $\frac{1}{2}$, the numerators 2 and 1 decreaſe by an unit, as

in the former caſes, from 2, which is the index of the power to which the binomial quantity $a+b$ has been raiſed, to 1, and the denominators 1 and 2 increaſe at the ſame time by an unit from 1 to the ſaid index 2.

22. It appears therefore that this law of the generating fractions of the numeral co-efficients of the terms of the ſerieſes that are equal to the powers of the binomial quantity $a+b$, takes place in the caſes of the ſquare, the cube, the fourth power, the fifth power, and the ſixth power, of the ſaid binomial quantity. This is a very ſtrong ground for conjecturing that the ſame law will take place in the generating fractions of the numeral co-efficients of the terms of the ſerieſes which are equal to the powers of the ſaid binomial quantity in all other caſes whatſoever; or that, if the index of the power to which the ſaid binomial quantity is raiſed be any whole number whatſoever, denoted by the letter m, the generating fractions, by the continual multiplication of which the numeral co-efficients of the ſecond and other following terms of the ſeries which is equal to $\overline{a+b}|^{m}$, or the mth power of the ſaid binomial quantity, may be derived from 1, or the co-efficient of the firſt term, a^m, or $1 \times a^m$, of the ſaid ſeries, will be $\frac{m}{1}$, or (as it is ſometimes called) $\frac{m-0}{1}$, and $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, till we come to the term $\frac{m-m}{m+1}$, which is $= 0$, or till the ſaid ſeries is terminated, or exhauſted.

23. And the ground for conjecturing that this is a general law that takes place among the generating fractions of the terms of theſe ſerieſes in all caſes, or when the index is equal to any whole number whatſoever, will become ſtill ſtronger if we try it in a few more examples of ſerieſes that are equal to higher powers of the binomial quantity $a+b$, than the ſixth power. I ſhall therefore now proceed to try

it in the ſerieſes which are equal to $\overline{a+b}|^{7}$, $\overline{a+b}|^{8}$, and $\overline{a+b}|^{9}$.

24. Now $\overline{a+b}|^{7}$ is ($= \overline{a+b}|^{6} \times \overline{a+b} = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \times \overline{a+b}) = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$; in which ſeries the numeral co-efficients of the terms are 1, 7, 21, 35, 35, 21, 7, and 1. The generating fractions, by the multiplication of which the ſecond of theſe co efficients, to wit, 7, is derived from the firſt co-efficient 1, and the third and other following co-efficients are derived from thoſe which immediately preceed them, are evidently $\frac{7}{1}$, $\frac{21}{7}$, $\frac{35}{21}$, $\frac{35}{35}$, $\frac{21}{35}$, $\frac{7}{21}$, and $\frac{1}{7}$; which are reſpectively equal to $\frac{7}{1}$, $\frac{6}{2}$, $\frac{5}{3}$, $\frac{4}{4}$, $\frac{3}{5}$, $\frac{2}{6}$, and $\frac{1}{7}$; in which laſt fractions the numerators 7, 6, 5, 4, 3, 2, and 1, regularly decreaſe by an unit from 7, (which is the index of the power to which the binomial quantity $a+b$ has been raiſed), to 1, and the denominators 1, 2, 3, 4, 5, 6, and 7, regularly increaſe at the ſame time by an unit from 1 to the ſaid index 7; agreeably to what was obſerved in the five former examples.

25. And $\overline{a+b}|^{8}$ is ($= \overline{a+b}|^{7} \times \overline{a+b} = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \times \overline{a+b}$) $= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$; in which ſeries the numeral co-efficients of the terms are 1, 8, 28, 56, 70, 56, 28, 8, and 1. The generating fractions of theſe co-efficients are evidently $\frac{8}{1}$, $\frac{28}{8}$, $\frac{56}{28}$, $\frac{70}{56}$, $\frac{56}{70}$, $\frac{28}{56}$, $\frac{8}{28}$, and $\frac{1}{8}$; which are reſpectively equal to $\frac{8}{1}$, $\frac{7}{2}$, $\frac{6}{3}$, $\frac{5}{4}$, $\frac{4}{5}$, $\frac{3}{6}$, $\frac{2}{7}$, and $\frac{1}{8}$; in which laſt fractions the numerators 8, 7, 6, 5, 4, 3, 2, and 1, decreaſe regularly by an unit from 8, (which is the index of the power to

which the binomial quantity $a + b$ has been raiſed), to 1, and the denominators 1, 2, 3, 4, 5, 6, 7, and 8, regularly increaſe at the ſame time by an unit from 1 to the ſaid index 8; agreeably to what has been obſerved in the ſix former examples.

26. And, laſtly, $\overline{a+b}|^{9}$ is $(= \overline{a+b}|^{8} \times \overline{a+b} = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8 \times \overline{a+b}) = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$; in which ſeries the numeral co-efficients of the terms are 1, 9, 36, 84, 126, 126, 84, 36, 9, and 1. The generating fractions of theſe co-efficients are evidently $\frac{9}{1}$, $\frac{36}{9}$, $\frac{84}{36}$, $\frac{126}{84}$, $\frac{126}{126}$, $\frac{84}{126}$, $\frac{36}{84}$, $\frac{9}{36}$, and $\frac{1}{9}$; which are reſpectively equal to $\frac{9}{1}$, $\frac{8}{2}$, $\frac{7}{3}$, $\frac{6}{4}$, $\frac{5}{5}$, $\frac{4}{6}$, $\frac{3}{7}$, $\frac{2}{8}$, and $\frac{1}{9}$; in which laſt fractions the numerators 9, 8, 7, 6, 5, 4, 3, 2, and 1, decreaſe regularly by an unit from 9, (which is the index of the power to which the binomial quantity $a + b$ has been raiſed), to 1, and the denominators 1, 2, 3, 4, 5, 6, 7, 8, and 9, regularly increaſe at the ſame time by an unit from 1 to the ſaid index 9; agreeably to what has been obſerved in all the former examples.

27. After obſerving this law of the co-efficients to take place in ſo many different examples, it would be impoſſible for our mathematical inveſtigator not to conclude with a very high degree of confidence that it would take place in all other caſes whatſoever; or that, when the index m is equal to any whole number whatſoever, the generating fractions of the numeral co-efficients of the terms of the ſeries that is equal to $\overline{a+b}|^{m}$, will be $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, &c, till we come to the fraction $\frac{m-m}{m+1}$, which is

$= 0$, or till the ſaid ſeries of fractions is terminated, or exhauſted. And then it would follow, from what has been ſhewn above concerning the literal parts of the terms of the ſaid ſeries, that the ſaid ſeries which is equal to $\overline{a+b}|^m$, or the *m*th power of the binomial quantity $a+b$, would be $a^m + \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 + \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 + \frac{m-4}{5} E a^{m-5} b^5 + \frac{m-5}{6} F a^{m-6} b^6 +$ &c, continued to b^m. Q. E. I.

28. This method of diſcovering (by a conjecture grounded on ſome trials in particular examples) that the generating fractions by which the numeral co-efficients of the third, and fourth, and other following terms of the ſeries that is equal to $\overline{a+b}|^m$ (or any integral power of the binomial quantity $a+b$), are derived from m (the index of the power to which the ſaid binomial quantity is raiſed), or from the co-efficient of the ſecond term of the ſaid ſeries (which is always equal to the ſaid index) are $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, is ſuggeſted by Profeſſor Saunderſon, in the ſecond volume of his Algebra, in the chapter on the Binomial Theorem; where the Reader will find a good explanation and illuſtration of the ſaid celebrated Theorem, by a variety of examples, both in the caſe of Integral powers, and in the caſe of Roots and other Fractional powers, and even in the caſe of Negative powers, and of powers that are both fractional and negative; but no demonſtration of it in any caſe, not even in that of Integral and Affirmative powers.

29. We have now ſhewn with demonſtrative certainty that the literal parts of the terms of the ſeries which is equal to $\overline{a+b}|^m$, or the *m*th power of the binomial quantity $a+b$,

when the letter m denotes any affirmative whole number whatſoever, are $a^{m} + a^{m-1}b + a^{m-2}b^{2} + a^{m-3}b^{3} + a^{m-4}b^{4} + a^{m-5}b^{5} + a^{m-6}b^{6} + a^{m-7}b^{7} +$ &c, till we come to the term $a^{m-m}b^{m}$, (or $a^{0} \times b^{m}$, or $1 \times b^{m}$), or b^{m}, and that the numeral co-efficient of the literal part of the firſt term of the ſaid ſeries is 1, and the numeral co-efficient of the literal part of the ſecond term of it is m, and conſequently that the two firſt whole terms of the ſaid ſeries are $1 \times a^{m}$, and $m \times a^{m-1}b$, or $a^{m} + m \times a^{m-1}b$, or $a^{m} + \frac{m}{1} \times a^{m-1}b$, or $a^{m} + \frac{m-0}{1} \times a^{m-1}b$. And we have alſo ſhewn that in the ſerieſes which are equal to $\overline{a+b}\vert^{2}$, $\overline{a+b}\vert^{3}$, $\overline{a+b}\vert^{4}$, $\overline{a+b}\vert^{5}$, $\overline{a+b}\vert^{6}$, $\overline{a+b}\vert^{7}$, $\overline{a+b}\vert^{8}$, and $\overline{a+b}\vert^{9}$, or when the index m is equal to 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the ſaid ſerieſes are derived from m, or the numeral co-efficient of the ſecond term, by the continual multiplication of the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, $\frac{m-8}{9}$, and $\frac{m-9}{10}$; which affords a very ſtrong ground for conjecturing that the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the ſerieſes that are equal to any higher powers whatſoever of the binomial quantity $a+b$, will, in like manner, be derived from m, the co-efficient of the ſecond term, by the continual multiplication of the ſame generating fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, $\frac{m-8}{9}$, $\frac{m-9}{10}$, $\frac{m-10}{11}$, $\frac{m-11}{12}$, &c, till we come to the term $\frac{m-m}{m+1}$, which is $= 0$, or till the ſaid ſeries of generating fractions is terminated, or exhauſted. Now this conjecture may be changed into abſolute certainty, and the ſaid law of the generation of the co-efficients may be ſhewn to take place in all the ſerieſes that are equal to the quantity $\overline{a+b}|^{m}$, or the powers of the binomial quantity $a+b$, when the index m of the ſaid quantity $\overline{a+b}|^{m}$, is equal to any whole number, how great ſoever, by ſhewing that, if it takes place when the index m is equal to any one particular number, (as we have ſeen that it does when the index m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9), it muſt alſo take place when the index m is greater by an unit than in the former caſe. For then it will follow that it muſt be true likewiſe when the index m is greater by any multitude of units than in the former caſe, or when it is equal to any other whole number, how great ſoever. This we ſhall now proceed to ſhew in the remaining part of this diſcourſe.

Of the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the Series that is equal to $\overline{a+b}|^{m}$, *and the law of the generation of the ſaid co-efficients from* m, *the co-efficient of the ſecond term of the ſaid ſeries, and from each other.*

30. In order to demonſtrate the law of the generation of theſe co-efficients, it will be convenient to get rid of the powers of a and b, in the terms of the ſeries that is equal to $\overline{a+b}|^{m}$, and to fix our attention only on the generation of the numeral co-efficients of the third, fourth, fifth, ſixth, and other following terms of the ſaid ſeries. This may be done by ſuppoſing a and b to be, each of them, equal to 1,

and conſequently $a+b$ to be equal to $1+1$, and $\overline{a+b}|^{m}$ to be equal to $\overline{1+1}|^{m}$. For, as all the powers of both a and b will, on this ſuppoſition, be equal to 1, the Binomial Theorem ſet forth above in art. 1, will then be reduced to this, to wit, that $\overline{1+1}|^{m}$ will be equal to the ſeries $1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}+$ &c, continued to the term $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times$ &c, $\times\frac{m-\overline{n-1}}{m}$, or to the term $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times$ &c, $\times\frac{m-m+1}{m}$, or to the term $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times$ &c, $\times\frac{1}{m}$, or to the term 1. For the laſt term of this ſeries muſt always be 1; becauſe the numerators of the ſeveral factors in it form a decreaſing progreſſion of numbers, decreaſing by an unit, from m to 1; and the denominators of the ſame factors form an increaſing progreſſion of numbers, increaſing by an unit, from 1 to m; and conſequently the product of the multiplication of all the denominators is equal to the product of the multiplication of all the numerators, and therefore the product of the multiplication of all the ſaid factors, or fractions, $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, into each other, or the laſt term of the ſeries, muſt always be equal to 1.

We are therefore now to demonſtrate that $\overline{1+1}|^{m}$ is equal to the ſeries $1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}$

$\times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c, + 1. And this we propoſe to do by ſhewing, by abſtract and general reaſonings, that, if this Theorem is true when the index m is of any particular value, as, for example, when it is equal to 9, it muſt likewiſe be true when the index m is increaſed by an unit, or that, if n be taken $= m+1$, the quantity $\overline{1+1}\rvert^{n}$, or the nth power of the binomial quantity $1+1$, will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$ + &c, continued to the term $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$ × &c, $\times \frac{1}{n}$, or to the term 1.

31. To facilitate the demonſtration of this propoſition, it will be convenient to premiſe the following Lemma.

A LEMMA.

If the terms of the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c, + 1 (in which m repreſents any whole number whatſoever), be ſet down twice together in two parallel lines, or rows, one under the other, but with the terms in the lower row advanced one ſtep further to the right-hand than the terms in the upper row, ſo that the firſt term in the lower row ſhall ſtand under the ſecond term of the upper row, and the ſecond term in the lower row ſhall ſtand under the third term in the upper row, and the third, fourth, fifth, ſixth, and other following terms in the lower row ſhall ſtand under the fourth, fifth, ſixth, ſeventh, and other following terms in the upper row, reſpectively; and both rows are continued to the ſame number of terms, namely, to the whole number of terms in the ſaid ſeries, or to $m+1$ terms; and then the terms in the lower row (each of which, it is evident, will conſiſt of one factor leſs than the correſponding term, or term ſtanding immediately above it in the upper row) be reduced to the ſame denomination as the terms that ſtand immediately above them in the upper row, and, after being ſo reduced, are added to the ſaid terms that ſtand immediately above them in the ſaid upper row;— upon theſe ſuppoſitions the new ſeries of terms ariſing from this addition of the ſaid two rows of terms to each other, will be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m+1}{5} +$ &c $+ 1$; in which ſeries the laſt term is 1, as well as in the two ſerieſes from the addition of which this ſeries ariſes; and the numerators of the laſt factors in all the terms, except the laſt, are always equal to $m+1$, inſtead of being equal to $m-1$, $m-2$, $m-3$, $m-4$, &c, as in the two foregoing ſerieſes; and the number of terms in the ſaid new ſeries is $m+2$, inſtead of $m+1$, which is the number of terms in each of the ſaid foregoing ſerieſes.

DEMON-

DEMONSTRATION.

32. This will appear by ſetting down the ſaid ſeries $1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+$ &c, $+$ 1 twice over, in the manner that has been juſt deſcribed; which may be done as follows:

$$1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+\&c.$$

$$1+\frac{m}{1}\qquad+\frac{m}{1}\times\frac{m-1}{2}\qquad+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\qquad+\&c.$$

In theſe two rows of terms it is evident, in the firſt place, that the terms in the upper row, after the two firſt terms 1 and $\frac{m}{1}$, conſiſt of two, three, and four, and more, factors, every new term having one more factor than the term next before it; and, 2dly, that the terms in the lower row that ſtand immediately under the third, fourth, fifth, and other following terms in the upper row, conſiſt of one factor leſs than the correſponding terms, or terms immediately over them in the upper row; and, 3dly, that the terms in the lower row conſiſt of the very ſame factors as the correſponding terms in the upper row, excepting that they want the laſt factors of the ſaid terms in the upper row. And hence it follows, that, in order to reduce the terms in the lower row to the ſame denomination as the terms in the upper row, we muſt multiply them by factors that ſhall have the ſame denominators as the laſt, or additional factors in the upper row, and which muſt have their numerators equal to their denominators, ſo as to make each of them equal to 1, to the end that the magnitudes of the ſaid lower terms may not be altered by the multiplication of them by the ſaid new factors. Thus, for example, the ſecond term of the lower row; to wit, $\frac{m}{1}$, muſt be multiplied into the factor $\frac{2}{2}$, in order to bring it to the ſame denomination as the third term in the upper row; to wit, $\frac{m}{1} \times \frac{m-1}{2}$, without altering its magnitude; and the third term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, muſt be multiplied into the factor $\frac{3}{3}$, in order to bring it to the ſame denomination as the fourth term of the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, without altering its magnitude; and the fourth term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, muſt be multiplied into the factor $\frac{4}{4}$, in order to bring it to the ſame denomination as the fifth term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, without altering its magnitude; and, for the like reaſon, the fifth, and ſixth, and ſeventh, and other following terms in the lower row muſt be multiplied into the ſeveral factors $\frac{5}{5}$, and $\frac{6}{6}$, and $\frac{7}{7}$, &c, reſpectively; after which multiplications the two rows of terms that are to be added to each other, will be as follows, to wit,

$$1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+\&c$$

$$+1+\frac{m}{1}\times\frac{2}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{3}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{4}{4}+\&c.$$

33. And, if theſe two rows of terms (being now brought to the ſame denominations) are added together in the manner above deſcribed; that is, every term in the lower row to the term that is immediately above it, the ſum thence reſulting will be the ſeries $1+\frac{m+1}{1}+\frac{m}{1}\times\frac{m+1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m+1}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m+1}{4}+$ &c, in which

the numerator of the laſt factor in every term is always $m+1$, inſtead of $m-1$, $m-2$, $m-3$, $m-4$, &c.

And "That this muſt be the caſe in all the following "terms of the ſaid new ſeries as well as in the few terms of "it that have been here ſet down," will be evident from this conſideration, to wit, That the denominator of the laſt factor of every term in the upper of the two rows of terms that are added together is always greater by an unit than the number which is ſubtracted from m in the numerator of the ſame factor. For from thence it follows that the denominator of the new multiplying fraction in the correſponding term of the lower row (which is always equal to the denominator of the ſaid laſt factor in the upper row,) muſt always be greater by an unit than the number which is ſubtracted from m in the numerator of the laſt factor of the ſaid upper term. And, therefore, the numerator of the ſaid new multiplying fraction in the lower row (which is always equal to its denominator,) muſt alſo always be greater by an unit than the number which is ſubtracted from m in the numerator of the laſt factor of the ſaid upper term; the conſequence of which, in adding the lower term to the upper term, is to convert the numerator of the laſt factor in the upper term from $m-1$, or $m-2$, or $m-3$, or the exceſs of m above ſome other number, into $m+1$. Q. E. D.

34. And the number of terms in the new ſeries, ariſing from the addition of the two former in the manner that has been deſcribed, will be greater by one than the number of the terms in either of the two added ſerieſes: becauſe the lower row of terms, conſiſting of the ſame number of terms as the upper row, and being placed one term further to the right-hand, muſt extend one term beyond it; and conſequently, as the number of terms in each of the two rows of terms is $m+1$, the number of terms in the new ſeries, ariſing from the addition of the two rows together, muſt be $m+2$. Q. E. D.

35. And, laſtly, the laſt term of the ſaid new ſeries muſt be the ſame as the laſt term of the old ſeries, or of the lower

row of terms; becauſe, as the lower row of terms extends one term beyond the upper row, the laſt term in the lower row will not have any term over it in the upper row to which it is to be added, and conſequently will continue the ſame in the new ſeries $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} +$ &c, as in the old ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c. But we have ſeen above, in art. 30, that the laſt term of the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, is 1. Therefore the laſt term in the new ſeries $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} +$ &c, will alſo be 1. Q. E. D.

36. Coroll. 1. Now let the order of the numerators m, $m-1$, $m-2$, $m-3$, $m-4$, &c, and $m+1$, of the factors of the third and other following terms of the laſt ſeries $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} +$ &c, be changed, by making $m+1$ the numerator of the firſt factor of every term inſtead of being the numerator of the laſt factor. The ſaid ſeries will then be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$ &c. Now this change in the order of the numerators of the ſeveral factors of the terms will create no change in the values,

or magnitudes, of the ſeveral terms themſelves; becauſe the products ariſing from the multiplication of the ſame numbers are always the ſame, in whatever order the numbers are multiplied. Therefore the foregoing ſeries, after this change in the order of the numerators of the ſeveral factors of its terms, will ſtill be of the ſame magnitude as before, and conſequently will be equal to the ſum that ariſes from the addition of the aforeſaid two rows of terms in the manner above deſcribed; that is, the ſeries $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$ &c, $+ 1$, will be equal to the ſum that ariſes from the addition of the aforeſaid two rows of terms in the manner above deſcribed.

37. Coroll. 2. Now let n be $= m + 1$. Then will $n-1$ be $= m$, and $n-2$ will be $= m-1$, and $n-3$ will be $= m-2$, and $n-4$ will be $= m-3$; and, in like manner, $n-5$, $n-6$, $n-7$, &c, will be equal to $m-4$, $m-5$, $m-6$, &c, reſpectively. And conſequently the ſeries obtained in the foregoing Corollary, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$ &c, $+ 1$, conſiſting of $m+2$ terms, will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c, $+ 1$, conſiſting of $n+1$ terms. Therefore the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c, $+ 1$, conſiſting of $n + 1$ terms, will be equal to the ſum that ariſes by adding the two aforeſaid rows of terms together in the manner above deſcribed.

The Demonſtration of the principal Propoſition.

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38. Theſe things being premiſed, the main propoſition ſtated at the end of art. 30, to wit, that, if m denote any whole number whatſoever, the quantity $\overline{1+1}|^{m}$, or the mth power of the binomial quantity $1+1$, will be equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c, continued to $m+1$ terms, or to the term 1, may be demonſtrated: in the manner following.

39. The product that ariſes by multiplying the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m-3}{4}$ + &c, into $1+1$ is the ſum that ariſes by ſetting down the ſaid ſeries twice following in two parallel rows, one under the other, with the terms in the lower row advanced one term further to the right-hand than the terms in the upper row, in the manner above deſcribed, and then adding the terms in the lower row to the correſponding terms in the upper row. And the $\overline{m+1}|$th power of $1+1$ is the product of the multiplication of the mth power of $1+1$ into $1+1$. Therefore, if in any particular value of m the mth power of $1+1$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ + &c, + 1, conſiſting of $m+1$ terms, the $\overline{m+1}|$th power of $1+1$ will be equal to the ſum that ariſes by ſetting down the ſaid ſeries twice following in two parallel rows in the manner above deſcribed, and adding the ſaid two rows of

terms together. But, by the ſecond Corollary of the foregoing Lemma, if n be $= m+1$, the ſum ariſing from the addition of the ſaid two rows of terms is the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c, + 1$, conſiſting of $n+1$ terms. Therefore, if in any particular value of m the mth power of $1+1$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c, + 1$, conſiſting of $m+1$ terms, it will follow that the $\overline{m+1}|$th, or nth, or next higher power, of $1+1$ will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, conſiſting of $n+1$ terms. But it has been ſhewn in art. 17, 18, 19, &c, - - - 26, that when m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9, the mth power of $1+1$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c + 1$, conſiſting of $m+1$ terms. Therefore, if n be equal to $9+1$, or 10, the $\overline{9+1}|$th power, or 10th power, or nth power, of $1+1$ will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, conſiſting of $n+1$, or $10+1$, or 11, terms. And in the ſame manner it may be proved that, ſince, when m is $= 10$, the mth power of $1+1$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c, + 1$, conſiſting of $m+1$, or $10+1$, or 11, terms, the $\overline{m+1}|$th, or $\overline{10+1}|$th, or 11th, or (putting $n = m+1 = 10+1 = 11$) the nth, power of $1 + 1$ will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, conſiſting of $n+1$, or $11+1$, or 12, terms. And ſo we may proceed from number to number *ad infinitum*. And conſequently, whatever be the whole number denoted by m, it will always be true that $\overline{1+1}|^{m}$ is equal to the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} +$ &c $+ 1$, conſiſting of $m + 1$ terms. Q. E. D.

The foregoing Demonſtration expreſſed in a more conciſe Manner.

40. The foregoing reaſonings may be expreſſed in a more conciſe manner as follows. If n be $= m + 1$, and it be true in any particular value of m that $\overline{1+1}|^{m}$ is $=$ the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, it will alſo be true that $\overline{1+1}|^{n}$ will be $= 1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c.

For $\overline{1+1}|^{n}$ is $= \overline{1+1}|^{m+1} = \overline{1+1}|^{m} \times \overline{1+1} =$ the ſeries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2}$

×

$\times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, multiplied into $1+1 =$

$$1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+\&c.$$

$$+1+\frac{m}{1} \qquad +\frac{m}{1}\times\frac{m-1}{2} \qquad +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3} \qquad +\&c.$$

$$=1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+\&c.$$

$$+1+\frac{m}{1}\times\frac{2}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{3}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{4}{4}+\&c.$$

$$=1+\frac{m+1}{1}+\frac{m}{1}\times\frac{m+1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m+1}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m+1}{4}+\&c.$$

$$=1+\frac{m+1}{1}+\frac{m+1}{1}\times\frac{m}{2}+\frac{m+1}{1}\times\frac{m}{2}\times\frac{m-1}{3}+\frac{m+1}{1}\times\frac{m}{2}\times\frac{m-1}{3}\times\frac{m-2}{4}+\&c$$

$$=1+\frac{n}{1}+\frac{n}{1}\times\frac{n-1}{2}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}+\&c.$$

But it has been ſhewn in art. 17, 18, 19, &c, - - - - 26, that, when m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9, $\overline{1+1}|^{m}$ is equal to the ſeries $1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+$ &c. Therefore, if n be $= 9+1$, or 10, $\overline{1+1}|^{n}$, or $\overline{1+1}|^{10}$, will be = the ſeries $1+\frac{n}{1}+\frac{n}{1}\times\frac{n-1}{2}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ + &c. And it may be ſhewn in like manner, that, if n be put for 11, 12, 13, 14, &c, *ad infinitum* ſucceſſively, $\overline{1+1}|^{n}$ will, in all theſe ſuppoſitions, be always equal to the ſeries $1+\frac{n}{1}+\frac{n}{1}\times\frac{n-1}{2}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ + &c; and therefore the propoſition is univerſally true, whatever be the whole number denoted by the letter n.

Q. E. D.

41. This demonſtration of the binomial theorem in the caſe of integral powers, is nearly the ſame with that given by Mr. John Stewart, of Aberdeen, in the 6th Section of his Commentary on Sir Iſaac Newton's curious little Tract, intitled, *Analyſis by Equations of an infinite number of Terms.* See his edition of Newton's Treatiſe *on the Quadrature of Curves*, and of the ſaid Tract intitled *Analyſis, &c*, with his learned Comments on both, in one volume, quarto, publiſhed at London, in the year 1745, page 471, Art. 155.

Of the Powers of a Reſidual Quantity a — b, when their Indexes are whole Numbers.

42. We have hitherto been conſidering the integral powers of a *binomial* quantity $a+b$, or of the *ſum* of two ſingle quantities a and b; and we have ſeen that, if the ſaid binomial quantity $a+b$ be raiſed to any power of which a whole number denoted by m is the index, the quantity $\overline{a+b}\rvert^{m}$, or the ſaid mth power of $a+b$, will be equal to the ſeries $a^{m}+\frac{m}{1}a^{m-1}b+\frac{m}{1}\times\frac{m-1}{2}a^{m-2}b^{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}a^{m-4}b^{4}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}a^{m-5}b^{5}+$ &c, $+\,b^{m}$, or (if we put $A=1$, $B=\frac{m}{1}A$, $C=\frac{m-1}{2}B$, $D=\frac{m-2}{3}C$, $E=\frac{m-3}{4}D$, $F=\frac{m-4}{5}E$, and G, H, I, K, L, &c, $=\frac{m-5}{6}F$, $\frac{m-6}{7}G$, $\frac{m-7}{8}H$, $\frac{m-8}{9}I$, $\frac{m-9}{10}K$, &c, reſpectively,) to the ſeries $a^{m}+\frac{m}{1}Aa^{m-1}b+\frac{m-1}{2}Ba^{m-2}b^{2}+\frac{m-2}{3}Ca^{m-3}b^{3}+\frac{m-3}{4}Da^{m-4}b^{4}+\frac{m-4}{5}Ea^{m-5}b^{5}+$ &c $+\,b^{m}$;

in which all the terms after the firſt term a^m are marked with the ſign +, or are added to the ſaid firſt term. We will now proceed to conſider the value of $\overline{a-b}|^{m}$, or the mth power of the *reſidual* quantity $a-b$, or of the *difference* of the two quantities a and b, upon a ſuppoſition that a is the greater of the two.

43. Now, if a be ſuppoſed to be greater than b, and m be any whole number whatſoever, the quantity $\overline{a-b}|^{m}$, or the mth power of the reſidual quantity, or difference, $a-b$, will be equal to the ſeries $a^m - \frac{m}{1} a^{m-1}b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2}b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4}b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5}b^5$ + &c, or (if we put A, as before, = 1, and B = $\frac{m}{1}$ A, and C = $\frac{m-1}{2}$ B, and D = $\frac{m-2}{3}$ C, and E = $\frac{m-3}{4}$ D, and F = $\frac{m-4}{5}$ E. and G, H, I, K, L, &c, = $\frac{m-5}{6}$ F, $\frac{m-6}{7}$ G, $\frac{m-7}{8}$ H, $\frac{m-8}{9}$ I, $\frac{m-9}{10}$ K, &c, reſpectively,) to the ſeries $a^m - \frac{m}{1} \mathrm{A} a^{m-1}b + \frac{m-1}{2} \mathrm{B} a^{m-2}b^2 - \frac{m-2}{3} \mathrm{C} a^{m-3}b^3 + \frac{m-3}{4} \mathrm{D} a^{m-4}b^4 - \frac{m-4}{5} \mathrm{E} a^{m-5}b^5$ + &c, which conſiſts of exactly the ſame terms as the ſeries that is equal to $\overline{a+b}|^{m}$, or the ſame power of the binomial quantity $a+b$, but with the ſign − prefixed to the ſecond, and fourth, and ſixth, and every following even term in the ſeries, which denotes that the ſaid terms are not to be added to the firſt term a^m, and to the third, and fifth, and other following odd terms, (as they were in the former ſeries, which was equal to $\overline{a+b}|^{m}$,) but to be ſubtracted from them.

44. That this muſt be ſo, will be evident from conſidering the manner in which the ſeveral powers of the reſidual quantity $a - b$ are generated from each other by the continual multiplication of $a-b$, of which we will now exhibit a ſpecimen with reſpect to a few of its loweſt powers. The ſecond, third, fourth, and fifth powers of $a-b$ are derived from $a-b$ itſelf by the following multiplications.

$$
\begin{array}{l}
a-b \\
a-b \\
\hline
aa-ab \\
\quad -ab+bb \\
\hline
aa-2ab+bb = \overline{a-b}|^{2}. \\
\qquad a-b \\
\hline
a^{3}-2a^{2}b+ab^{2} \\
\quad -a^{2}b+2ab^{2}-b^{3} \\
\hline
a^{3}-3a^{2}b+3ab^{2}-b^{3} = \overline{a-b}|^{3}. \\
\qquad\qquad a-b \\
\hline
a^{4}-3a^{3}b+3a^{2}b^{2}-ab^{3} \\
\quad -a^{3}b+3a^{2}b^{2}-3ab^{3}+b^{4} \\
\hline
a^{4}-4a^{3}b+6a^{2}b^{2}-4ab^{3}+b^{4} = \overline{a-b}|^{4} \\
\qquad\qquad\qquad a-b \\
\hline
a^{5}-4a^{4}b+6a^{3}b^{2}-4a^{2}b^{3}+ab^{4} \\
\quad -a^{4}b+4a^{3}b^{2}-6a^{2}b^{3}+4ab^{4}-b^{5} \\
\hline
a^{5}-5a^{4}b+10a^{3}b^{2}-10a^{2}b^{3}+5ab^{4}-b^{5} = \overline{a-b}|^{5}.
\end{array}
$$

45. From theſe operations it is evident that, wherever the odd powers of b occur in the ſaid powers of $a-b$, the terms are marked with the ſign —, and that, wherever the even powers of b occur in the ſaid powers of $a-b$, the terms are marked with the ſign +. And the ſame thing, it is evident, muſt happen in all higher powers of $a-b$ whatſoever, as well in thoſe that have been here ſet down, becauſe

b is marked with the ſign — in the two original factors *a* — *b* and *a* — *b*; whence it follows, from the nature of algebraick multiplication, that, whenever *b* is multiplied into itſelf an even number of times, the product will be marked with the ſign + ; and, whenever it is multiplied into itſelf an odd number of times; the product will be marked with the ſign —. And it is further evident, from the foregoing multiplications, that the odd powers of *b* occur in the ſecond, and fourth, and ſixth, terms of the foregoing products, and that the even powers of *b* occur in the third and fifth terms of them. And it is eaſy to ſee that the odd powers of *b* will occur in like manner in the eighth, and tenth, and twelfth, and other following even terms of all higher powers of *a* — *b* whatſoever, and that the even powers of *b* will occur in like manner in the ſeventh, and ninth, and eleventh, and other following odd terms of the ſaid higher powers of *a* — *b*. And it is alſo evident, from the foregoing multiplications, that the terms themſelves of which the ſeveral powers of *a* — *b* will be compoſed, are exactly the ſame with the terms of which the ſame powers of *a* + *b* are compoſed. And hence it follows that the ſeries which is equal to $\overline{a-b}|^{m}$ will be the ſame with the ſeries which is equal to $\overline{a+b}|^{m}$, when the ſign — has been prefixed to the ſecond, and fourth, and ſixth, and other following even terms of it, inſtead of the ſign +, or that $\overline{a-b}|^{m}$, or the *m*th power of the reſidual quantity *a* — *b*, will be equal to the ſeries $a^{m} - \frac{m}{1}a^{m-1}b + \frac{m}{1} \times \frac{m-1}{2}a^{m-2}b^{2} - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}b^{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}a^{m-4}b^{4} - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}a^{m-5}b^{5} +$ &c, or $a^{m} - \frac{m}{1}\,\mathrm{A}\,a^{m-1}b + \frac{m-1}{2}\,\mathrm{B}\,a^{m-2}b^{2} - \frac{m-2}{3}\,\mathrm{C}\,a^{m-3}b^{3} + \frac{m-3}{4}\,\mathrm{D}\,a^{m-4}b^{4} - \frac{m-4}{5}\,\mathrm{E}\,a^{m-5}b^{3} +$ &c.

Q. E. D.

A

DISCOURSE

OF

COMBINATIONS,

ALTERNATIONS,

AND

ALIQUOT PARTS.

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OF

COMBINATIONS,

ALTERNATIONS,

AND

ALIQUOT PARTS.

CHAP. I.

Of the variety of Elections, or Choice, in taking or leaving One or more, out of a certain Number of things propoſed.

FOR the better underſtanding of what is propoſed; ſuppoſe we a certain number of counters or other things expoſed; as, for inſtance, 7; *a b c d e f g*: The queſtion is, what variety, or how many caſes there may be, of taking from thence one, or two of them; as *a*, *b*, *c*, *d*, &c. Or, *ab*, *ac*, *ad*, *bc*, *bd*, &c. Or, three's, as *abc*, *abd*, *acd*, *bdc*, &c. Or, fours, fives, &c. Or all, or none? And the like if any other number of things were ſo expoſed.

In order to the ſolution whereof, I ſhall here inſert a table, borrowed from my Arithmetick of Infinites, Prop. 132, 169, 183, 189, &c, (becauſe there will be often occaſion of having recourſe to it,) And then proceed to propoſitions thereunto relating.

To be left.

Numbers.	0	1	2	3	4	5	6	7	8	9	10	To be taken.
Monadicks.	1	1	1	1	1	1	1	1	1	1	1	0
Laterals.	1	2	3	4	5	6	7	8	9	10		1
Triangulars.	1	3	6	10	15	21						2
Pyramidals.	1	4	10	20	35							3
Triang. Triang.	1	5	15	35								4
Triang. Pyram.	1	6	21									5
Pyram. Pyram.	1	7										6
&c.	1	8										7
	1	9										8
	1	10										9
	1											10

Now, as to the conſtruction of this table, we are to obſerve, that, (the firſt line being all units,) the following numbers are, in every place, the aggregate of all thoſe in the line next above it, ſo far. As for example; for the three firſt in the uppermoſt line, 1, 1, 1, we have in the ſecond line (under the laſt of them) the number 3, which is the aggregate of them. And, in like manner, we have in the next place 4, which is the aggregate of 1, 1, 1, 1. (And ſo of the reſt.) And, in the lines following, likewiſe: for 1, 2, 3, (the three foremoſt of the ſecond line,) we have in the third line (under the laſt of them) the number 6, equal to all of them: and ſo every where. This premiſed, the propoſitions follow.

1. It is manifeſt, that, if we would *take none*, that is, if we would *leave all*, there can be but one caſe thereof, whatever be the number of things expoſed. (For this admits of no variety.) Which (in the table) is expreſſed in the firſt (tranſverſe) line, where the numbers are all *Monadicks*, or units.

2. The ſame happens, if we would *take all*, (or *leave none*.) For here alſo there can be no variety of choice, whatever be the number of things expoſed, *a*, *b*, *c*, &c. And this, in the table, we expreſs in the firſt (erect) column, where alſo the numbers are all *Monadicks*.

3. If we would *take One*, it is manifeſt, that there are as many caſes or varieties of choice, as is the number of things. For that One may be any one of them, as *a*, *b*, *c*, *d*, *e*, *f*, *g*; which is expreſſed in the ſecond line, where the numbers are in their natural order or conſecution, 1, 2, 3, &c, which I call *Laterals*.

4. The ſame happens, if, taking all the reſt, we *leave One*; that is, if we take *All but One*. For it is manifeſt, there is the ſame variety of *leaving One* as of *taking One*, as *abcdef*, *abcdeg*, *abcdfg*, *abcefg*, *abdefg*, *acdefg*, *bcdefg*, which is ſignified in the ſecond column, where the numbers are alſo *Laterals*.

5. If we would take *Two*; It is manifeſt, that we may firſt take *a*, combined with any other of the reſt; as *ab*, *ac*, *ad*, *ae*, *af*, *ag*; the number of which combinations are therefore as many as the number of things wanting one. We may then take *b* (omitting its combination with *a*, as being already taken,) combined with every of thoſe which follow it; as *bc*, *bd*, *be*, *bf*, *bg*; the number of which combinations are therefore as many as the number of things expoſed, *wanting Two*. In like manner, *c*, (omitting its combinations with *a* and *b*, becauſe *ca*, *cb*, are but the ſame with *ac*, *bc*, already taken,) may be further combined with every of thoſe which follow it, (which are ſo many as is the number of things expoſed *wanting Three*,) as *cd*, *ce*, *cf*, *cg*; and the fourth *d*, (omitting *da*, *db*, *dc*, as being the ſame with *ad*, *bd*, *cd*, already taken,) may be further combined with every of thoſe which follow, (which are as many as the number of things *wanting Four*,) as *de*, *df*, *dg*. And in like manner for the fifth, ſixth, &c; each of which affords new combinations fewer by one than that next before it, till at length we come to 1, as *ef*, *eg*, and *fg*. So that the number of all theſe combinations, is the aggregate of all the numbers in the ſame line ſo far; that is, in the preſent caſe, (the num-

ab	*ac*	*ad*	*ae*	*af*	*ag*	6
	bc	*bd*	*be*	*bf*	*bg*	5
		cd	*ce*	*cf*	*cg*	4
			de	*df*	*dg*	3
				ef	*eg*	2
					fg	1
						21

ber of things expoſed being 7,) the combinations are, 6 + 5 + 4 + 3 + 2 + 1 = 21. To which anſwers (in the third tranſverſe, or horizontal line of the foregoing Table of the Figurate numbers,) the *Triangular* number 21, juſt under the number 6, (which is leſs by one than the number of things expoſed.) Such *Triangular* numbers, being the aggregate of all the *Laterals* ſo far. And univerſally, (whatever be the number of things expoſed) the number of Two's, is *a Triangular number, whoſe ſide is leſs by one than the number of things expoſed.*

6. The ſame happens, if we are to take *All but Two*; for there is the ſame variety of leaving Two, as of taking Two; that is, in both caſes, ſo many as is the triangular number, whoſe ſide is leſs by one than the number of things expoſed, which (in the table,) is ſignified in the third column, whoſe numbers are the ſame with thoſe of the third line.

7. If we would take *Three*, it is manifeſt, that firſt, *ab*, (the firſt and ſecond,) may be combined with every of thoſe that follow; the number of which are as many as the things expoſed *wanting Two*, (which therefore afford us ſo many different Triads, or Three's,) as *abc*, *abd*, *abe*, *abf*, *abg*. Then that *ac* (the firſt and third,) may be combined (omitting *acb*, as being the ſame with *abc* already taken,) with every of thoſe that follow, (which therefore afford us ſo many new *Three's*, as is the number of things *wanting Three*,) as *acd*, *ace*, *acf*, *acg*. And, in like manner, *a* coupled with thoſe that follow, (as *ad*, *ae*, *af*,) may each of them be further combined with their reſpective ſubſequents, affording each of them new Triads, fewer by one than that next before it, till at length we come to 1, as *ade*, *adf*, *adg*, and *aef*, *aeg*, and *afg*. (But *ag* affords none, becauſe *g* being the laſt, there is none remaining with which it might be combined.) The aggregate of all which, is a Triangular number (as being an aggregate of Laterals,) whoſe ſide is leſs by two, than the number of things expoſed; that is, in the

abc abd abe abf abg	5
acd ace acf acg	4
ade adf adg	3
aef aeg	2
afg	1
	15

preſent caſe, 5 + 4 + 3 + 2 + 1 = 15; which is a triangular number of the ſide 5, which is leſs by two, than 7, the number of things expoſed, in all which, *a* is one of the Ingredients.

In like manner (omitting all the Triads wherein *a* is an Ingredient, as being already taken,) *bc* (the ſecond and third) may be further combined with each of thoſe that follow *d*, *e*, *f*, *g*, affording us as many new Triads as did *ac*, (which was before ſo compounded,) that is, ſo many as is the number of things *wanting Three*. And then again *bd*, *be*, *bf*, afford as many as *ad*, *ae*, *af*, did before. Which afford us a new Triangular number, whoſe ſide is leſs by one than that we had before; that is, 4 + 3 + 2 + 1 = 10, whoſe ſide is 4; in all which Triads *b* is the leader.

bcd	*bce*	*bcf*	*bcg*	4
	bde	*bdf*	*bdg*	3
		bef	*beg*	2
			bfg	1
				10

In the ſame manner may be ſhewed, that (omitting the combinations of *a* and *b*,) thoſe Triads wherein *c* is the leader, will give another Triangular number, whoſe ſide is yet leſs by one, and ſo onward continually till we come at 1: as 3 + 2 + 1 = 6, a triangular number whoſe ſide is 3; and 2 + 1 = 3, a Triangular number, whoſe ſide is 2; and 1, a Triangular number whoſe ſide is alſo 1.

cde	*cdf*	*cdg*	3
	cef	*ceg*	2
		cfg	1
			6

def	*deg*	2
	dfg	1
		3

efg	1
	1

And then the aggregate of theſe Triangulars is 35, a *Pyramidal* number, which (in the fourth line,) ſtands next under 15, the greateſt of them, whoſe ſide is leſs by two, than the number of things expoſed; that is, a *Pyramidal number whoſe ſide is leſs by two than the number of things expoſed*. And ſo many are the ſeveral Triads which may be had in ſuch number of things expoſed; that is, in the preſent caſe, 15 +

5, 4, 3, 2, 1	15
4, 3, 2, 1	10
3, 2, 1	6
2, 1	3
1	1
	35

$10 + 6 + 3 + 1 = 35$, which is repreſented in the fourth line, which is of pyramidal numbers.

8. The ſame happens, if inſtead of taking Three, we take *All but Three*. For the ſame variety of caſes happens, if now we take what were before left, and leave what were then taken. And as that is repreſented in the fourth line, ſo this is in the fourth column.

9. If we would take *Four*; then, with a, may be made ſo many Fours (or Quaternions,) as may be formed Triads of thoſe that follow, (as b, c, d, e, f, g,) that is, (by art. 7,) a pyramidal number whoſe ſide is leſs by Two than the number of theſe; that is, leſs by Three than the number of things expoſed; that is, in the preſent caſe, 20; which is a Pyramidal number of the ſide 4, which is leſs by Three, than 7.

In like manner, (omitting a,) there may with b, be ſo many Quaternions formed, as may be Triads of thoſe that follow it, (as c, d, e, f, g;) that is, a Pyramidal number whoſe ſide is leſs by 1, than that foregoing; that is, 10; whoſe ſide 3 is leſs by 4 than 7.

And (omitting a, b,) there may with c be formed ſo many Quaternions, as may of thoſe that follow it (d, e, f, g,) be formed Triads; that is, a Pyramidal number whoſe ſide is yet leſs by 1: that is, 4, whoſe ſide is 2. And ſo onward, till we come at 1.

And then the aggregate of all theſe Pyramidals, that is,
the number in the fifth line, next under the greateſt of
20 them, is (what they call) a *Trianguli-triangular* number,
10 whoſe ſide is leſs by three than the number of things
4 expoſed. That is, in the preſent caſe, (where the
1 number of things is 7,) $20 + 10 + 4 + 1 = 35$, (a
35 trianguli-triangular number, of the ſide $4 = 7 - 3$,)
is the number of different Quaternions which may be
had when the things expoſed are 7.

(If any like not the name of *trianguli-triangular*, and ſo of the reſt that follow; I am content to change them. For I am not fond of them, but uſe them becauſe I find them.)

Which number is the ſame which before we had for Three's; which hence comes to paſs, becauſe, when the number of things is 7, the number 4, is the ſame with *All wanting* 3; where the variety is the ſame as if 3 were taken; as is ſhewed in the preceeding article.

10. The ſame happens, (for the reaſons already ſhewed,) if we were to take *All wanting Four*. And as that is to be found in the fifth line, ſo this, in the fifth column, whoſe numbers are the ſame with thoſe of the fifth line.

11. In the ſame manner will be ſhewed, that, if we would take *Five* (or *All but Five,*) the varieties are then ſo many as is the aggregate of the numbers in the fifth line, ending with that whoſe ſide is *leſs by Four* than the number of things expoſed. That is, the number in the ſixth line (which is *Triangulipyramidals*) next under the greateſt of thoſe, whoſe ſide is *leſs by Four* than the number of things expoſed. That is, in the preſent caſe, 15 + 5 + 1 = 21, a *Triangulipyramidal* number, whoſe ſide is 3 = 7 — 4. And ſo, if *Six* are to be taken, (or *All but Six,*) the varieties are ſo many, as is the aggregate of numbers in the ſixth line (or the number anſwering thereunto in the ſeventh,) ending with that whoſe ſide is *leſs by Five* than the number of things expoſed. And ſo for *Seven, Eight,* &c, (or all but ſeven, eight, &c,) we are to take the numbers of the following lines, ending with that whoſe ſide is leſs by one, than that for the line next above. As, in the preſent, (where 7 is the number of things expoſed,) the number of ſixes is 7; the number of ſevens is 1.

$$\begin{array}{r}15\\5\\1\\\hline 21\end{array}$$

$$\begin{array}{r}6\\1\\\hline 7\end{array}$$

12. All theſe varieties of choice, for any number of things expoſed, are found in the Table foregoing, in a rank of numbers obliquely deſcending; in which that number which is the number of things expoſed, is to be found in the ſecond line, and again in the ſecond column, both which are of Laterals. As, in the preſent caſe (where 7 is the number of things expoſed,) in the oblique deſcent paſſing

by 7 in the ſecond line, and again in the ſecond column; we have the numbers 1, 7, 21, 35, 35, 21, 7, 1; which repreſent the variety of caſes for taking, 0, 1, 2, 3, 4, 5, 6, 7. And the like for any other number of things expoſed.

13. And theſe numbers (as appears upon view,) are the ſame with thoſe which are called *unciæ*, prefixed to the proportionals that conſtitute the reſpective powers of a binomial root; or, (which is the ſame) the reſpective powers of 1 + 1 conſidered as a binomial root. That is, the root, ſquare, cube, fourth, fifth power, &c, of 1 + 1, according as the number of things expoſed are 1, 2, 3, 4, 5, 6, &c.

14. The table thus begun, is eaſily continued as far as there is occaſion: for the number of each place, is the aggregate of two numbers, whereof one is next above it, and the other next before it, as 15 = 5 + 10, 20 = 10 + 10, 35 = 20 + 15. And ſo every where.

15. Having therefore any number of things expoſed, let that number be ſought in the ſecond line, (which is of Laterals,) and again in the ſecond column; and then, in the ſloping rank of numbers paſſing through theſe two, we have the number of caſes for taking 0, 1, 2, 3, 4, &c, in ſuch order as the index on the ſide directs; and likewiſe for taking *All but* 0, 1, 2, 3, 4, &c, in ſuch order as the index on the top directs.

16. And if we would have the ſum of all theſe varieties (for any ſuch number of things propoſed) all together, it is had by adding the numbers of ſuch ſloping rank; as in the preſent caſe, 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.

17. Which number is always that power of the number 2, (that is, of 1 + 1) which is of ſo many dimenſions as is the number of things expoſed, (or that power whoſe exponent

is ſuch number,) that is, the product of ſo many two's continually multiplied, (as, in the preſent caſe, 2 × 2 × 2 × 2 × 2 × 2 × 2 = 128;) or, 1 ſo many times doubled as is the number of things expoſed. That is, for 0, it is 1. (For here, to take all, or to leave all, is but one and the ſame caſe.) For 1, it is (the ſide) 1 + 1. For 2, (the ſquare) 1 + 2 + 1 = 4. For 3, (the cube) 1 + 3 + 3 + 1 = 8. For 4, (the biquadrate) 1 + 4 + 6 + 4 + 1 = 16. And ſo of the reſt.

18. And thus far we have conſidered the variety of caſes concerning taking or leaving, *None*, *One*, *Two*, *Three*, &c, of any number of things expoſed, without regarding the order of them, ſo that *abc*, *acb*, *bac*, *bca*, &c, are reputed for one and the ſame caſe. But if the different alternations, or changes of order, in the ſame things, be accounted as different caſes; this we are to conſider in the next Chapter. And if therein, ſome two or more are indifferently reputed as one and the ſame, or indifferently to be taken each for other; what abatement of the former number will hereupon ariſe, is conſidered in the ſame Chapter.

19. If, by *Combination*, we underſtand the taking of two or more, (but not of one, or none;) then, out of the number of caſes before found, we muſt abate ſo many as is the number of things expoſed, and one more. For, of thoſe, ſo many as is the number of things expoſed, anſwers to the caſes of 1. And one more, anſwers to the caſe of taking *None*. But all the reſt are combinations in that ſenſe. For though *Combination* (as coming from *Bini*,) in its proper ſignification extend only to the taking of *couples*, (or Two's;) yet in common acceptation the word is now uſed of greater numbers. And, in Engliſh alſo, we ſcruple not to ſay, that Three, or Four, (or more than ſo,) are *coupled* together, that is, connected.

20. If, out of the former number of caſes, we pleaſe to exclude that of *taking None*, or 0, (becauſe, *to take none*, is *not to take*,) then is the number of caſes fewer by one,

than is above expreſſed. And ſo we have the caſes of taking one or more. And ſo many are the number of *Diviſors* of a number compounded of ſo many different Prime numbers continually multiplied, as are the caſes of taking one or more of ſo many things expoſed.

21. And if further we abate one more (which anſwers to the caſe of taking all;) then have we the number of *Aliquot Parts* of a number ſo compoſed of different Primes or Incompoſite numbers. The number of *Aliquot Parts* being fewer by one, than is the number of diviſors.

I ſhall ſubjoin to this Chapter (as properly appertaining to this place,) an Explication of the *Rule of Combination*, which I find in *Buckley*'s Arithmetick, at the end of *Seton*'s Logick, (in the Cambridge edition;) which (becauſe obſcure,) Mr. George Fairfax (a Teacher of the Mathematicks then in Oxford,) deſired me to explain; to whom (Sept. 12, 1674,) I gave the explication under written; Conſonant to the doctrine of this Treatiſe, (which had been long before written, and was the ſubject of divers public Lectures in Oxford, in the years 1671, 1672.)

REGULA COMBINATIONIS.

Quot fuerint Numeri, quos Combinare velimus;
Tot ſint et ſeries, quibus eſt proportio dupla;
Quarum principium ducatur ſemper ab Uno.
Omnes has ſeries conjunge per Additionem.
Producto, numerum quot Combinatio conſtat,
Aufer. Quod ſupereſt, numerum citat; unde patebit,
Quot faciant numeros diſtinctos, undique ſiquis
Propoſitos numeros velit in ſe Multiplicare.
Si nihil à ſummâ prædictâ ſurripiatur;
Reſtabunt partes Aliquotæ, quæ numerabunt
Illum, qui numeros eſt inter Maximus omnes,
Ex ductu in ſeſe numerorum provenientem.

I have taken the liberty, to alter the pointing (ſo as to make the ſenſe the clearer,) and to reſtore (in the ſecond verſe)

verſe) *ſint*, for *ſunt*; and (in the third verſe) *principium*, for *principio*; which had been miſprinted. And (in the fifth verſe) I have reſtored *numerum*, for *numeros*; for it is but one Number that is to be ſubducted, namely, the Number of thoſe Numbers which are to be Combined. My Explication was this:

" Let as many Numbers as you pleaſe, be propoſed to be " Combined: Suppoſe *Five*, which we will call *a b c d e*.

1	*a*	*ab*	*abc*	*abcd*	*abcde*	
2	*b*	*ac*	*abd*	*abce*	—	
4	*c*	*ad*	*abe*	*abde*	1	
8	*d*	*ae*	*acd*	*acde*		
16	*e*	*bc*	*ace*	*bcde*		
—	—	*bd*	*ade*	—	5	
31	5	*be*	*bcd*	5	10	10
-5		*cd*	*bce*		10	10
—		*ce*	*bde*		5	5
26		*de*	*cde*		1	1
		—	—		—	—
		10	10		31	26

" Put, in ſo many Lines, Numbers, in duple proportion, " beginning with 1.

" The Sum (31) is the Number of Sumptions, or Elections; " wherein, one or more of them, may ſeveral ways be " taken.

" Hence ſubduct (5) the Number of the Numbers pro- " poſed; becauſe each of them may once be taken ſingly.

" And the Remainder (26) ſhews how many ways they " may be taken in Combination; (namely, Two or more at " once.)

" And, conſequently, how many Products may be had " by the Multiplication of any two or more of them ſo " taken.

" But the ſame Sum (31) without ſuch Subduction, ſhews " how many Aliquot Parts there are in the greateſt of thoſe " Products, (that is, in the Number made by the continual " Multiplication of all the Numbers propoſed,) *a b c d e*. " For every one of thoſe Sumptions, are Aliquot Parts of

" *abcde*, except the laſt, (which is the whole,) and inſtead thereof, 1 is alſo an Aliquot Part; which makes the number of Aliquot Parts, the ſame with the Number of Sumptions.

" Only here is to be underſtood, (which the Rule ſhould have intimated;) that, all the Numbers propoſed, are to be Prime Numbers, and each diſtinct from the other. For if any of them be Compound Numbers, or any Two of them be the ſame, the Rule for Aliquot Parts will not hold."

CHAP. II.

Of Alternations; or the different Change of Order, in any Number of Things propoſed.

SUPPOSE we a certain Number of things expoſed, different each from other, as *a*, *b*, *c*, *d*, *e*, &c. The queſtion is, how many ways the order of theſe may be varied? as, for inſtance, how many changes may be Rung upon a certain Number of Bells; or, how many ways (by way of Anagram) a certain Number of (different) Letters may be differently ordered?

$$\left.\begin{matrix}a\end{matrix}\right\}1$$

1. If the thing expoſed be but *One*, as *a*, it is certain, that the order can be but one. That is 1.

$$\left.\begin{matrix}ab\\ ba\end{matrix}\right\}2$$

$$\overline{1\times2=2}$$

2. If *Two* be expoſed, as *a*, *b*, it is alſo manifeſt, that they may be taken in a double order, as *ab*, *ba*, and no more. That is 1 × 2 = 2.

3. If *Three* be expoſed; as *a*, *b*, *c*: Then, beginning with *a*, the other two *b*, *c*, may (by art. 2,) be diſpoſed according to Two different orders, as *bc*, *cb*; whence ariſe Two Changes (or varieties of order) beginning with *a*, as *abc*, *acb*: And, in like manner it may be ſhewed, that there be as many beginning with *b*; becauſe the other two, *a*, *c*, may be ſo varied, as *bac*, *bca*. And again as many beginning with *c*, as *cab*, *cba*. And therefore, in all, Three times Two. That is, $1\times2,\times3=6$.

$$\begin{array}{l}\left.\begin{array}{l}abc\\acb\end{array}\right\}2\\ \left.\begin{array}{l}bca\\bac\end{array}\right\}2\\ \left.\begin{array}{l}cab\\cba\end{array}\right\}2\\ \hline 2\times3=6\end{array}$$

4. If *Four* be expoſed, as *a*, *b*, *c*, *d*; Then, beginning with *a*, the other Three may (by art. preceeding) be diſpoſed ſix ſeveral ways. And (by the ſame reaſon) as many beginning with *b*, and as many beginning with *c*, and as many beginning with *d*. And therefore, in all, Four times ſix, or 24. That is, the Number anſwering to the caſe next foregoing, ſo many times taken as is the Number of things here expoſed. That is, $1\times2\times3,\times4=6\times4=24$.

5. And in like manner it may be ſhewed, that this Number 24 Multiplied by 5, that is $120=24\times5=1\times2\times3\times4\times5$, is the number of alternations (or changes of order) of *Five* things expoſed. (Or, the Number of Changes on Five Bells.) For each of theſe five being put in the firſt place, the other four will (by art. preceeding) admit of 24 varieties, that is, in all, five times 24. And, in like manner, this Number 120 Multiplied by 6, ſhews the Number of Alternations of 6 things expoſed; and ſo onward, by continual Multiplication by the conſequent Numbers 7, 8, 9, &c.

$$\begin{array}{l}\left.\begin{array}{l}abcd\\abdc\\acbd\\acdb\\adbc\\adcb\end{array}\right\}6\\ \left.\begin{array}{l}bacd\\badc\\bcad\\bcda\\bdac\\bdca\end{array}\right\}6\\ \left.\begin{array}{l}cabd\\cadb\\cbad\\cbda\\cdab\\cdba\end{array}\right\}6\\ \left.\begin{array}{l}dabc\\dacb\\dbac\\dbca\\dcab\\dcba\end{array}\right\}6\\ \hline 4\times6=24\end{array}$$

6. That is, how many ſo ever of Numbers, in their natural Conſecution, beginning from 1, being continually Multiplied,

tiplied, give us the Number of Alternations (or Change of order) of which ſo many things are capable as is the laſt of the Numbers ſo Multiplied. As for inſtance, the Number of Changes in Ringing Five Bells, is 1 × 2 × 3 × 4 × 5 = 120. In Six Bells, 1 × 2 × 3 × 4 × 5 × 6 = 120 × 6 = 720. In Seven Bells, 720 × 7 = 5040. In Eight Bells, 5040 × 8 = 40320. And ſo onward, as far as we pleaſe.

Thus *Voſſius* tells us, (*Cap.* 7, *De Scientiis Mathematicis,*) That if an Hoſt promiſe to entertain ſeven Gueſts ſo long as they ſit every day in a different order, this extends to 14 years. He means, *almoſt* ſo many years, namely, 5040 days, which of 14 years wants 73 or 74 days, according as the Leap-years may chance to fall.

7. This Number of Alternations, according as the Number of things expoſed doth increaſe, will proceed to a vaſt Multitude beyond what at firſt one would expect. As for Example, the 24 Letters will admit of ſo many Varieties or Alternations in Changing their order, as that if ſo many Bells were to be Rung according to all thoſe Changes, it could not have been diſpatched (as the Learned *John Gerard Voſſius*, in the place laſt cited, doth obſerve,) from the beginning of the World to this day. I add; no, nor if for every Minute of an hour which hath paſſed, there had paſſed *Ten Thouſand Thouſand Years*; as will appear by the following Computation.

1	1	×
2	2	×
6	3	×
24	4	×
120	5	×
720	6	×
5,040	7	×
40,320	8	×
362,880	9	×
3,628,800	10	×
39,916,800	11	×
479,001,600	12	×
6,227,020,800	13	×
87,178,291,200	14	×
1,307,674,368,000	15	×
20,922,789,888,000	16	×
355,687,428,096,000	17	×
6,402,373,705,728,000	18	×
121,645,100,408,832,000	19	×
2,432,902,008,176,640,000	20	×
51,090,942,171,709,440,000	21	×
1,124,000,727,777,607,680,000	22	×
25,852,016,738,884,976,640,000	23	×
620,448,401,733,239,439,360,000	24	×

```
    In 1 year.
    365¼ days.
     × 24
    ----
    1460
    730
      6
    ----
    8766 hours.
    × 60
   -------
  525,960 Minutes
  In 6000 years.
  -------------
3,155,760,000 Minutes
         × 5
  -------------
15,778,800,000 Changes.
  525,960 Min. in 1 year.
  -------------------
 9467280000000
1420092
788940
315576
788940
--------------------
8,299,017,648,000,000
         10,000,000
--------------------------------
82,990,176,480,000,000,000,000
```

For, ſuppoſing in one year, 365¼ days; and, from the beginning of the World, to have paſſed 6000 years; (both of which ſuppoſitions are at the largeſt,) and therefore the Number of minutes in all that time, 3,155,760,000. Suppoſe we then, in every Minute of an hour, 5 Changes to be diſpatched, that is, (becauſe of 24 Bells) 120 ſtrokes ſucceſſively one after another, (which allowance is alſo at the largeſt;) and therefore, in 6000 years, 15,778,800,000 Changes, which Number if we Multiply by 525,960, (the Number of Minutes in one year,) we have 8,299,017,648,000,000 for the Number of Changes to be diſpatched in ſo many years as there have been Minutes, which Multiplied by 10,000,000, (Ten Thouſand Thouſand, or Ten Millions,) will be but 82,990,176,480,000,000,000,000, which is leſs than 620,448,401,733,239,439,360,000, the Number of Changes whereof 24 Bells are capable.

Nay, if we ſhould proceed no further than to 14 Bells, and allow 10 Changes (that is, 140 ſtrokes) to every Minute, the Number of Minutes requiſite to Ring them all would be 8,717,829,120, (a tenth part of the Number of Changes,) which is more than double (almoſt treble) the Number of Minutes in 6000 Years; and would require more than 16 Thouſand Years (yea, more than 16,575 Years) to Ring them all.

8. Hence it may appear, how many ways the Letters of a Name or Word, (ſuppoſing them to be all ſeveral,) may be differently diſpoſed by way of Anagram, (out of which thoſe that are of uſe may be ſelected, neglecting the reſt,) by art. 6. For Example, the Word *R O M A*, (conſiſting of four different Letters) may admit of Changes $24 = 1 \times 2 \times 3 \times 4$.

Roma
roam
rmoa
rmao
raom
ramo

orma
oram
omra
omar
oarm
oamr

mroa
mrao
mora
moar
maro
maor

arom
armo
aorm
aomr
amro
amor

Of which (in Latin) theſe ſeven are only uſeful; *Roma, ramo, oram, mora, maro, armo, amor.* The other forms are uſeleſs, as affording no Latin Word of known ſignification.

9. But in caſe ſome one or more of the Letters do occur more than once, the Number of Alternations ſo found as before, muſt be divided by ſuch Number or Numbers as ſuch repetitions do require: Namely, if the ſame Letter do twice occur, we are to divide by 2; if three times, by 6; if four

times, by 24; and ſo onward, according to the varieties that ſuch a Number is capable of. For, if the Letters *a* and *b* be reputed for the ſame; then, whereas (the reſt remaining as before) *ab* and *ba* would every where afford two varieties, they are in this caſe to paſs for one, and therefore the Number of caſes will be only half ſo many as otherwiſe they would be. In like manner (the reſt remaining as before) *abc* would every where (according as they may change places one with another) afford ſix varieties; but in caſe the three Letters *a*, *b*, *c* be conſidered as being all the ſame, or as being *a*, *a*, *a*, theſe Six caſes muſt then paſs but for one. And in like manner, if *abcd* be ſeverals, they afford (the reſt remaining as before) 24 varieties; but, if the ſame, theſe 24 muſt paſs but for one: And the like in other caſes. And, if more Letters be ſo repeated, there muſt be for each of them ſuch diviſion.

For Example, the Word *MESSES* having 6 Letters, if they were all different, the Alternations would be $720 = 1\times2\times3\times4\times5\times6$. But becauſe the Letter *e* comes twice, that Number is to be divided by 2. (For if inſtead of *ee*, we put *εη*, then *mεssηs* and *mηssεs* would be two forms, both which are now Co-incident in *messes*: And ſo every where.) Again, becauſe the Letter *s* comes three times, we are (for the like reaſon) to divide by 6. (For if thoſe three were three different, they would in every poſition of the reſt, afford 6 caſes, all which are now Co-incident in *sss*.) And therefore, (becauſe both happen,) 720 being divided by 2, and again by 6, the different varieties will be $\frac{720}{2\times6} = 60$.

meesss
mesess
messes
messse
mseess
mseses
msesse
mssees
mssese
msssee

emesss
emsess
emsses
emssse
eemsss
eesmss
eessms
eesssm
esmess
esmses

esmsse
esemss
esesms
esessm
essmes
essmse
essems
essesm
esssme
esssem

smeess
smeses
smesse
smsees
smsese
smssee
semess
semses
semsse
seemss

seesms
seessm
sesmes
sesmse
sesems
sesesm
sessme
sessem
ssmees
ssmese

ssmsee
ssemes
ssemse
sseems
sseesm
ssesme
ssesem
sssmee
ssseme
ssseem

Of all which varieties, there is none beſide *meſſes* itſelf, that affords an uſeful Anagram.

In like manner we may ſhew, that the Letters *abbcccdddd* will admit of $\frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10=3628800}{2\times 6\times 24=288}=12600$ ſeveral varieties: And *abbccdd*, of, $\frac{1\times 2\times 3\times 4\times 5\times 6\times 7=5040}{2\times 2\times 2=8}=630$: And *aaabbccc*, of $\frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8=40320}{6\times 2\times 6=72}=560$. And the like in other caſes however varied.

10. The converſe of this, is of like uſe, when what was conſidered but as one and the ſame ſeveral times repeated, comes afterward to be diſtinguiſhed. For then the Number before found, is to be ſo often Multiplied as the Number of things ſo diſtinguiſhed ſhall require.

As, in the Word *meſſes* before mentioned, where *sss* are conſidered but as one Letter thrice repeated, and *ee* as the ſame twice repeated, the Number of different poſitions is 60; but if *sss* be diſtinguiſhed as three ſeverals, and *ee* as two ſeverals, the Number of all will be $60\times 6\times 2=720$.

Thus *Voſſius*, Cap. 7, *De Scientiis Mathematicis*, tells us that this verſe,

Rex, lex, ſol, lux, dux, fons, mons, ſpes, pax, petra, Chriſtus.

which (conſiſting of 11 Words) may be turned (abſolutely) 39,916,800 ways; and ſo as to preſerve the Rules of an Hexameter verſe, be turned 3,628,800 ways, he ſhould rather have ſaid 3,265,920. That is, the 9 Monoſyllables (which may promiſcuouſly take each other's place) 362,880 times; and *Chriſtus* is capable of 9 (not 10) different poſitions; that is, in the firſt, ſecond, third, fourth, fifth, ſixth, ſeventh, eighth, (but not in the ninth, and tenth,) and in the laſt place; (and *petra* confined, by the nature of the verſe, to the place next before the laſt ſpondee,) That is, $362,880\times 9=3,265,920$ ways:

He ſays alſo that the verſe

Tot tibi ſunt dotes, virgo, quot ſidera cælo;

may be turned abſolutely 40320 ways; and, ſo as to preſerve the verſe 1022 ways; which is very true, (and I have been told, of ſomebody, who, in praiſe of the Virgin *Mary*, had made a Book of that verſe turned ſo many ways, which was wont to be reputed the Number of the Fixed Stars, according to the ancient Catalogue of them.) But it is true alſo, that it may be turned many more ways than ſo, and yet preſerve the verſe true: Namely, 2628, retaining the quantity of the laſt Syllables in *tibi* and *virgo* as before; and 468, Changing their quantity in *virgo tibi*. That is, in all 3096 ways. As will appear by the Scheme adjoyned, and the brief Explication, (or Demonſtration) of it: which is thus to be underſtood.

Tot, *ſunt*, *quot*, which may promiſcuouſly ſupply each other's place, are (in verſe 1, 2, 3, &c,) ſet down only in this order, and ſo paſs but for one caſe; but are capable of ſix varieties; which caſe I call $a = 6$. And the like for *dotes*, *virgo*, *cælo*; which caſe I call $b = 6$. And again, *tot tibi* may change place with *ſidera*; which caſe I call $c = 2$: And, becauſe all theſe happen in verſe 1, the varieties thereby repreſented, are $abc = 72 = 6 \times 6 \times 2$. And ſo of the reſt, as the Scheme directs.

1.	*Tot*	*tibi*	*ſunt*	*dotes*	*virgo*	*quot*	*ſidera*	*cælo.*	abc	= 72
2.					*quot*	*virgo*			$abcd$	= 144
3.				*quot*	*dotes*				ace	= 1152
4.			*dotes*	*ſunt*	*virgo*	*quot*			$abcf$	= 144
5.		*ſunt*	*dotes*	*quot*	*virgo*	*tibi*			agh	= 180
6.			*quot*	*dotes*	*tibi*	*virgo*			abi	= 324
7.					*virgo*	*tibi*			abi	= 324
8.		*dotes*	*tibi*	*ſunt*	*virgo*	*quot*			ab	= 36
9.					*quot*	*virgo*			ak	= 108
10.			*ſunt*	*virgo*	*quot*	*tibi*			$ablm$	= 144
									virgo tibi	2628

11.	*Virgo tibi tot ſunt dotes quot ſidera cælo.*	*an* =	36
12.	*quot dotes*	*an* =	36
13.	*dotes ſunt quot*	*ao* =	36
14.	*Tot ſunt virgo tibi dotes*	*an* =	36
15.	*quot dotes*	*an* =	36
16.	*dotes virgo tibi quot*	*an* =	36
17.	*Tot dotes ſunt*	*ap* =	12
18.	*ſunt dotes ſidera cælo virgo tibi quot*	*aq* =	144
19.	*dotes ſunt*	*apr* =	24
20.	*cælo ſunt ſidera*	*ap* =	12
21.	*ſidera tot dotes ſunt cælo*	*apr* =	24
22.	*cælo ſunt*	*ap* =	12
23.	*dotes tot*	*apr* =	24
		virgo tibi	468
		tibi virgo	2628
			3096

Tot, ſunt, quot, $a = 6$. *dotes, virgo, cælo,* $b = 6$.

tot tibi, ſidera, $c = 2$. *tot tibi, virgo,* $d = 2$.

Tot tibi, ſunt quot, dotes, virgo, cælo; $e = 120 - 24 = 120 \times \frac{4}{5} = 96$.

(Becauſe *tot tibi* cannot ſupply the place of *cælo*, as of the reſt.) *Tot tibi, dotes,* $f = 2$. *Tot ſunt, dotes, virgo, cælo,* $g = 24$. *Quot tibi, ſidera,* $k = 1\frac{1}{4}$.

(Becauſe when *tot ſunt*, or its Equivalent *ſunt quot*, comes next before *tibi*, which is a fourth part of the caſes contained in g, then will *quot tibi*, change with *ſidera*; which adds $\frac{1}{4}$ of what was before.) *Tot ſunt*, (and *ſunt quot*,) *dotes, virgo, cælo,* $i = 9$.

(Becauſe *dotes, virgo, cælo,* contained in b, may each of them change with *tot ſunt*, which Multiplies by 4, or adds a Triple to what was before, as at g, and $\frac{1}{3}$ of that Triple, or $\frac{1}{4}$ of that Quadruple, as at b; that is, it adds a Quadruple or Multiplies by 5: And again, each of them with *ſunt quot*, which, for the ſame reaſon, adds another Quadruple: Therefore both together, add an Octuple, or Multiply by 9.)

Dotes, ſunt quot, virgo, cælo, $k = 24 - 6 = 24 \times \frac{3}{4} = 18.$

(Becauſe, if *ſunt quot* ſupply the place of *dotes*, it will be Co-incident with ſome of the caſes of ver. 3.)

Quot tibi, ſidera, $e = 2.$
tot ſunt, dotes, cælo, $n = 6.$
dotes, cælo, $p = 2.$
ſidera, cælo, $r = 2.$

virgo, quot tibi, $m = 2.$
dotes, ſunt quot, cælo, $o = 6.$
tot ſunt, dotes, ſidera, cælo, $q = 24.$

I will not be poſitive that there may not be ſome other Changes: (and then, thoſe may be added to theſe:) Or, that moſt of theſe be twice repeated, (and if ſo, thoſe are to be abated out of the number:) But I do not, at preſent, diſcern either the one and other *.

CHAP. III.

Of the Diviſors, and Aliquot Parts, of a Number propoſed.

1. BY *Number*, I here underſtand only Integer Numbers, as 1, 2, 3, 4, 5, &c. Not Fractions, as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, &c. Or Mixed, as $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{2}{5}$, &c. Much leſs Surds, as $\surd 2$, $\surd 5$, $\surd 6$, &c.

2. By the *Diviſor* of a Number, I here underſtand, ſuch Integer as doth meaſure ſuch Number; that is, being once or oftener taken doth equal it. So, of the Number 6, the Diviſors are, 1, 2, 3, 6: Becauſe 6, once taken; and 3, twice taken; and 2, thrice; and 1, ſix times taken; do equal 6.

$$1)6(6;\ 2)6(3;\ 3)6(2;\ 6)6(1. \quad 6 = 1 \times 6 = 2 \times 3 = 3 \times 2 = 6 \times 1.$$

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* The number of all the poſſible variations of the words in this Hexameter Verſe, without deſtroying the meaſure of it, has been inveſtigated with greater accuracy by Mr. *James Bernouilli*, in the ſecond part of his excellent Treatiſe, *De Arte Conjectandi*, and is there found to be 3312. See above, pages 8, 9, and 10.

3. By *Aliquot Part* of a Number, I underſtand ſuch a Diviſor as is leſs than it. As of 6, the Aliquot Parts are 1, 2, 3; but not 6. For, though 6 be alſo a Diviſor of itſelf; yet not an Aliquot Part; becauſe the Word *Part* implies ſomewhat leſs than the whole.

4. The Number of Aliquot Parts, therefore, is always leſs by one than the Number of Diviſors. Becauſe all the Diviſors except one, are Aliquot Parts; all the Aliquot Parts are Diviſors, and there is likewiſe one more Diviſor of the Number, to wit, the whole Number itſelf.

5. So that, the Number of Diviſors being given, the Number of Aliquot Parts is given alſo. And contrarywiſe; if this, then that. As, of the Number 6, the Diviſors being 4, the Aliquot Parts are 3, (that is, $4 - 1$.) And, theſe being 3, the Diviſors are $4 = 3 + 1$.

6. It is manifeſt, that the Number 1, hath no Aliquot Part, and but one Diviſor, that is 1. Becauſe there is no Number leſs than itſelf that may be a part of it: But it meaſures itſelf; and therefore is its own Diviſor.

7. Any other Prime Number hath one Aliquot Part, and Two Diviſors. For a *Prime Number*, we call, ſuch as is meaſured (beſide itſelf) by no other Number but an Unit. As 2, 3, 5, 7, 11, &c. Each of which are meaſured by 1, and by itſelf; but not by any other Number. And hath therefore 2 Diviſors, and 1 Aliquot Part; but no more.

8. Every *Power* of a *Prime Number* (other than of 1, which here is underſtood to be excluded,) hath ſo many Aliquot Parts as are the dimenſions of ſuch Power; and one Diviſor more than ſo. As (ſuppoſing a, b, c, &c, to be ſo many Prime Numbers;) a hath two diviſors (1 and a;) a^2 or aa hath three, (1, a, aa;) a^3, or aaa, hath four, (1, a, aa, aaa;) and ſo of the reſt. That is, the Number of Diviſors is one more than the Number of Dimenſions. Becauſe 1, and all the Degrees of ſuch Power (not higher than itſelf) are Diviſors; but not any other Number, if a be a Prime. That is, one more than the Number of Dimenſions: Of which the greateſt Diviſor (being the whole) is not an Aliquot Part;

and therefore the Aliquot Parts are juſt ſo many as are the Dimenſions. Thus of 8 (the Cube of 2) the Diviſors are four, (1, 2, 4, 8 ;) the Aliquot Parts are three, (1, 2, 4 ;) Of 81 (the Biquadrate of 3) the Diviſors are five, (1, 3, 9, 27, 81 ;) the Aliquot Parts are four, (1, 3, 9, 27,) juſt ſo many as are the Dimenſions. That is, (of ſuch Biquadrate) the Diviſors are 1, *a*, *aa*, *aaa*, *aaaa* ; the Aliquot Parts 1, *a*, *aa*, *aaa* ; and ſo every where : For, though the higheſt Dimenſion came not into the Number of Aliquot Parts, yet 1 being ſupernumerary, makes the Aliquot Parts juſt as many as the Dimenſions.

9. If a Prime Number, or any Power thereof, be Multiplied by any other Prime Number, or any Power hereof ; the Product hath ſo many Diviſors, as is the Number of Diviſors in That, Multiplied by the Number of Diviſors in This ; and, therefore, the Aliquot Parts fewer by one than ſo.

For Example : Let *a*, *b*, be two different Prime Numbers, (ſuppoſe 2, 3 ;) and certain Powers thereof, as a^3, b^2, (that is 8, 9,) the Product a^3b^2, (that is, 72 = 8 × 9.) Now for as much as the Diviſors of the former 1, *a*, *aa*, *aaa* ; (that is, 1, 2, 4, 8,) divide a^3 (that is 8 ;) not only theſe, or (which is the ſame) every of theſe Multiplied by 1 ; but alſo every of them Multiplied by *b*, and by *bb*, (that is by 3, and by 9,) will divide a^3b^2. That is, every of the Diviſors of a^3, Multiplied into every of the Diviſors of b^2 ; will divide a^3b^2.

$$\left.\begin{array}{rr} 1 & 1\\ a & 2\\ aa & 4\\ aaa & 8\end{array}\right\} 4. \quad \text{The Diviſors of } a^3 \text{ Multiplied by } 1.$$

$$\left.\begin{array}{rr} b & 3\\ ab & 6\\ aab & 12\\ aaab & 24\end{array}\right\} 4. \quad \text{The ſame Multiplied by } b.$$

$$\left.\begin{array}{rr} bb & 9\\ abb & 18\\ aabb & 36\\ aaabb & 72\end{array}\right\} 4. \quad \text{The ſame Multiplied by } bb.$$

$$12 = 4 \times 3.$$

The Number therefore of all; is the Number of 1, *a*, *aa*, *aaa*, (that is 4,) ſo many times taken as is the Number of 1, *b*, *bb*, (that is, 3 times;) That is, 4 × 3 = 12: The Number of Diviſors therefore is 12; and of Aliquot Parts, 11.

10. If a Product made by the Multiplication of different Prime Numbers, or of their Powers by one another, be further Multiplied by another Prime Number different from every of thoſe: The Number of Diviſors in this new Product, will be ſo many as is the Number of Diviſors in that firſt Product Multiplied by the Number of Diviſors in the new Multiplier.

For Example: The Number of Diviſors in the Product but now mentioned a^3b^2, is 12; (as is already ſhewed:) if therefore this be Multiplied by any other Prime Number, as *c* (ſuppoſe 5,) different from *a*, *b*; (whoſe Diviſors therefore are two, 1 and *c*;) the Diviſors of the Product a^3b^2c (that is, of 72 × 5 = 360) will be 12 × 2 = 24. Namely, all thoſe (before found) which divided a^3b^2, will alſo divide a^3b^2c; or (which is the ſame) all thoſe Multiplied by 1 (which is one of the Diviſors of *c*;) and the ſame alſo Multiplied by *c*, (which are as many more;) and therefore both together are twice as many; that is, 12 × 2 = 24. Namely, 1, *a*, *aa*, *aaa*; *b*, *ab*, *aab*, *aaab*; *bb*, *abb*, *aabb*, *aaabb*; *c*, *ac*, *aac*, *aaac*; *bc*, *abc*, *aabc*, *aaabc*; *bbc*, *abbc*, *aabbc*, *aaabbc*. That is, 1, 2, 4, 8; 3, 6, 12, 24; 9, 18, 36, 72: 5, 10, 20, 40; 15, 30, 60, 120; 45, 90, 180, 360.

And if, for the new Multiplier *c* = 5, where taken *cc* = 25, or *ccc* = 125; (the Number of whoſe Diviſors are 3 or 4;) the Number of Diviſors of the Product $a^3b^2c^2$, or $a^3b^2c^3$, would (accordingly) be 12 × 3 = 36, or 12 × 4 = 48. (And, in like manner, for any other Power of *c*.) For now not only the Diviſors of a^3b^2 Multiplied, by 1, and by *c*; but the ſame alſo Multiplied by *cc*, (which is a third time ſo many,) will be Diviſors of $a^3b^2c^2$; and the ſame Multiplied by *ccc*, (which is as many a fourth time,) will be Diviſors of $a^3b^2c^3$.

$$\underbrace{\begin{array}{cccc}
\left.\begin{array}{rr} 1 & 1\\ a & 2\\ aa & 4\\ aaa & 8\end{array}\right\}4 &
\left.\begin{array}{rr} c & 5\\ ac & 10\\ aac & 20\\ aaac & 40\end{array}\right\}4 &
\left.\begin{array}{rr} cc & 25\\ acc & 50\\ aacc & 100\\ aaacc & 200\end{array}\right\}4 &
\left.\begin{array}{rr} ccc & 125\\ accc & 250\\ aaccc & 500\\ aaaccc & 1000\end{array}\right\}4 \\
\left.\begin{array}{rr} b & 3\\ ab & 6\\ aab & 12\\ aaab & 24\end{array}\right\}4 &
\left.\begin{array}{rr} bc & 15\\ abc & 30\\ aabc & 60\\ aaabc & 120\end{array}\right\}4 &
\left.\begin{array}{rr} bcc & 75\\ abcc & 150\\ aabcc & 300\\ aaabcc & 600\end{array}\right\}4 &
\left.\begin{array}{rr} bccc & 375\\ abccc & 750\\ aabccc & 1500\\ aaabccc & 3000\end{array}\right\}4 \\
\left.\begin{array}{rr} bb & 9\\ abb & 18\\ aabb & 36\\ aaabb & 72\end{array}\right\}4 &
\left.\begin{array}{rr} bbc & 45\\ abbc & 90\\ aabbc & 180\\ aaabbc & 360\end{array}\right\}4 &
\left.\begin{array}{rr} bbcc & 225\\ abbcc & 450\\ aabbcc & 900\\ aaabbcc & 1800\end{array}\right\}4 &
\left.\begin{array}{rr} bbccc & 1125\\ abbccc & 2250\\ aabbccc & 4500\\ aaabbccc & 9000\end{array}\right\}4 \\
\hline
4\times3=12 & 12 & 12 & 12
\end{array}}_{12\times4=48}$$

The ſame will in like manner be ſhewed, if this new Product a^3b^2c, (whoſe Diviſors are 24,) be further Multiplied by d, or dd, &c. Namely, the Diviſors of a^3b^2cd will be $24\times2 = 48$; and, of $a^3b^2cd^2$, $24\times3 = 72$. And ſo forward.

Or (which comes to the ſame paſs) if a^3b^2 (whoſe Diviſors are $12 = 4\times3$,) be Multiplied by cd, (whoſe Diviſors are $4 = 2\times2$,) or by cdd; (whoſe Diviſors are $2\times3 = 6$;) for then will the Diviſors of a^3b^2cd be $12\times4 = 48$; and of $a^3b^2cd^2$, $12\times6 = 72$; as before.

And in like manner, the ſame will hold, how many ſoever Prime Numbers, and what ever Powers of ſuch Primes, be ſo continually Multiplied; provided always (which is heedfully to be attended,) that ſuch Primes a, b, c, d, &c, be all different each from other.

11. If any Number however Compounded, be further multiplied by any of thoſe Primes of which it was before Compounded, or by any Power of ſuch Prime; the Number of Diviſors thence ariſing, will be ſuch as would have been by advancing that Prime ſo many Degrees higher, as is the Degree of ſuch Multiplier.

As, for inſtance, if c, d, were the ſame Prime; then inſtead of cd, whoſe Diviſors, if different, would have been

4 =

$4 = 2 \times 2$, $(1, c, d, cd,)$ we are to take cc, whoſe Diviſors are but 3, $(1, c, cc,)$ becauſe c, d, which would otherwiſe have been two different Diviſors, are now but one and the ſame. And accordingly, the Diviſors of a^3b^2cd, that is, (becauſe $c = d$,) of $a^3b^2c^2$, will now be (not $12 \times 4 = 48$, as before,) but $12 \times 3 = 36$. So if a^3b^2c be Multiplied by d^2, and $d = b$. For then $a^3b^2cd^2$ is the ſame with a^3b^4c; and the number of Diviſors (not $4 \times 3 \times 2 \times 3 = 72$, but) $4 \times 5 \times 2 = 40$. And the like in other caſes, as is of itſelf manifeſt.

12. And, univerſally: *If a Number be made, by continual Multiplication of how many ſoever Prime Numbers, (different each from other,) or of any Powers of ſuch Primes: The Number of Diviſors of ſuch Compound Number, is Compounded (by continual Multiplication) of the exponents of the Degrees of ſuch Primes or their Powers ſo Compounded, increaſed (each of them) by* 1. *And ſuch Number of Diviſors, wanting* 1, *is the Number of Aliquot Parts.* (Which Theorem contains the main ſubſtance of the Doctrine of Aliquot Parts.

As, for the Number a^3b^2cd; the exponents of the Degrees or Dimenſions of the Primes a, b, c, d, are 3, 2, 1, 1; and theſe increaſed by 1, are 4, 3, 2, 2. Theſe, continually Multiplied, give us the Number of Diviſors $4 \times 3 \times 2 \times 2 = 48$; and, of Aliquot Parts $48 - 1 = 47$. (And, in like manner, for any other Number however Compounded.) As is evident by what is before Demonſtrated.

Hence we may gather the ſolution of the following Problems.

13. Any Number being propoſed; to find how many Diviſors it hath; and, how many Aliquot Parts.

Divide the Number propoſed (and the Quotients ariſing from ſuch Diviſion) continually, by Prime Numbers (or the Powers of ſuch) according as it is capable, till we come to 1. And we ſhall thereby find, of how many different Prime Numbers, and what Powers of them, the Number propoſed is Compounded: which being done, we have the Number of

Diviſors, and of Aliquot Parts, by the propoſition foregoing.

As for Example; Let the Number ſo propoſed be 5940; we ſhall find, upon Tryal, that it may be divided by 2, twice; by 3, three times; by 5, once; (by 7, not at all;) and by 11, once.

11)5)3)3)3)2)2) 5940(2970(1485(495(165(55(11(1

And may therefore be thus deſigned, a^2b^3cd; where a, b, c, and d, denote the Numbers 2, 3, 5, and 11, reſpectively, and the exponents of a, b, c, d, are 2, 3, 1, 1; and theſe increaſed by 1, are 3, 4, 2, 2; which continually multiplied, are $3 \times 4 \times 2 \times 2 = 48$. So many therefore (by the propoſition foregoing) are the Number of Diviſors; and 47 the Number of Aliquot Parts.

14. Any Number being propoſed; to find, what are the Diviſors, and the Aliquot Parts thereof.

Firſt find (as in the preceeding article) of what Prime Numbers, and what Powers of them, the Number propoſed is Compounded. Then, taking any one of thoſe Prime Numbers to whatever Degree it be advanced, ſet down in order all the Diviſors of ſuch Degree. Then Multiply every of theſe by every Diviſor of ſuch Degree as ſome other of thoſe Primes is advanced to. And every of the Diviſors hitherto found, by every Diviſor of the Degree, to which a third Prime is advanced. And all theſe, by thoſe of a fourth; and ſo onward, if yet there be more Primes. (In ſuch manner as is to be ſeen above in art. 10.) And the Number ariſing from all thoſe Multiplications, is the Number of the Diviſors of the Number propoſed: And all theſe Diviſors, except itſelf, are the Aliquot Parts of it.

Thus for the Number $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 8 \times 9 \times 5$; ſuppoſe a^3b^2c. All the Diviſors of $a^3 = 8$, are 1, a, aa, aaa; that is, 1, 2, 4, 8. Let theſe be multiplied by all the Diviſors of $b^2 = 9$; which are 1, b, bb; that is, 1, 3, 9. And all the reſults of theſe, by the Diviſors of c; which are 1, c; that is, 1, 5. So have we all the Diviſors of 360.

1	*a*	*aa*	*aaa*	1	2	4	8
b	*ab*	*aab*	*aaab*	3	6	12	24
bb	*abb*	*aabb*	*aaabb*	9	18	36	72
c	*ac*	*aac*	*aaac*	5	10	20	40
bc	*abc*	*aabc*	*aaabc*	15	30	60	120
bc	*abbc*	*aabbc*	*aaabbc*	45	90	180	360

And in like manner we may proceed, whatever Number be propoſed, and howſoever Compounded.

But the ſame may alſo be done in divers other methods, (for we are not confined to proceed always in the ſame order,) which in the reſult will be the ſame with this. Provided always, in whatever order we proceed, that we be ſure to take all the Prime Numbers, that are Ingredients of ſuch Compound, with all the Degrees of them, and all the poſſible Combinations that may be made of them, not exceeding (in any) the Number of Dimenſions which they have in the Compound. And, that we may be ſure not to miſs any, it will be convenient to proceed, if not in this, at leaſt in ſome other regular order, that we may know when we have all. And ſome other forms of proceſs we may after have occaſion to mention.

15. To find a Number, which ſhall have juſt ſo many Diviſors, or ſo many Aliquot Parts, as is propoſed: And, in how many forms the ſame may be had; and, the leaſt in each form; or the leaſt of all, that may have ſo many.

The Number of Aliquot Parts propoſed, increaſed by 1, is the Number of Diviſors. This Number, we are to conſider, how many ways it may be expreſſed in Integers; whether by one alone, or by the Multiplication of two or more: (As is to be after ſhewed in art. 17, 18.) And, as many ways as this may be done, ſo many forms there are of Numbers which have juſt ſo many Diviſors: Namely, for every of the Integers by which ſuch Number is to be expreſſed, ſo many different Prime Numbers are to be aſſigned; and ſuch Degrees or Powers thereof, whoſe exponents are leſs by one than the reſpective Integers which they repreſent; and thoſe Powers or Degrees, (continually multiplied, if there be more of them,) will have ſuch Number of Diviſors as is required.

As for Example: If a Number be required which ſhall have 99 Aliquot Parts, or, (which is the ſame) 100 Diviſors. This Number 100, may be expreſſed by Integers (ſingle, or multiplied into one another,) nine ſeveral ways: 100 = 50 × 2 = 25 × 4 = 25 × 2 × 2 = 20 × 5 = 10 × 10 = 10 × 5 × 2 = 5 × 5 × 4 = 5 × 5 × 2 × 2: And ſo many ſeveral forms there are of Numbers which ſhall have 100 Diviſors, or 99 Aliquot Parts. Namely, if (for every of the forms wherein the Number 100 may be ſo deſigned) we take ſo many different Primes, as there are Integers in ſuch deſignation; and each of them advanced reſpectively to ſuch Degree whoſe exponent is leſs by one than the Integer it repreſents. As a^{99}, $a^{49}b$, $a^{24}b^3$, $a^{24}bc$, $a^{19}b^4$, a^9b^9, a^9b^4c, $a^4b^4c^3$, a^4b^4cd; whatever be thoſe Prime Numbers a, b, c, d, different each from other. (As appears from art. 12.) But not any other forms: As may be thence ſhewed, in caſe any other form be aſſigned. As, for inſtance, if any form be aſſigned wherein (whatever be the other Ingredients) there is the bare Square of a Prime Number, (ſuch as in none of theſe appears) as e^2. For whatever be the Number which the reſt of the ingredients deſign, that Number (becauſe of e^2) is to be Tripled (by art. 9.) But 100 is not the Triple of any Integer (as not being diviſible by 3:) And therefore cannot be ſo deſigned. And in like manner may be ſhewed, (with ſuch variation as the caſe ſhall require,) concerning any other form, different from thoſe aſſigned.

100	a^{99}
50×2	$a^{49}b$
25×4	$a^{24}b^3$
25×2×2	$a^{24}bc$
20×5	$a^{19}b^4$
10×10	a^9b^9
10×5×2	a^9b^4c
5×5×4	$a^4b^4c^3$
5×5×2×2	a^4b^4cd

Now for finding the leaſt Number in each form, that ſhall have ſo many Diviſors; no more is to be done, but for a, b, c, d, &c, or ſo many of them as occur in each form reſpectively, to take ſo many of the ſmalleſt Primes, 2, 3, 5, 7, &c. And, of theſe, ſtill to aſſign the leſſer for that which is to have the greater Number of Dimenſions.

(As is of itſelf manifeſt.) So for the form a^9b^4c, it is manifeſt, that if for a, b, c, we take 2, 3, 5, the number muſt needs be leſs, than if we take 2, 3, 7, or 3, 7, 11, or any other numbers: And, (ſuppoſing thoſe three to be taken,) it muſt needs be leſs if we aſſign $a = 2$, $b = 3$, $c = 5$, than if we aſſign them any otherwiſe. Becauſe, in the compoſition, a is oftener to be repeated than b, and b than c.

Now when it appears, which is the leaſt in each form; it is eaſily determined upon view, which is the leaſt of all. As, in the preſent caſe, putting $a = 2$, $b = 3$, $c = 5$, $d = 7$; it is eaſy to judge that a^4b^4cd, that is, $16 \times 81 \times 5 \times 7 = 45360$, is the ſmalleſt number that can have 100 Diviſors. For it is, to $a^4b^4c^3$; as $d = 7$, to $cc = 9$: And it is, to a^9b^4c; as $d = 7$, to $a^5 = 32$: And, to a^9b^9; as $cd = 35$, to $a^5b^5 = 7776$. And ſo of the reſt.

And, for the moſt part, thoſe are the ſmaller numbers wherein more Primes be ingredients; than where fewer Primes, but in higher Degrees; as $ab = 2 \times 3 = 6$, is leſs than $a^3 = 8$; though each of them have four Diviſors. But it is not always ſo; for $a^3b = 8 \times 3 = 24$, is leſs than $abc = 2 \times 3 \times 5 = 30$; (though the number of Diviſors be eight in each.) For here one Degree of a greater Prime $c = 5$, doth over-balance two Degrees of a leſſer $aa = 4$.

16. It appears moreover, That, wherever the number of Diviſors is odd, ſuch Number is a Square: And, contrarywiſe, of every Square Number, the number of Diviſors is odd. And, of every Non quadrate Number, the number of Diviſors is even: And, wherever the number of Diviſors is even, ſuch Number is a Non-quadrate Number.

For every Diviſor divides the Number propoſed by ſome other Diviſor, (whereof when one is the Diviſor, the other is the Quotient;) except only the Square Root, (where the Diviſor and Quotient are the ſame.) All other Diviſors therefore go by couples, and make an even Number: To which when the Square Root is to be added (which is the caſe of all Square Numbers, and of theſe only;) this being ſolitary, makes the number of Diviſors odd.

1	36	1	*aabb*
2	18	*a*	*abb*
3	12	*b*	*aab*
4	9	*aa*	*bb*
6		*ab*	

1	72	1	*aaabb*
2	36	*a*	*aabb*
3	24	*b*	*aaab*
4	18	*aa*	*abb*
6	12	*ab*	*aab*
8	9	*aaa*	*bb*

1	360	1	*aaabbc*
2	180	*a*	*aabbc*
3	120	*b*	*aaabc*
4	90	*aa*	*abbc*
5	72	*c*	*aaabb*
6	60	*ab*	*aabc*
8	45	*aaa*	*bbc*
9	40	*bb*	*aaac*
10	36	*ac*	*aabb*
12	30	*aab*	*abc*
15	24	*bc*	*aaab*
18	20	*abb*	*aac*

17. A Number being propoſed; to find, how many different ways it may be deſigned by Integers; whether ſingly or by the continual Multiplication of more than one.

Firſt find out (by art. 14,) what are all the Diviſors of ſuch propoſed Number. Then, conſidering them all ſingly (beginning at the greateſt and ſo proceeding to the leſſer; that, by keeping ſuch order, we may be the more ſure not to miſs any;) inquire, what Number doth with every of theſe compoſe the Number propoſed; and if this chance to be a Compound, let this in like manner be reſolved into its Components, (and ſo onward as long as the Component is itſelf a Compound;) whereby, having thus run through them all, we ſhall meet with all the ways whereby the Number propoſed may ſo be deſigned by Integers.

As for Example: Let ſuch Number propoſed, be 360; whoſe Diviſors (found by art. 14,) are 360, 180, 120, 90, 72, 60, 45, 40, 36, 30, 24, 20, 18, 15, 12, 10, 9, 8, 6, 5, 4, 3, 2, 1, where we ſhall find the firſt deſignation to be 360, (or 360×1.) Then 180×2, 120×3, 90×4, and (becauſe $4 = 2\times2$,) $90\times2\times2$. Then 72×5, 60×6; and (becauſe $6 = 3\times2$)

360
180×2
120×3
90×4
$90\times2\times2$
72×5
60×6
$60\times3\times2$
45×8
$45\times4\times2$
$45\times2\times2\times2$
40×9
$40\times3\times3$
36×10

36×10
36×5×2
30×12
30×6×2
30×4×3
30×3×2×2
24×15
24×5×3
20×18
20×9×2
20×6×3
20×3×3×2
18×10×2
18×5×4
18×5×2×2
15×12×2
15×8×3
15×6×4
15×6×2×2
15×4×3×2
15×3×2×2×2
12×10×3
12×6×5
12×5×3×2
10×9×4
10×9×2×2
10×6×6
10×6×3×2
10×4×3×3
10×3×3×2×2
9×8×5
9×5×4×2
9×5×2×2×2
8×5×3×3
6×6×5×2
6×5×4×3
6×5×3×2×2
5×4×3×3×2.
5×3×3×2×2×2

60×3×2. Then 45×8; and (becauſe 8 = 4×2 = 2×2×2) 45×4×2, 45×2×2×2. Then 40×9; and (becauſe 9 = 3×3,) 40×3×3. Then 36×10; and (becauſe 10 = 5×2,) 36×5×2. Then 30×12; and (becauſe 12 = 6×2 = 4×3 = 3×2×2.) 30×6×2, 30×4×3, 30×3×2×2. Then 24×15, and (becauſe 15 = 5×3,) 24×5×3. Then 20×18, and (becauſe 18 = 9×2 = 6×3 = 3×3×2,) 20×9×2, 20×6×3, 20×3×3×2. Then, (omitting 18×20, as being the ſame with 20×18; and reſolving 20 = 10×2 = 5×4 = 5×2×2;) 18×10×2, 18×5×4, 18×5×2×2. Then (omitting 15×24, as being the ſame with 24×15; and ſo every where when a greater follows a leſs, as being had before; and reſolving 24 = 12×2 = 8×3 = 6×4 = 6×2×2 = 4×3×2 = 3×2×2×2;) 15×12×2, 15×8×3, 15×6×4, 15×6×2×2, 15×4×3×2, 15×3×2×2×2. In like manner (omitting ſuch Combinations of 12 as have been already,) 12× (30 = 15×2 =) 10×3, 12×6×5, 12×5×3×2. In like manner, 10× (36 = 18×2 = 12×3 =) 9×4, 10×9×2×2, 10×6×6, 10×6×3×2, 10×4×3×3, 10×3×3×2×2. Then 9× (40 = 20×2 = 10×4 =) 8×5, 9×5×4×2, 9×5×2×2×2. Then 8× (45 =) 5×3×3. Then 6× (60 =) 6×5×2, 6×5×4×3, 6×5×3×2×2. Laſtly, 5× (72 =) 4×3×3×2, 5×3×3×2×2×2. (The Diviſors 4, 3, 2, 1, afford no new caſes; becauſe every of them is leſs than 5, and cannot without it, or ſome greater Number, make up 360.) Which forms (in Number 52) are all the forms in which 360, may thus be expreſſed by Integers. And how, to every of theſe forms, we may fit ſo many forms of Numbers which ſhall have 360 Diviſors, is before ſhewed in art. 15. As, for $5\times3\times3\times2\times2\times2$, $a^4b^2c^2def$: And ſo of the reſt.

But, why I have here omitted (for inſtance) 5×72, $5\times36\times2$, $5\times24\times3$, $5\times18\times4$, $5\times18\times2\times2$, $5\times12\times6$, $5\times12\times3\times2$, $5\times9\times8$, $5\times9\times4\times2$, $5\times9\times2\times2\times2$, $5\times8\times3\times3$, $5\times6\times6\times2$, $5\times6\times4\times3$, $5\times6\times3\times2\times2$, and others of like kind; the cauſe is evident: Becauſe, the Numbers 72, 36, 24, 18, 12, 9, 8, 6, being greater than 5, all the Combinations which have theſe ingredients were had before. For 5×72, is but the ſame with 72×5; and ſo of the reſt. And it is ſo ordered all along, that whenever a greater Number comes to follow a leſſer, we may know that that caſe was (or ſhould have been) had before.

But it is no way neceſſary that we ſhould always obſerve this order; for the ſame will hold, in whatever method we proceed: provided we be ſure to take them all, in whatever order.

18. The ſame alſo may be thus had, if the Number itſelf (of Diviſors required) or the form thereof, be ſo expreſſed in Species, as it may thence appear in what form itſelf is Compounded of the ingredient Primes: As if we put a^3b^2c, for the Number $360 = 2\times2\times2\times3\times3\times5$; or for any other Number which is Compounded of the Third Degree of one Prime, Multiplied by the Second Degree of another Prime, and this by a Third Prime.

For, however we are not by this directed how to proceed (as before) from the greater to the leſſer in a continual order, (becauſe the Second or Third Degree of a leſſer Prime, may poſſibly be greater than the firſt of ſome greater Prime;) yet we may thus, though in another order, meet with them all.

And it will be then convenient (beginning with 1,) to take the Species or Symbols, firſt ſingly, one by one, (as a, b, c,) in ſuch order as they follow in the Alphabet. And then by Two's, (as aa, ab, ac, bb, &c,) and here, firſt thoſe that begin with *a*; and here again *aa* before *ab*, and this before *ac*, 1, &c, and then thoſe that begin with *b*; and here (omitting *ba*, as being the ſame with *ab* which was had before,) beginning with *bb*, or (in caſe there be not a ſecond *b*) with *bc*, and ſo onward: And then by Threes, and Fours, and ſo onward as there is occaſion; obſerving all along, as the caſe will permit, the Alphabetical order, (that we may be the more ſure not to miſs any.) Placing always, over againſt each, the correſpondent Diviſor; which doth, with it, conſtitute the Number propoſed. As, againſt *aa*, putting *abbc*, which, with it, compleats *aaabbc*.

1	a^3b^2c
a	a^2b^2c
b	a^3bc
c	a^3b^2
a^2	ab^2c
ab	a^2bc
ac	a^2b^2
b^2	a^3c
bc	a^3b
a^3	b^2c
a^2b	abc
a^2c	ab^2

Or thus rather,

1	*aaabbc*
a	*aabbc*
b	*aaabc*
c	*aaabb*
aa	*abbc*
ab	*aabc*
ac	*aabb*
bb	*aaac*
bc	*aaab*
aaa	*bbc*
aab	*abc*
aac	*abb*

And this we are to purſue ſo far, till, in that oppoſite rank, we meet with the ſame (in the caſe of a Square Number propoſed,) or, (if not a Square Number,) that which was next to follow, in the firſt rank. (As here, againſt *aaac*, we have *abb*; which was next to have followed if the firſt rank had proceeded.) For, when we be come ſo far, thoſe which were to have followed in the continuation of the firſt rank, do follow (in the ſame order, but going backward,) in the latter rank, till we come to the greateſt of all.

And having thus diſpoſed all the Diviſors in due order; we may then (beginning with the greateſt, and ſo proceeding backward to the leaſt,) compound each with its oppoſite, which ſtands againſt it. (As *cbbaaa*, *cbbaa* × *a*, *cbaaa* × *b*, &c.) And when that ſecond Component is itſelf a Compound, we are to reſolve it into its Components; (as *cba*×*aa*, *cba*×*a*×*a*, &c,) and ſo continually till it be reſolved into Primes.

When we have thus diſpatched all the Diviſors of the latter rank (for till then, there is no danger,) we are to take heed, that ſome of the Compoſitions already taken, be not taken a ſecond time in another order; and when they do ſo occur a ſecond time, we are to paſs them by. And accordingly, when I come at *caa*, I do not Compound this with the whole of *bba* which ſtands againſt it; (becauſe this hath been already conſidered, and there joined in all the Compoſitions that it is capable of;) but with all theſe Components of *b b a*, which had not before been fully conſidered. And when I come at *cb:* I omit, not only the whole of *baaa*, (which ſtands againſt it) but all the Components of it which have three Members, (becauſe not only thoſe of Four, but even of Three Components, have been fully diſpatched, before we come at *cb* which hath but

cbbaaa	360
cbbaa×*a*	180×2
cbaaa×*b*	120×3
bbaaa×*c*	72×5
cbba×*aa*	90×4.
×*a*×*a*	90×2×2
cbaa×*ba*	60×6
×*b*×*a*	60×3×2
bbaa×*ca*	36×10
×*c*×*a*	36×5×2
caaa×*bb*	40×9
×*b*×*b*	40×3×3
baaa×*bc*	24×15
×*b*×*c*	24×5×3
cbb×*aaa*	45×8
×*aa*×*a*	45×4×2
×*a*×*a*×*a*	45×2×2×2
cba×*baa*	30×12
×*ba*×*a*	30×6×2
×*aa*×*b*	30×4×3
×*b*×*a*×*a*	30×3×2×2
bba×*caa*	18×20
×*ca*×*a*	18×10×2
×*aa*×*c*	18×4×5
×*c*×*a*×*a*	18×5×2×2
caa×*bb*×*a*	20×9×2
×*ba*×*b*	20×6×3
×*b*×*b*×*a*	20×3×3×2
baa×*cb*×*a*	12×15×2
×*ca*×*b*	12×10×3
×*ba*×*c*	12×6×5
×*c*×*b*×*a*	12×5×3×2
aaa×*cb*×*b*	8×15×3
×*bb*×*c*	8×9×5
×*c*×*b*×*b*	8×5×3×3
cb×*ba*×*aa*	15×6×4
×*a*×*a*	15×6×2×2
×*aa*×*b*×*a*	15×4×3×2
×*b*×*a*×*a*×*a*	15×3×2×2×2

two Components.) And when I come at *ca*, I omit *ca*×*bb*×*aa*, &c, becauſe *bb* had been before conſidered. And in like manner, at *ba*, I omit all the Compoſitions wherein *cb*, *bb*, *ca*, were ingredients; becauſe theſe had been before conſidered. And in like manner, at *aa*, and *c*, I omit all thoſe of two Members which might be Compounded with them; becauſe already had. As is to be ſeen in the order adjoined.

bb×*ca*×*aa*	9×10×4
×*a*×*a*	9×10×2×2
×*aa*×*c*×*a*	9×4×5×2
×*c*×*a*×*a*×*a*	9×5×2×2×2
ca×*ba*×*ba*	10×6×6
×*b*×*a*	10×6×3×2
×*aa*×*b*×*b*	10×4×3×3
×*b*×*b*×*a*×*a*	10×3×3×2×2
ba×*ba*×*c*×*a*	6×6×5×2
×*aa*×*c*×*b*	6×4×5×3
×*c*×*b*×*a*×*a*	6×5×3×2×2
aa×*c*×*b*×*b*×*a*	4×5×3×3×2
c×*b*×*b*×*a*×*a*×*a*	5×3×3×2×2×2

And over againſt the forms thus expreſſed in Species; I have ſet the Numbers anſwering to them; which are the ſame with thoſe at art. 17, but not in the ſame order. Becauſe here I was guided by the forms of Compoſition, in directing the order; but, there, by the bigneſs of the Numbers.

Having thus laid the Foundation of this Doctrine of Diviſors and Aliquot Parts; I ſhall give ſome Examples of Operations concerning them.

Examples of the foregoing Operations.

19. Of the Number 110,880: How many are the Diviſors, and Aliquot Parts? And which be they?

The Number 110,880 divided, as is directed at art. 13, is reſolved into theſe Primes; 2, 2, 2, 2, 2, 3, 3, 5, 7, 11. And is therefore in this form a^5b^2cde.

11) 5) 7) 3) 3) 2) 2) 2) 2) 2) 110880 (55440 (27720 (13860
(6930 (3465 (1155 (385 (77 (11 (1

Or, I might at firſt cut off the Cypher; and, for it, ſet down two Diviſors 2, 5: And then, becauſe it is obvious to view, that 11,088 is diviſible by 11; I might next ſet down 11 for another Diviſor. (Becauſe by this means we come the ſooner to ſmall Numbers.) And then divide the Quotient 1008 by 2, and 3, as oft as I can; which done, we ſhall have 7 for the laſt Diviſor. Or, I might have divided 11,088 by 9; (and for it ſet down two Diviſors 3, 3:) For it is obvious alſo to view that it may be ſo divided; becauſe the Figures put together without regard had to the places, (as is uſual in the proofs of Multiplication and Diviſion,) may be ſo divided; or, caſting away 9 as oft as may be, nothing remains; or, I may ſo do, for the ſame reaſon, with 1008; or, take any the like advantage for expedition, as the view ſhall direct. For it matters not, in what order we find the Component Primes, ſo we have them all.

The Number therefore appearing in this form a^5b^2cde; it is manifeſt (by art. 12,) that the Number of Diviſors is $6\times3\times2\times2\times2 = 144$; and, of Aliquot Parts, $144 - 1 = 143$. And thoſe, (according to the method of art. 18,) are found to be theſe that follow.

1	1	110880	*aaaaabbcde*
a	2	55440	*aaaabbcde*
b	3	36960	*aaaaabcde*
c	5	22176	*aaaaabbde*
d	7	15840	*aaaaabbce*
e	11	10080	*aaaaabbcd*
aa	4	27720	*aaabbcde*
ab	6	18480	*aaaabcde*
ac	10	11088	*aaaabbde*
ad	14	7920	*aaaabbce*
ae	22	5040	*aaaabbcd*
bb	9	12320	*aaaaacde*
bc	15	7392	*aaaaabde*
bd	21	5280	*aaaaabce*
be	33	3360	*aaaaabcd*
cd	35	3168	*aaaaabbe*
ce	55	2016	*aaaaabbd*
de	77	1440	*aaaaabbc*
aaa	8	13860	*aabbcde*
aab	12	9240	*aaabcde*
aac	20	5544	*aaabbde*

aad	28	3960	*aaabbce*
aae	44	2520	*aaabbcd*
abb	18	6160	*aaaacde*
abc	30	3696	*aaaabde*
abd	42	2640	*aaaabce*
abe	66	1680	*aaaabcd*
acd	70	1584	*aaaabbe*
ace	110	1008	*aaaabbd*
ade	154	720	*aaaabbc*
bbc	45	2464	*aaaaade*
bbd	63	1760	*aaaaace*
bbe	99	1120	*aaaaacd*
bcd	105	1056	*aaaaabe*
bce	165	672	*aaaaabd*
bde	231	480	*aaaaabc*
cde	385	288	*aaaaabb*
aaaa	16	6930	*abbcde*
aaab	24	4620	*aabcde*
aaac	40	2772	*aabbde*
aaad	56	1980	*aabbce*
aaae	88	1260	*aabbcd*
aabb	36	3080	*aaacde*
aabc	60	1848	*aaabde*
aabd	84	1320	*aaabce*
aabe	132	840	*aaabcd*
aacd	140	792	*aaabbe*
aace	220	504	*aaabbd*
aade	308	360	*aaabbc*
abbc	90	1232	*aaaade*
abbd	126	880	*aaaace*
abbe	198	560	*aaaacd*
abcd	210	528	*aaaabe*
abce	330	336	*aaaabd*
abde	462	240	*aaaabc*
acde	770	144	*aaaabb*
bbcd	315	352	*aaaaae*
bbce	495	224	*aaaaad*
bbde	693	160	*aaaaac*
bcde	1155	96	*aaaaab*

aaaaa

aaaaa	32	3465	*bbcde*
aaaab	48	2310	*abcde*
aaaac	80	1386	*abbde*
aaaad	112	990	*abbce*
aaaae	176	630	*abbcd*
aaabb	72	1540	*aacde*
aaabc	120	924	*aabde*
aaabd	168	660	*aabce*
aaabe	264	420	*aabcd*
aaacd	280	396	*aabbe*
aaace	440	252	*aabbd*
aaade	616	180	*aabbc*

The ſame, ordered according to the greatneſs of the Numbers, will ſtand thus :

1	1	110880	*aaaaabbcde*
a	2	55440	*aaaabbcde*
b	3	36960	*aaaaabcde*
aa	4	27720	*aaabbcde*
c	5	22176	*aaaaabbde*
ab	6	18480	*aaaabcde*
d	7	15840	*aaaaabbce*
aaa	8	13860	*aabbcde*
bb	9	12320	*aaaaacde*
ac	10	11088	*aaaabbde*
e	11	10080	*aaaaabbcd*
aab	12	9240	*aaabcde*
ad	14	7920	*aaaabbce*
bc	15	7392	*aaaaabde*
aaaa	16	6930	*abbcde*
abb	18	6160	*aaaacde*
aac	20	5544	*aaabbde*
bd	21	5280	*aaaaabce*
ae	22	5040	*aaaabbcd*
aaab	24	4620	*aabcde*
aad	28	3960	*aaabbce*
abc	30	3696	*aaaabde*
aaaaa	32	3465	*bbcde*

be

be	33	3360	*aaaaabcd*
cd	35	3168	*aaaaabbe*
aabb	36	3080	*aaacde*
aaac	40	2772	*aabbde*
abd	42	2640	*aaaabce*
aae	44	2520	*aaabbcd*
bbc	45	2464	*aaaaade*
aaaab	48	2310	*abcde*
ce	55	2016	*aaaaabbd*
aaad	56	1980	*aabbce*
aabc	60	1848	*aaabde*
bbd	63	1760	*aaaaace*
abe	66	1680	*aaaabcd*
acd	70	1584	*aaaabbe*
aaabb	72	1540	*aacde*
de	77	1440	*aaaaabbc*
aaaac	80	1386	*abbde*
aabd	84	1320	*aaabce*
aaae	88	1260	*aabbcd*
abbc	90	1232	*aaaade*
aaaaab	96	1155	*bcde*
bbe	99	1120	*aaaaacd*
bcd	105	1056	*aaaabe*
ace	110	1008	*aaaabbd*
aaaad	112	990	*abbce*
aaabc	120	924	*aabde*
abbd	126	880	*aaaace*
aabe	132	840	*aaabcd*
aacd	140	792	*aaabbe*
aaaabb	144	770	*acde*
ade	154	720	*aaaabbc*
aaaaac	160	693	*bbde*
bce	165	672	*aaaabd*
aaabd	168	660	*aabce*
aaaae	176	630	*abbcd*
aabbc	180	616	*aaade*
abbe	198	560	*aaaacd*
abcd	210	528	*aaabe*
aace	220	504	*aaabbd*

aaaaad

aaaaad		224		495		*bbce*
bde		231		480		*aaaaabc*
aaaabc	.	240	.	462	.	*abde*
aabbd		252		440		*aaace*
aaabe		264		420		*aabcd*
aaacd		280		396		*aabbe*
aaaaabb		288		385		*cde*
aade	.	308	.	360	:	*aaabbc*
bbcd		315		352		*aaaaae*
abce		330		336		*aaaabd*

20. Of Numbers (for inſtance) which have 12 Diviſors: To exhibit all the forms; and, all the Numbers in each form; not exceeding the Number 2048; (which is the loweſt Number of the higheſt form;) according to art. 15. 18.

All the ways according to which 12 may be expreſſed by Integers (as in art. 17, 18,) are $12 = 6\times2 = 4\times3 = 3\times2\times2$: Which affords us theſe forms, a^{11}, a^5b, a^3b^2, a^2bc. And in each of theſe, the Numbers are as follow; being in all 211.

a^{11}.

2048.

a^5b.

32 × 3 = 96
32 × 5 = 160
× 7 = 224
× 11 = 352
× 13 = 416
× 17 = 544
× 19 = 608
× 23 = 736
× 29 = 928
× 31 = 992
× 37 = 1184
× 41 = 1312
× 43 = 1376
× 47 = 1504
× 53 = 1696
× 59 = 1888
× 61 = 1952
243 × 2 = 486
× 5 = 1215
× 7 = 1701

a^3b^2.

8 × 9 = 72
× 25 = 200
× 49 = 392
× 121 = 968
× 196 = 1352
27 × 4 = 108
× 25 = 675
× 49 = 1323
125 × 4 = 500
× 9 = 1125

343 × 4 = 1372
a^2bc.
4 × 3 × 5 = 60
× 7 = 84
× 11 = 132
× 13 = 156
× 17 = 204
× 19 = 228
× 23 = 276
× 29 = 348
× 31 = 372
× 37 = 444
× 41 = 492
× 43 = 516
× 47 = 564
× 53 = 636
× 59 = 708
× 61 = 732
× 67 = 804
× 71 = 852
× 73 = 876
× 79 = 948
× 83 = 996
× 89 = 1068
× 97 = 1164
× 101 = 1212
× 103 = 1236
× 107 = 1284
× 109 = 1308
× 113 = 1356
× 127 = 1524
× 131 = 1572
× 137 = 1644
× 139 = 1668
× 149 = 1788
× 151 = 1812
× 157 = 1884
× 163 = 1956
× 167 = 2004

4 × 5 × 7 = 140
× 11 = 220
× 13 = 260
× 17 = 340
× 19 = 380
× 23 = 460
× 29 = 580
× 31 = 620
× 37 = 740
× 41 = 820
× 43 = 860
× 47 = 940
× 53 = 1060
× 59 = 1180
× 61 = 1220
× 67 = 1340
× 71 = 1420
× 73 = 1460
× 79 = 1580
× 83 = 1660
× 89 = 1780
× 97 = 1940
× 101 = 2020
4 × 7 × 11 = 308
× 13 = 364
× 17 = 476
× 19 = 532
× 23 = 644
× 29 = 812
× 31 = 868
× 37 = 1036
× 41 = 1148
× 43 = 1204
× 47 = 1316
× 53 = 1484
× 59 = 1652
× 61 = 1708
× 67 = 1876
× 71 = 1988

4×7 × 73 = 2044
4×11 × 13 = 572
× 17 = 748
× 19 = 836
× 23 = 1012
× 29 = 1276
× 31 = 1364
× 37 = 1628
× 41 = 1804
× 43 = 1892
4×13 × 17 = 884
× 19 = 988
× 23 = 1196
× 29 = 1508
× 31 = 1612
× 37 = 1924
4×17 × 19 = 1292
× 23 = 1564
× 29 = 1972
4×19 × 23 = 1748
9×2 × 5 = 90
× 7 = 126
× 11 = 198
× 13 = 234
× 17 = 306
× 19 = 342
× 23 = 414
× 29 = 522
× 31 = 558
× 37 = 666
× 41 = 738
× 43 = 774
× 47 = 846
× 53 = 954
× 59 = 1062
× 61 = 1098
× 67 = 1206
× 71 = 1278
× 73 = 1314

9×2 × 79 = 1422
× 83 = 1498
× 89 = 1602
× 97 = 1746
× 101 = 1818
× 103 = 1854
× 107 = 1926
× 109 = 1962
× 113 = 2034
9×5 × 7 = 315
× 11 = 495
× 13 = 585
× 17 = 765
× 19 = 855
× 23 = 1035
× 29 = 1305
× 31 = 1395
× 37 = 1665
× 41 = 1845
× 43 = 1935
9×7 × 11 = 693
× 13 = 819
× 17 = 1071
× 19 = 1197
× 23 = 1449
× 29 = 1827
× 31 = 1953
9×11 × 13 = 1287
× 17 = 1683
× 19 = 1881
9×13 × 17 = 1989
25×2 × 3 = 150
× 7 = 350
× 11 = 550
× 13 = 650
× 17 = 850
× 19 = 950
× 23 = 1150
× 29 = 1450

25×2 × 31 = 1550
× 37 = 1850
25×3 × 7 = 525
× 11 = 825
× 13 = 975
× 17 = 1275
× 19 = 1425
× 23 = 1725
25×7 × 11 = 1925
49×2 × 3 = 294
× 5 = 490
× 11 = 1078
× 13 = 1274

49×2 × 17 = 1666
× 19 = 1862
49×3 × 5 = 735
× 11 = 1617
× 13 = 1911
121×2 × 3 = 726
× 5 = 1210
× 7 = 1694
121×3 × 5 = 1815
169×2 × 3 = 1014
× 5 = 1690
289×2 × 3 = 1734

Theſe digeſted according to their natural order, ſtand thus:

60
72
84
90
96
108
126
132
140
150
156
160
198
200
204
220
224
228
234
260
276
294

306
308
315
340
342
348
350
352
364
372
380
392
414
416
444
460
476
486
490
492
495
500

516
522
525
532
544
550
558
564
572
580
585
608
620
636
644
650
666
675
693
708
726
732

735
736
738
740
748
765
774
804
812
819
820
825
836
846
850
852
855
860
868
876
884
928

940
948
950
954
968
975
988
992
996
1012
1014
1035
1036
1060
1062
1068
1071
1078
1098
1125
1148
1150

1164
1180
1184
1196
1197
1204
1206
1210
1212
1215
1220
1236
1274
1275
1276
1278
1284
1287
1292
1305
1308
1312
1314
1316
1323
1340
1352
1356
1364
1372
1376
1395
1420
1422
1425
1449
1450
1460
1484
1494
1504
1508
1524
1550
1564
1572
1580
1602
1612
1617
1628
1644
1652
1660
1665
1666
1668
1683
1690
1694
1696
1701
1708
1725
1734
1746
1748
1780
1788
1804
1812
1815
1818
1827
1845
1850
1854
1862
1876
1881
1884
1888
1892
1911
1924
1925
1926
1935
1940
1952
1953
1956
1962
1972
1988
1989
2004
2020
2034
2044
2048

Of the Numbers that are moſt convenient for the Purpoſe of dividing Large Quantities into Leſſer equal Parts.

21. Thoſe Numbers which (for the bigneſs of them) have the greateſt Number of Diviſors, and Aliquot Parts; have been wont to be made choice of, as moſt convenient for uſe; eſpecially when there may be frequent occaſion of dividing things ſo deſigned.

Hence it is, that the *Engliſh* Penny is divided into Four Farthings, (and almoſt all things in Four Quarters of a different Name,) becauſe there is often occaſion to divide into halves, and then again into halves. Hence alſo the *Roman*

1	1	1
2	2	a
4	3	a^2
6	4	ab
12	6	a^2b
24	8	a^3b
36	9	a^2bc
48	10	a^4b
60	12	a^2bc
120	16	a^3bc
180	18	a^2b^2c
240	20	a^4bc
360	24	a^3b^2c
720	30	a^4b^2c
840	32	a^3bcd
1260	36	a^2b^2cd
1680	40	a^4bcd
2520	48	a^3b^2cd
5040	60	a^4b^2cd
7560	64	a^3b^3cd
10080	72	a^5b^2cd
15120	80	a^4b^3cd
20160	84	a^6b^2cd
25200	90	$a^4b^2c^2d$
27720	96	a^3b^2cde
45360	100	a^4b^4cd
50400	108	$a^5b^2c^2d$
55440	120	a^4b^2cde
83160	128	a^3b^3cde
110880	144	a^5b^2cde
166320	160	a^4b^3cde
221760	168	$a^6b^2c^2de$
277200	180	$a^4b^2c^2de$
332640	192	a^5b^3cde
498960	200	a^4b^4cde
554400	216	$a^5b^2c^2de$
665280	224	a^6b^3cde

Pound, (and that which we now call the Pound *Troy Weight*,) is divided into 12 Ounces; and the *Engliſh* Shilling, into 12 pence; the Foot, into 12 Inches; the Zodiack, into 12 Signs; the Year, into 12 Months; becauſe, beſide the Diviſion into Quarters, it is further diviſible by 3. And for a like reaſon *Ptolemy* (and others after him) makes uſe of the Sexagenary diviſion, of Integers into firſt Minutes, or ſmall, or *minute*, parts of the firſt order; and of theſe, into Second Minutes, or Seconds, or ſmall, or *minute*, parts of the Second order; and ſo onward: becauſe 60 is diviſible by 2 and 3, and likewiſe by 5. And the *Chineſes* (or *Cathaians*) Number their Years (and other things) by Revolutions of 60. After this; 360 is looked on as moſt conſiderable, becauſe it may be further divided by 2 and 3 once more: Which therefore is made the Number of Degrees in a Circle; admitting of 24 Diviſors. And if this be not enough, each of theſe is divided into 60 Minutes; (that is, by 4, 3, 5, once more;) and theſe into Seconds, and ſo forth. And the *Engliſh* Pound Sterling, is divided into 20 Shillings; which Number is diviſible by 4 and 5, (as 12, the Number of Pence in a Shilling, is diviſible by 4 and 3;) which was accounted more convenient than to make another Collection of Shillings by 12; becauſe this would not afford a diviſion by 5. So that now 960 the Number of Farthings in a Pound Sterling, is for the firſt ſtep (from Farthings to Pence) diviſible by 4; for the ſecond ſtep (from Pence to Shillings) by 4 and 3; for the third ſtep (from Shillings to Pounds) by 4 and 5. And (without taking notice of the diviſion of Pence into Farthings) the Number of Pence in a Pound Sterling, 240, is capable of 20 Diviſors; and, of more than ſo many, no Number is capable which is not greater than it.

Of the foregoing Table of Numbers in page 316.

In purſuance of which notion, I have here Collected a Table of all thoſe Numbers, which (of all not greater than themſelves) have the greateſt Number of Diviſors; (together with the Number of Diviſors in each of them, and the Form of their Compoſition;) as far as 665,280, which hath 224 Diviſors. All which (except 1,) are made by the Compoſition of 2, 3, 5, 7, 11, (which I call *a*, *b*, *c*, *d*, *e*,) and the Powers of theſe, without admitting any other Prime. (But, if we would proceed to a greater Number of Diviſors, we muſt further take in $f = 13$.) And, of theſe, ſome are of that nature, that none can have a greater Number of Diviſors, which is not at leaſt the double of them. Such are 1, 2, 6, 12, 60, 360, 2520: But not any after theſe for a great way.

Of the Uſe of a Table of Prime Numbers.

22. For reſolution of ſome of the Queſtions above mentioned, (as in art. 13, 14, 17, 18, &c,) it is very convenient to have at hand a Table of Prime Numbers: That we may know, by what Numbers to make trial of the Diviſions therein directed. And, becauſe, in great Numbers, it would be tedious to make trial of all the Prime Numbers in order, it is convenient alſo to know, by what Prime ſuch greater Numbers may be divided.

In order to which, it is evident, in the firſt place, that all even Numbers may be divided by 2; and, if the Quotient of ſuch diviſion be even alſo, it may be again divided by 2, and ſo continually as long as the Quotient is an even Number.

It is alſo evident, in the ſecond place, that all Numbers ending in 5, are diviſible by 5; and, if they end in 0, then by 2 and 5. And ſo continually, as long as the Quotient of ſuch diviſion ends in 0, or 5.

It is known alſo, in the 3d place, that, if the Figures of any Number being added promiſcuouſly (without regarding the places wherein they ſtand) are diviſible by 9, (or, caſting away 9 as oft as may be, nothing remains,) ſuch Number is alſo diviſible by 9. As in 29097; when (the Nines being left out, and) 2 + 7 = 9 being caſt away, nothing remains; whence we may conclude, 'tis diviſible by 9 = 3 × 3. And I add further, as a fourth obſervation, (though I do not find that others have taken notice of it,) that the ſame holds alſo as to the Number 3: That is, from the Figures ſo promiſcuouſly added, if 3 being caſt away as oft as may be, nothing remain, ſuch Number is diviſible by 3; Otherwiſe, it is not. As in 530,967; where, all the threes, nines and ſixes being left out (as manifeſtly diviſible by 3,) the reſt 5 + 7 = 12, is ſo alſo, (or, which is the ſame, $1 + 2 = 3$;) ſo that all the threes being caſt away, nothing remains; whence we may conclude, that the whole Number is diviſible (though not by 9) at laſt by 3.

The ground of this and the former Obſervation is one and the ſame: Becauſe, the places increaſing in decuple proportion, if from 10, or any Number of tens, we caſt away all the nines or all the threes, there remains 1, or ſo many ones. So that, in caſe of ſuch caſting away of nines and threes, 1 and 10, have the ſame remainders; and ſo 2 and 20; 3 and 30, &c. And conſequently 1, 10, 100, 1000, &c, 2, 20, 200, 2000, &c. So that the ſame Figure, as to this, is of the ſame influence in whatever place it ſtand.

Of Dr. Pell's *Table of Prime Numbers.*

23. Beſide this, we have at the end of Dr. *Pell's Algebra*, (Tranſlated and Publiſhed by *Thomas Branker*, in the Year 1668, with Dr. *Pell's* directions,) a Compendious Table of all odd Numbers (not ending in 5) as far as 100,000; ſhewing not only, which of them are Prime Numbers; but alſo by what ſmalleſt Prime Number every other of them may be divided.

So that, whatever Number be propoſed, having divided it firſt by 2 and 5, (and if you will by 3 alſo,) as oft as may be, if it be capable of ſuch diviſion: If the reſult of ſuch diviſion do not exceed 100,000, we have direction in that Table, by what Prime it may be next divided; and then, by what Prime to divide the Quotient of ſuch Diviſion; and ſo continually, 'till we come to a Prime Number.

The reaſon why, in that Table, he omits all even Numbers, and all Numbers ending in 5, is obvious: to wit, Becauſe it appears to view (without the help of a Table) that ſuch are accordingly diviſible by 2, or 5.

He might, for a like reaſon, have omitted alſo all that are diviſible by 3, (becauſe this would preſently appear upon ſuch promiſcuous adding of the Figures as was but now mentioned ;) but that he could not well omit theſe, without diſordering the Form of the Table.

Now, becauſe, in ſuch Tables, it is of great moment that they be carefully Computed, and exactly Printed, (becauſe miſtakes therein are not eaſily obſerved and Corrected by the Readers Eye,) I have taken care to examine that whole Table very exactly, (in the ſame method and with the ſame pains as if I were to compute it anew ;) and find that, though it had been Computed and Printed with great care, yet ſome few miſtakes (and but a few) have eſcaped the Corrector's Eye. Moſt of which are noted in the Table of *Errata*, Printed with it. Beſide which I have obſerved theſe that follow : Which (to ſave another Reader the like labour) I have thought fit (for his eaſe and ſatisfaction) here to note. And, theſe being alſo amended as is here directed (beſide thoſe noted in the Printed *Errata*,) the Table will then be very accurate ; and (I think,) without any Error.

Pag.	*Numb.*	*For*	*Set*	*Pag.*	*Numb.*	*For*	*Set*
3	5579	P	7	28	55609	3	P
5	9287	19	37	31	60701	01	101
8	14873	73	107		60799	63	163
11	20983	3	P	33	64499	13	P
16	30167	71	97		65479	3	P
	31001	-29	29	34	67993	1	P
17	33409	47	P	38	75653	151	P
19	37583	13	7	41	80561	17	13
21	40049	19	29	43	85909	137	P
	40599	P	3	44	86993	79	P
	40759	3	P	47	93719	7	P
	41581	41	43	48	94769	41	97
24	46199	73	P	49	96109	3	13
27	53941	13	17		97487	3	13
28	54449	71	P				

Pag. 7, in the margin (after 43) for 37 ſet 47.

By the help of this Table, if we had the Number propoſed 539,454,600, it is eaſy to reſolve it into the Primes of which it is Compoſed. For firſt, (becauſe of two Cyphers at the end) it is manifeſt that it may be divided twice by 2, and twice by 5. And then (becauſe theſe Cyphers being cut off, the Remainder is yet an even Number) it may be a third time divided by 2; and the reſult will be 2,697,273. And, if this Number were not beyond the reach of the Table, I ſhould ſeek it there; to ſee by what Prime it may be next divided. But, becauſe it is too big for it; I find, upon conſideration, that, the Figures being promiſcuouſly added, and 9 caſt away as oft as may be, nothing remains; and therefore that it may be divided by 9: Which being done, the next Quotient 299,697, may (for a like reaſon) be again divided (not by 9, but) by 3. And the Quotient 99899, is now come within the reach of this Table. And (without aſſaying the Prime Numbers 7, 11, 13, &c, in their order, till I come to a Prime Number by which it may be divided,) I find, by the Table, that it may be divided by 283, but not by any ſmaller Prime; and the Quotient of ſuch diviſion will be 353, another Prime. And therefore the Number propoſed 539,454,600 is $= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 283 \times 353$.

But if, inſtead of 99,899, I had come to a Number greater than this Table, and yet not diviſible by 2, 5, or 3; I muſt then (for want of ſuch Table large enough) have been fain to make tryal of the conſequent Prime Numbers 7, 11, 13, &c, 'till by help of ſuch I had brought it within the Compaſs of the Table; And, if no ſuch can be found, before I come at a Prime as great as the Square Root of ſuch Number; I may then conclude ſuch Number to be a Prime.

CHAP. IV.

Monſieur FERMAT's *Problems concerning Diviſors and Aliquot Parts.*

IT is here proper to conſider of ſuch Queſtions (concerning Aliquot Parts) as thoſe on account of which Monſieur *Fermat* and Monſieur *Frenicle* did value themſelves; as is to be ſeen in my *Commercium Epiſtolicum*, Epiſt. 1, 11, 12, 22, 25, 26, 31, 33. And in a Treatiſe purpoſely Publiſhed on this occaſion by Monſieur *Frenicle*, intituled, *Solutio duorum Problematum, circà numeros Cubos & Quadratos, quæ tanquam inſolubilia univerſis Europæ Mathematicis à Clariſſimo Viro D. Fermat ſunt propoſita*, &c, *à* D. B. F. D. B. *inventa*, &c, (that is, *à Domino Bernardo Frenicle de Beſſy.*) *Pariſiis apud Jacobum Langlois*, &c, 1657, in which he glories much that he was able to ſolve them. And amongſt Monſieur *Fermat's poſthumous* Works, (Publiſhed ſince his death) the Publiſher is pleaſed to inſert his formal Challenge of me to ſolve them (with ſome others Letters to and from Monſieur *Fermat*, concerning the ſame) in theſe Words:

Problemata propoſita à D. Fermat.

Proponatur (ſi placet) Walliſio, & reliquis Angliæ Mathematicis, ſequens Quæſtio Numerica.

Invenire Cubum, qui, additus omnibus ſuis partibus aliquotis, conficiat Quadratum. Exempli gratiâ, Numerus 343 *eſt Cubus à latere* 7. *Omnes ipſius partes aliquotæ ſunt* 1, 7, 49, *quæ, adjunctæ ipſi* 343, *conficiunt numerum* 400, *qui eſt Quadratus à latere* 20. *Quæritur alius Cubus numerus ejuſdem naturæ.*

Quæritur etiam numerus Quadratus, qui, additus ſuis partibus aliquotis, conficiat numerum Cubum.

Has ſolutiones expectamus: Quas, ſi Anglia aut Galliæ Belgicæ & Celtica non dederint, Dabit Gallia Narbonenſis; eáſque, in pignus naſcentis amicitiæ, Domino Digby offeret & dicabit.

But was not ſo kind (though he there inſert alſo divers Letters to and from Monſieur *Fermat*, concerning the ſame) as to inſert thoſe of mine, wherein I ſolved theſe (and others of) his Problems: Nor thoſe of Monſieur *Fermat*, wherein he acknowledgeth that I had ſo done. Which are to be ſeen in my *Commercium Epiſtolicum*, in Epiſt. 23, 28, 29, 47, and elſewhere.

To thoſe two Problems, I added a third of a like nature:

Invenire duos numeros Quadratos, qui, partibus ſuis aliquotis additi, eandem efficiant ſummam. Exempli gratiâ, 16 + 8 + 4 + 2 + 1 = 31 = 25 + 5 + 1. *Inveniantur iſtiuſmodi alii duo.*

The whole Myſtery of ſolving theſe (and ſuch like) Queſtions, I there diſcover in Epiſt. 23, which depends on what is here delivered in art. 8, 9, 10, 11, 12, of the Chapter here next preceeding.

For, 1. A Number added to all its Aliquot Parts, is all one as the Aggregate of its Diviſors. 2. The Diviſors of any Power of a Prime Number, (as of *a*) is a Geometrical Progreſſion from 1 to ſuch Power; as, for inſtance, of a^5, the Diviſors are 1, *a*, *aa*, a^3, a^4, a^5. 3. And therefore the ſum of ſuch Geometrical Progreſſion is the Aggregate of thoſe Diviſors. 4. This Aggregate is conveniently expreſſed by the Primes which Compoſe it. 5. The Diviſors of any Power, or Degree, of one ſuch Prime, ſeverally Multiplied into all thoſe of any Power, or Degree, of any other Prime, give all the Diviſors of the Compound of thoſe Powers. 6. And therefore the Aggregate of thoſe firſt into the Aggregate of thoſe ſecond, give the Aggregate of the Diviſors of ſuch Compound. (For, by the common practice of Multiplication, all the Members of one Number, or Aggregate, Multiplied ſeverally into all the Members of another, are equivalent to the whole of the one, multiplied into the whole of the other.) 7. And therefore the Primes Compoſing this laſt Aggregate, are the ſame with thoſe of both the Aggregates which Compoſe it. 8. And the ſame is in like manner to be argued, in caſe any Power, or Degree, of a third, fourth, or further, Prime, be continually Multiplied with thoſe foregoing: provided always, that they be all ſeveral Primes, and not any of the former Primes repeated; for, in ſuch caſe we are to follow the direction of art. 11, Chap. preceeding.

As, for inſtance; ſuppoſing $a = 2$, and therefore $a^5 = 32$: All the Diviſors hereof (or the Aggregate of ſuch Diviſors) are $1 + a + aa + a^3 + a^4 + a^5 = 1 + 2 + 4 + 8 + 16 + 32 = 63 = 3 \times 3 \times 7$. And ſuppoſing $b = 3$, and therefore $b^4 = 81$: The Aggregate of the Diviſors hereof are $1 + b + bb + b^3 + b^4 = 1 + 3 + 9 + 27 + 81 = 121 = 11 \times 11$: And therefore, of a^5b^4, the Aggregate of Diviſors is $63 \times 121 = 3 \times 3 \times 7, \times 11 \times 11$. And ſuppoſing further $c = 5$, and therefore $c^3 = 125$: The Aggregate of the Diviſors hereof are $1 + c + cc + c^3 = 1 + 5 + 25 + 125 = 156 = 2 \times 2 \times 3 \times 13$: And therefore, of $a^5b^4c^3$, the Aggregate of Diviſors is $63 \times 121 \times 156 = 3 \times 3 \times 7, \times 11 \times 11, \times 2 \times 2 \times 3 \times 13$, or $2 \times 2 \times 3 \times 3 \times 3 \times 7 \times 11 \times 11 \times 13$. And ſo onwards, in caſe of further Compoſitions.

Now, this being univerſal; it will be eaſy to make application thereof, to the particular caſes propoſed; or to any other of like nature.

As for Example.

I. The firſt Queſtion, is, *To find a Cube Number, which added to all its Aliquot Parts will make a Square*; (that is, the Aggregate of whoſe Diviſors ſhall be a Square Number.)

Here it is manifeſt, that ſuch Cube Number muſt be either the Cube of ſome Prime, (or at leaſt the ſecond, third, fourth, or further Cube of ſuch Prime; that is, ſome Power thereof whoſe exponent is diviſible by 3;) or elſe Compounded by the continual Multiplication of ſuch Cubes (firſt, ſecond, third, and ſo forth,) of two or more ſuch Prime Numbers. (For all ſuch, will be Cube Numbers, and no other but ſuch.)

Now, if we can find any ſuch Cube (firſt, ſecond, third, &c,) of any one Prime Number, whereof the Aggregate of Diviſors being expreſſed in Primes, thoſe Primes will be all Pairs, (that is, each of them occurring an even Number of times;) ſuch Aggregate ('tis manifeſt) will be a Square Number; and therefore ſuch Cube, will be ſuch as is required.

And ſuch Cube is $343 = 7 \times 7 \times 7$; whoſe Diviſors are $1 \times 7 \times 49 \times 343 = 400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5$; which is the Square of $2 \times 2 \times 5 = 20$.

When the Cubes (firſt, ſecond, third, or others,) of ſeveral Primes, have not their Aggregate of Diviſors expreſſable by Pairs of Primes; yet may the Compound of Two, Three, or more of ſuch Cubes continually Multiplied (which will alſo be a Cube Number,) have its Aggregate of Diviſors (which is the Compound of the ſeveral Aggregates continually Multiplied) ſo expreſſed: Namely, if the Cubes ſo to be Compounded be ſo choſen as that, what Primes in expreſſing ſome of the Aggregates be ſingle, may be Paired by like ſingle Primes in ſome other of them.

Thus, for the Cube of 47, the Aggregate of Diviſors (expreſſed in Primes) is $2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 13 \times 17$; where (beſide Pairs) we have 2, 3, 5, 13, 17; ſingly: And, for the Cube of 5, the Aggregate is $2 \times 2 \times 3 \times 13$, where (beſide Pairs) we have 3, 13, ſolitary; which (joined to thoſe before) ſerve to Pair 3, 13, but leave 2, 5, 17, yet ſolitary: And, for the Cube of 13, the Aggregate is $2 \times 2 \times 5 \times 7 \times 17$, which afford fellows to 5, 17, but leaves us 2, 7, yet ſolitary: And, for the Cube of 41, the Aggregate is $2 \times 2 \times 3 \times 7 \times 29 \times 29$; where (beſide Pairs) we have 3, 7, ſolitary; which afford a fellow to 7, but leave 2, 3, ſolitary. So that for the Cube of $47 \times 5 \times 13 \times 41$, we have (beſide Pairs) 2, 3, ſolitary. Which may thus be Paired.

For the Cube of 11, the Aggregate of Diviſors is, $2 \times 2 \times 2 \times 3 \times 61$, where (beſide Pairs) we have 2, 3, 61, ſolitary; which afford fellows to 2, 3, but leave 61, ſolitary: And, for the Cube of 27 (or the third Cube of 3, the Aggregate is $2 \times 2 \times 11 \times 11 \times 61$; which (beſide Pairs) affords a fellow to 61. So that, for the Cube of $47 \times 5 \times 13$

×41×11×27, (or 27×5×11×13×41×47) the Aggregate of Diviſors, is 2×2×2×2×2×3×5×13×17, ×2 ×2×3×13, ×2×2×5×7×17, ×2×2×3×7×29× 29, ×2×2×2×3×61, ×2×2×11×11×61: Or (putting the Primes in order) 2×2×2×2×2×2×2×2×2×2×2 ×2×2×2×2×2×3×3×3×3×5×5×7×7×11×11 ×13×13×17×17×29×29×61×61; where we have 2, ſixteen times; 3, four times; and 5, 7, 11, 13, 17, 29, 61, twice; which therefore (being all continually Multiplied) muſt needs afford a Square Number. Which was the thing required to be found in Monſieur Fermat's firſt Queſtion.

In like manner; if with the Cube of 47 × 5 × 13 × 41 (as before) we Compound the Cubes of 2, and of 3, where we have the Aggregates 3 × 5, and 2 × 2 × 2 × 5, which (beſide Pairs) afford us 2, 3, ſolitary; which afford fellows to 2, 3, that were ſolitary before. And therefore for the Compound Cube of 47 × 5 × 13 × 41 × 2 × 3 (or 2 × 3 × 5 × 13 × 41 × 47) we ſhall have (in the Compound Aggregate of Diviſors) theſe Primes Components, 2, fourteen times; 3 and 5, four times; 7, 13, 17, and 29, twice: Which being all continually Multiplied will alſo make a Square Number. Which was the thing required to be found in Monſieur Fermat's firſt Queſtion.

Theſe two Compound Cubes, if they be further Compounded with the Cube of 7 (which is no ingredient in either of them) will afford two more; whoſe Aggregate of Diviſors will (beſide the Primes in each of them reſpectively) have theſe farther Primes Components, 2, four times; and 5, twice: Which, being Compounded with the fore-mentioned Squares, will ſtill afford Square Numbers.

So have we five Cubes, whoſe Aggregate of Diviſors are Squares.

Roots of the Cubes.

7.
27 × 5 × 11 × 13 × 41 × 47.
2 × 3 × 5 × 13 × 41 × 47.
27 × 5 × 7 × 11 × 13 × 41 × 47.
2 × 3 × 5 × 7 × 13 × 41 × 47.

Roots of the Squares.

2×2×5.
2× (Eight-times) ×3×3×5×7×11×13×17×29×61.
2× (Seven-times) ×3×3×5×5× 7×13×17×29.
2× (Ten-times) ×3×3×5×5× 7×11×13×17×29×61.
2× (Nine-times) ×3×3×5×5× 5× 7×13×17×29.

In all which I make uſe of no Cube of a Prime which is not leſs than 100. And, in like manner, may other ſuch Cubes be found; as is there ſhewed in Epiſt. 23, and 28, Such as theſe:

Roots of the Cubes.

2×3×5×13×17×31×41×191.
2×3×5× 7×13×17×31×41×191.
3×3×3× 5×11×13×17×31×41×191.
3×3×3× 5× 7×11×13×17×31×41×191,
17×31×47×191.
7×17×31×47×191.

Roots of the Squares.

2×(Twelve-times) 3×3×3×5×5×7×13×17×29×29×37.
2×(Fourteen-times)3×3×3×5×5×5×7×13×17×29×29×37.
2×(Thirteen-times)3×3×3×5×7×11×13×17×29×29×37×61.
2×(Fifteen-times)3×3×3×5×5×7×11×13×17×29×29×37×61.
2×(Ten-times) 3×3×5×13×17×29×37.
2×(Twelve-times) 3×3×5×5×13×17×29×37.

In all which I make uſe of no Cube of a Prime Number which is not leſs than 200.

But, in order to make theſe Inquiries for ſuch Cubes; it is expedient to have at hand a Table of the Cubes of Prime Numbers (and of the ſecond, third, or further Cubes, of the leſſer of them,) or of the Roots of ſuch Cubes; with the Aggregate of Diviſors (in each of thoſe Cubes) expreſſed in Primes.

And, to ſave the Reader the labour of computing ſuch a-new, I here ſubjoin what I have at hand.

Roots of the Cubes.	Aggregate of their Diviſors.
1	1
2	3×5
4	127
8	3×11×31
16	8191
32	3×5×17×257
3	2×2×2×5
9	1093
27	2×2×11×11×61
81	797161
243	2×2×2×2×2×5×17×41×193
5	2×2×3×13
25	19531
125	2×3×11×71×521
7	2×2×2×2×5×5
11	2×2×2×3×61
13	2×2×5×7×17
17	2×2×3×3×5×29
19	2×2×2×5×181
23	2×2×2×2×3×5×53
29	2×2×3×5×421
31	2×2×2×2×2×2×13×37
37	2×2×5×2603
41	2×2×3×7×29×29
43	2×2×2×5×5×11×37
47	2×2×2×2×2×3×5×13×17
53	2×2×3×3×3×5×281
59	2×2×2×3×5×1741
61	2×2×31×1861
67	2×2×2×5×17×449
71	2×2×2×2×3×3×2521
73	2×2×5×13×37×41

Roots of the Cubes.	Aggregates of their Diviſors.
79	2 × 2 × 2 × 2 × 2 × 5 × 3121
83	2 × 2 × 2 × 3 × 5 × 7 × 13 × 53
89	2 × 2 × 3 × 3 × 5 × 17 × 233
97	2 × 2 × 5 × 7 × 7 × 941
101	2 × 2 × 3 × 17 × 5101
103	2 × 2 × 2 × 2 × 5 × 13 × 1061
107	2 × 2 × 2 × 3 × 3 × 3 × 5 × 5 × 229
109	2 × 2 × 5 × 11 × 13 × 457
113	2 × 2 × 3 × 5 × 19 × 1277
127	2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 5 × 1613
131	2 × 2 × 13 × 19 × 2293
137	2 × 2 × 3 × 5 × 23 × 1877
139	2 × 2 × 2 × 5 × 67627
149	2 × 2 × 3 × 5 × 5 × 11 × 101
151	2 × 2 × 2 × 2 × 13 × 19 × 877
157	2 × 2 × 5 × 5 × 17 × 29 × 79
163	2 × 2 × 41 × 2657
167	2 × 2 × 2 × 3 × 7 × 2789
173	2 × 2 × 3 × 5 × 29 × 41 × 73
179	2 × 2 × 2 × 3 × 3 × 5 × 37 × 433
181	2 × 2 × 7 × 13 × 16381
191	2 × 2 × 2 × 2 × 2 × 2 × 2 × 3 × 17 × 29 × 37
193	2 × 2 × 5 × 5 × 5 × 14453
197	2 × 2 × 3 × 3 × 5 × 42691
199	2 × 2 × 2 × 2 × 5 × 5 × 19801

If, in the Queſtion propoſed, it had been required that the Aggregate of Diviſors (of the Cube ſought) ſhould be (not a Square Number, but) the *Double*, *Treble*, or otherwiſe *Multiple*, of a Square Number: The proceſs would be juſt the ſame, (and the ſame Table will ſerve,) ſave that, then, the Aggregate is to be diviſible by 2, 3, or ſuch other Number as is the exponent of the propoſed Multiple, and the reſt of the Primes compoſing it to be all Pairs.

Thus, if the *Decuple* of a Square be required; the Cube of 3 will anſwer it; where the Aggregate is 2 × 2 × 2 × 5; that is, beſide 2 × 5 = 10, the other Components are Pairs.

If the *Quadruple* of a Square (which muſt therefore itſelf be a Square;) the Cube 7 Anſwers it; whoſe Aggregate is $2 \times 2 \times 2 \times 2 \times 5 \times 5$: Out of which, if we exempt $2 \times 2 = 4$, the reſt are Pairs. And ſo will any other Cube whoſe Aggregate of Diviſors is an even Square, and therefore diviſible by 4.

If the *Sextuple* be required: The Cube of 27×11 anſwers it; where the Aggregate is $2 \times 2 \times 11 \times 11 \times 61, \times 2 \times 2 \times 2 \times 3 \times 61$. Whence if we exempt $2 \times 3 = 6$, the reſt are Pairs: And ſo will alſo (for the ſame reaſon) the Cube of 2×3; where the Aggregate is $3 \times 5, \times 2 \times 2 \times 2 \times 5$. And the like in other caſes.

But if ſuch Multiple ſhould be required, as that no Aggregate can be found (or not within certain limits) which, being divided by the Exponent of that Multiple, will leave the reſt of the Prime Components Pairs; ſuch caſe (at leaſt within ſuch limits) is an impoſſible caſe.

As, if we demand a Square's Multiple by 23, 43, or 47; and confine ourſelves to the Cubes of the Table foregoing; it is manifeſt that (without aſſuming the Cube of ſome other Prime, or ſome further Cube of ſome of theſe,) it cannot be done. For here, amongſt all the Prime Components of the Aggregates, the Numbers 43, and 47, come not at all; and though 23 come once (at the Cube of 137) yet it is there joyned with 1877, which (coming no more) cannot be Paired by any ſuch Compoſition of the propoſed Aggregates. (Remembring always, what was before noted, that the Aggregates for two or more Cubick Powers of the ſame Prime, are not here to be Compounded.) So that (within the limits of the Table) the caſe is not poſſible. And the like may be ſhewed of many others: I ſay, not poſſible *within the limits of this Table.* But, to ſay it is *not at all poſſible*, through the whole extent of all poſſible Numbers; is (I think) too bold an aſſertion for any to make out.

Of the Second Queſtion propoſed by Monſieur FERMAT. *See above, page* 322.

II. The Second Queſtion is, (*To find a Square Number, which*

which added to all its Aliquot Parts will make a Cube; that is, the Aggregate of whoſe Diviſors ſhall be a Cubick Number.)

And here the proceſs is much the ſame as before; ſave that here we ſhall need a Table of Square Numbers, (as there of Cubes,) with their Aggregate of Diviſors expreſſed in Primes: And here we are to find out, or ſo to Compound, the Aggregates, as that the Primes expreſſing them may be (not Couples or Duplicates, as there, but) Triplicates: That is, that each Prime may occur three, ſix, nine, or other Number of times diviſible by three.

But, though the proceſs be much the ſame, yet the ſucceſs will not be altogether ſo ready as there; becauſe Triplicates of the Components will not be ſo eaſily adjuſted as Duplicates. (And, for the ſame reaſons, if Biquadrates, or Surſolids, or ſome higher Powers, were required; the proceſs would ſtill be much the ſame, but the trouble of finding ſuch would ſtill be increaſed.)

Such Table of Squares (becauſe I have it at hand) I ſhall here ſubjoin; to ſave the Reader (who ſhall think fit to give himſelf the trouble of inquiring into ſuch Queſtions) the labour of Computing the ſame again.

Roots of the Squares.	Aggregate of their Diviſors.
1	1
2	7
4	31
8	127
16	511 = 7 × 73
32	2047 = 23 × 89
64	8191
128	32767 = 7 × 31 × 151
256	131071
3	13
9	121 = 11 × 11
27	1093
81	9841 = 13 × 757
243	88573 = 23 × 3851

Roots of the Squares.	Aggregate of their Diviſors.
5	31
25	781 = 11 × 71
125	19531
625	488281 = 19 × 31 × 829
7	57 = 3 × 19
49	2801
343	137257 = 29 × 4733
2401	6725601 = 3 × 3 × 19 × 37 × 1063
11	133 = 7 × 19
121	16105 = 5 × 3221
13	183 = 3 × 61
169	30941
17	307
289	88741
19	381 = 3 × 127
361	137561 = 151 × 911
23	553 = 7 × 79
29	871 = 13 × 67
31	993 = 3 × 331
37	1407 = 3 × 7 × 67
41	1723
43	1893 = 3 × 631
47	2257 = 37 × 61
53	2863 = 7 × 409
59	3541
61	3783 = 3 × 13 × 97
67	4557 = 3 × 7 × 7 × 31
71	5113
73	5403 = 3 × 1801
79	6321 = 3 × 7 × 7 × 43
83	6973 = 19 × 367
89	8011
97	9507 = 3 × 3169
101	10303
103	10713 = 3 × 3571
107	11557 = 7 × 13 × 127
109	11991 = 3 × 7 × 571

Roots of the Squares	Aggregate of their Diviſors.
113	12883 = 13 × 991
127	16257 = 3 × 5419
131	17293
137	18907 = 7 × 37 × 73
139	19461 = 3 × 13 × 499
149	22351 = 7 × 3193
151	22953 = 3 × 7 × 1093
157	24807 = 3 × 8269
163	26733 = 3 × 7 × 19 × 67
167	28057
173	30103
179	32221 = 7 × 4603
181	32943 = 3 × 79 × 139
191	36673 = 7 × 13 × 13 × 31
193	37443 = 3 × 7 × 1783
197	39007 = 19 × 2053
199	39801 = 3 × 13267
211	44733 = 3 × 13 × 31 × 37
223	49953 = 3 × 16651
227	51757 = 73 × 709
229	52671 = 3 × 97 × 181
233	54523 = 7 × 7789
239	57361 = 19 × 3019
241	58323 = 3 × 19441
251	63253 = 43 × 1471
257	66307 = 61 × 1087
263	69433 = 7 × 7 × 13 × 109
269	72631 = 13 × 37 × 151
271	73713 = 3 × 24571
277	77007 = 3 × 7 × 3667
281	79243 = 109 × 727
283	80373 = 3 × 73 × 367
293	86143
307	94557 = 3 × 43 × 733
311	97033 = 19 × 5107
313	98283 = 3 × 181 × 181
317	100807 = 7 × 14401

Roots of the Squares.	Aggregate of their Diviſors.
331	109893 = 3 × 7 × 5233
337	113907 = 3 × 43 × 883
347	120757 = 7 × 13 × 1327
349	122151 = 3 × 19 × 2143
353	124963 = 19 × 6577
359	129241 = 7 × 37 × 499
367	135037 = 7 × 101 × 191
373	139503 = 3 × 7 × 7 × 13 × 73
379	144021 = 3 × 61 × 787
383	147073
389	151711 = 7 × 21673
397	158007 = 3 × 31 × 1699
401	161203 = 7 × 23029
409	167691 = 3 × 55897
419	175981 = 13 × 13537
421	177663 = 3 × 59221
431	186193 = 7 × 67 × 397
433	187923 = 3 × 37 × 1693
439	193161 = 3 × 31 × 31 × 67
443	196693
449	202051 = 97 × 2083
457	209307 = 3 × 7 × 9967
461	213083 = 13 × 37 × 443
463	214833 = 3 × 19 × 3769
467	218557 = 19 × 11503
479	229921 = 43 × 5347
487	237657 = 3 × 7 × 11317
491	241573 = 37 × 6529
499	249501 = 3 × 7 × 109 × 109

Now it is manifeſt, upon view, that (if we confine ourſelves to the limits of this Table) many of theſe Numbers are not of uſe, to the preſent purpoſe. Becauſe many of the Primes (amongſt the Aggregates) come but once; as 5. 29. 71. 89. 101. 139. 191. 307. 331. 397. 409. 443. 571. 631. 709. 727. 733. 757. 787. 829. 883. 911. 991. 1063. 1087. 1327. 1471. 1693. 1699. 1723. 1783. 1801. 2053. 2083. 2143.

2143. 2801. 3019. 3169. 3193. 3221. 3541. 3571. 3667. 3769. 3851. 4603. 4733. 5107. 5113. 5233. 5347. 5419. 6529. 6577. 7789. 8011. 8191. 8269. 9967. 10303. 11317. 11503. 13267. 13537. 14401. 16651. 17293. 19441. 19531. 21673. 23029. 24571. 28057. 30103. 30941. 55897. 59221. 86143. 88741. 131071. 147073. 196693. Others but twice (not thrice) as 23. 79. 367. 499. 1093. And therefore cannot by any Compoſition (within theſe limits) make a Cube. And, conſequently, all the Squares to which any of them belong, are to be laid aſide as not of uſe. And thoſe are, the Squares of 32, 64, 256, 27, 81, 243, 25, 125. 625. 49, 343, 2401, 121, 169, 17, 289, 361, 23, 31, 41, 43, 53, 59, 71, 73, 83, 89, 97, 101, 103, 109, 113, 127, 131, 139, 149, 151, 157, 167, 173, 179, 181, 193, 197, 199, 223, 227, 233, 239, 241, 251, 257, 271, 277, 281, 283, 293, 307, 311, 317, 331, 337, 347, 349, 353, 359, 367, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 443, 449, 457, 461, 463, 467, 479, 487, 491. (And the Square of 1, is, in this caſe, inſignificant; becauſe a Multiplication by 1 makes no alteration.) And, theſe being laid aſide, we muſt alſo lay aſide the Squares of 128, 9, 13, 47, 61, 79, 229, 269. Becauſe, in thoſe that remain, 43 occurs but once; and 11, 61, 97, 151, but twice. And, thoſe being laid aſide, we muſt alſo lay aſide the Squares of 137, 211, 313, becauſe, in thoſe now remaining, 37, 181, occur but twice. And (137 being laid aſide) the Squares of 16, 373, muſt alſo be laid aſide; becauſe now 73 comes but twice.

So that we have now but theſe few left for conſideration, to wit, the Squares of 2, 4, 8, 3, 5, 7, 11, 19, 29, 37, 67, 107, 163, 191, 263, 439, 499. Which, with their Aggregates, ſtand thus:

2	7	5	31	29	13 × 67	163	3 × 7 × 19 × 67
4	31	7	3 × 19	37	3 × 7 × 67	191	7 × 13 × 13 × 31
8	127	11	7 × 19	67	3 × 7 × 7 × 31	263	7 × 7 × 13 × 109
3	13	19	3 × 127	107	7 × 13 × 127	439	3 × 31 × 31 × 67
						499	3 × 7 × 109 × 109

In which there is no Prime (amongſt the Aggregates) which doth not occur at leaſt three times. That is, 3 ſeven times; 7 eleven times; 13 and 31 ſix times; 67 four times; 19, 109, 127, three times.

Of theſe I will firſt conſider 127; which, becauſe it comes but thrice, we muſt take all or none of them. If all, then this (at 107) brings in 13; which muſt therefore be trebled. And it muſt be done one of theſe three ways, either by taking in the Squares of 3 and 29; or of 3 and 263; or of 191 alone.

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31

If the firſt way, this (at 29) brings in 67. Which (that it may be trebled) brings in two of theſe 3 Squares 37, 163, 439. Of which, if 163 be one, this (becauſe of 19) brings in the Squares 7 and 11. And if, for the other, we take the Square of 37; this brings in 3 and 7 a fourth time, and therefore either each of them muſt come in twice more (that we may have them ſix times) or elſe 37 muſt here be laid aſide. Now if, for 3 twice, we take (for one of them) the Square of 439, this brings in a fourth 67; which muſt not be (unleſs we could have it ſix times, which we cannot.) Therefore, if at all, this 3 twice, muſt be ſupplied by the Squares of 67 and 499 (for there is no other ſupply;) which brings in 109 twice; and this (that it may be tripled) requires the Square of 263. But, with this, comes in 13 a fourth time; and therefore (that we may have it ſix times) we muſt take in the Square of 191. But, by this time, we have 7 ten times; which muſt not be, unleſs we could (which we cannot) have it twelve times. Therefore the Square of 37 muſt here be laid aſide. If then (retaining that of 163) we take (inſtead of 37) the Square of 439; this brings in 3 a fourth time; which therefore we muſt have twice more. But not from the Square of 37 (becauſe already laid by, and becauſe it would bring in a fourth 67;) therefore,

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
439	3, 31, 31, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31
4	31
5	31

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
5	31

therefore, if at all, from the Squares of 67 and 499 (as before,) which requires that of 263; and, this, that of 191, as before. But now we have 31 a fourth time, which requires it twice more; which is not to be had, ſave at the Squares of 4 and 5; whereof that of 4 is not to be admitted, as being included in that of 8 already taken. So that the Square of 163 cannot be taken either with that of 37 or of 439, and muſt therefore be laid aſide; (and, with it, the Squares of 7 and 11.) And conſequently (retaining that of 3 and of 29,) we muſt (for trebling of 67) take the Squares of 37 and 439. And here we have 31 twice, and muſt therefore have it a third time: But not from the Square of 4; (becauſe included in that of 8:) Therefore either from that of 5, or of 191. If from that of 5; we ſhall want a third 7 (having yet but two;) which we cannot have from the Square of 2 (becauſe included in 8;) nor from 163 (becauſe already rejected;) nor from that of 11 (becauſe already excluded with that of 163;) nor from that of 191, becauſe this would bring in a fourth 31, (which may not be, becauſe we cannot have it ſix times without the Square of 4, which is included in that of 8;) nor from that of 69 (for the ſame reaſon;) nor from that of 499, becauſe this cannot ſtand without that of 263; nor from both theſe together; becauſe then we ſhall have it five times, but cannot have it a ſixth; (all the reſt wherein 7 is found, being already excluded.) Therefore (omitting that of 5) we muſt (if at all) have a third 31 from the Square of 191. But this brings in a fourth and fifth 13; which (for a ſixth) will require the Square of 263; and this (becauſe of 109) the

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
191	7, 13, 13, 31
263	7, 7, 13, 109
499	3, 7, 109, 109

Square of 499. And this (beſide Triplicates) brings in a fourth 3; (which therefore will afford, not a Cube, but the Triple of a Cube, if that had been required;) we want therefore 3 twice more (to make it up ſix times;) but can have neither of them from the Squares of 7 or 163 (as being already excluded,) nor from that of 67, (as bringing in a fourth 31,) and therefore not at all. And, conſequently, this firſt way (by the Squares of 3 and 29) doth not ſucceed.

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
7	3, 19
163	3, 7, 19, 67
11	7, 19
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31

The ſecond way of ſupplying 13 twice, (which at the Square of 107 were wanting;) is, from the Squares of 3 and 263: Which (becauſe of 109) requires that of 499. And, becauſe (amongſt the Aggregates) we have 3 twice; we muſt have it a third time. If, for this, we take in the Square of 7, or of 163; either of theſe (becauſe of 19) brings in the other, and that of 11. And now, becauſe of 67 once, we muſt have it twice more. But not from the Square of 29 (being already excluded as not to be taken with that of 3;) and therefore from the Squares of 37, and 439. And, by this time we have 3 ſix times (and more than ſo, we may not have it, unleſs we could have it nine times;) and 7 we have 7 times, and therefore muſt have it twice more: But, not from the Square of 2 (as being included in that of 8;) nor from that of 191, (becauſe this would bring in 13 a fourth and a fifth time, which would require a ſixth, from the Square of 29 already rejected;) therefore, if at all, from the Square of 67. But neither can this be, (becauſe it brings in a ſeventh 3; which may not be, there being no more to make it up nine times:) And, conſequently, the third 3 (wanting at the Square of 499) is not to be ſupplied from the Squares of 7, or of 163. If then (omitting theſe two) we ſhould take (for a third 3) the Square of 37 or of 439, either of theſe (becauſe of 67) would bring in the other, and alſo require that of 29, or of 163, already rejected. If then (omitting theſe of 37 and 439) we take (for a third 3) the Square of 67; this brings in 31, which is therefore to be Tripled. But not from the Square of 4 (as included in that of 8;) nor from the Square of 191 (becauſe that would bring in a fourth and fifth 13, which would require a ſixth from the Square of 29 already rejected;) nor from the Square of 439 (becauſe of 67 there, which would bring in that of 29, or 37, or 163, already rejected) nor from the Square of 5, becauſe (though that would afford a ſecond 31,) a third would yet be wanting, and not to be had. And, conſequently, (there being no other place from whence to fetch a third 3) this ſecond way will not ſucceed.

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
37	3, 7, 67
439	3, 31, 31, 67

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
67	3, 7, 7, 31

The third way for ſupplying 13 twice, (which at the Square of 107 were wanting) is (omitting the Squares of 3, 29, 263,) from the Square of 191. And, becauſe here we have 31 once, this muſt be Tripled. But not from the Square of 4: (as included in 8;) And therefore, if at all, either from that of 439 (where it is twice,) or from the Squares of 5 and 67. If from that of 439; then 67 (here found) muſt be Tripled; but not from the Square of 29 (as already excluded,) therefore from thoſe of 37, and 163; and this laſt (becauſe of 19) calls in thoſe of 7 and 11. But, by this time, we have 3 five times, and therefore

8	127
19	3, 127
107	7, 13, 127
191	7, 13, 13, 31
439	3, 31, 31, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31

ſhould have it a ſixth time; but not from the Square of 499 (for that would recall that of 263 already rejected;) therefore, if at all, from that of 67; but we ſhall then have 7 ſeven times; which is not to be admitted, ſince we cannot have it nine times. Therefore (omitting that of 439, and therefore thoſe of 37 and 163) take we thoſe of 5 and 67.

And, by this time; we have 7 four times; and therefore, if at all, we muſt have it twice more. But not from the Square of 2 (as included in 8;) nor from that of 37 or 163 (as already rejected, with that of 439;) nor from that of 11 (which, becauſe of 19, would bring us back to that of 163 already rejected;) nor from 499 (which, becauſe of 109, would bring us back to that of 263 already laid aſide;) and therefore not at all. So that this third way fails alſo: And, conſequently, the Squares of 8, 19, 107, (where we meet with 127,) muſt all be laid aſide.

8	127
19	3, 127
107	7, 13, 127
191	7, 13, 13, 31
5	31
67	3, 7, 7, 31

We have then but theſe left to be further conſidered.

2	7	5	31	29	13, 67	163	3,7,19,67	439	3,31,31,67
4	31	7	3,19	37	3,7,67	391	7,13,13,31	499	3,7,109,109
3	13	11	7,19	67	3,7,7,31	263	7,7,13,109		

And here we will begin with the Prime 109; which, becauſe it comes but once at the Square of 263, and twice at that of 499; theſe muſt either both be taken, or both omitted.

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31
439	3, 31, 31, 67

And becauſe, in theſe, we have 13 once; this muſt be taken twice more. And therefore either from the Squares of 3 and 29, or from that of 191 above; (ſince we have it now but five times in all.)

If the firſt way; then, becauſe of 67 once, we muſt take it twice more; from two Squares of theſe three, 37, 163, 439. Firſt, let thoſe be the Squares

Squares of 37 and 163; therefore (becauſe of 19) we muſt take alſo thoſe of 7 and 11. And, by this time, we have 3 four times, (and this affords us, not a Cube, but the Triple of a Cube, if that were required;) we muſt therefore take it twice more; which is only to be had at the Squares of 67 and 439, (for now we have it but ſix times in all,) but this brings in a fourth 67 which cannot be admitted. Secondly, let it be the Squares of 37 and of 439: which brings in 31 twice, and we muſt therefore have it a third time. Which if we take from the Square of 67; this brings in a fourth 3; which will require two more, from the Squares of 7 and 163; which will bring in a fourth 67. If from the Square of 191; this brings in a fourth and fifth 13, which cannot be admitted, becauſe we have not a ſixth. If from the Square either of 4, or of 5; either of theſe (beſide Triplicates) would leave us 7 four times (which would afford, not a Cube, but the Septuple of a Cube, if that had been required;) but this requires 7 twice more. Neither of which can be had from the Squares of 67, or 191, (as being already rejected;) nor from that of 163 (as bringing in a fourth 67;) and therefore, if at all, from the Squares of 2 and 11. But this would bring in 19; and therefore (to Triple it) will call in the Squares of 7 and 163; (which laſt is already rejected, and would bring in a fourth 67;) therefore not at all. Thirdly, (omitting that of 37) let this 67 twice, be taken from the Squares of 163 and 439. But this (becauſe of 19) calls in the Squares of 7 and 11; and conſequently, (becauſe then we have 3 four times) the Squares of 37 and 67 already rejected. So that this firſt way ſucceeds not.

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31
7	3, 19
163	3, 7, 19, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 3, 67
191	7, 13, 13, 31

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
4, 5	31
2	7
11	7, 19
7	3, 19
163	3, 7, 19, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
163	3, 7, 19, 67
439	3, 31, 31, 67
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
439	3, 31, 31, 67
37	3. 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31
4	31
5	31

If we take the ſecond way of ſupplying 13 twice, (which at the Squares of 263 and 499 were wanting) by the Square of 191 (omitting thoſe of 3 and 29;) then, becauſe here we have 31 once, which muſt therefore be ſupplied twice more: We will firſt try whether it may be done by the Square of 439 (where it comes twice;) and then whether it can be done without this.

If we ſupply it from the Square of 439; this brings in 67, which muſt therefore be Tripled: But not by the Square of 29 (as already rejected, and as bringing in a fourth 13;) therefore from thoſe of 37 and 163. Where becauſe we have 19 once, we muſt have it twice more, from the Squares of 7 and 11. And by this time we have 7 ſeven times, and muſt therefore have it twice more; And we have 3 five times, and muſt therefore have it once more. Both which we may have from the Square of 67 (and from thence only, becauſe 3 is to be had no where elſe;) and now we have 31 a fourth time; which requires it twice more (that it may be ſix times;) and theſe we have at the Squares of 4 and 5. So that now we have a Cube compleated; whoſe Components are, 7, nine times; 3 and 31, ſix times; 13, 67, and 109, three times. And the Square whence it ariſeth, is that of 4 × 5 × 7 × 11 × 37 × 67 × 163 × 191 × 263 × 439 × 499.

2	7
3	13
29	13, 67

The remaining Squares which are not ingredients into this, are thoſe of 2, 3, 29.

Now if from theſe (without the other) we could form another Cube, ſuch Cube would not only be another ſuch Cube as is deſired, but (being a Prime to that already found) might be Compounded with that found, to make a third. But this cannot be: Becauſe (for theſe) we have no Prime that comes three times.

It remains to ſee, if (omitting the Square of 439) we can otherwiſe ſupply 31 twice; which at the Square of 191 were wanting. Where, firſt, it is manifeſt, that (the Square of 439 being laid aſide) thoſe of 37 and 163 (becauſe of 67) muſt alſo be laid aſide, unleſs we can have a third 67 from the Square of 29. Which cannot be, becauſe this would introduce a fourth 13, and we have not two more to make up ſix. Then, having laid by that of 163, we muſt (becauſe of 19) lay by thoſe of 7 and 11. So that there remain only the Squares of 2, 4, 5, 67, to ſupply 31 twice (becauſe we have it once) and 7 twice (becauſe we have it four times) and 3 twice (becauſe we have it once.) Now 31 might be ſupplied twice from the Squares of 4 and 5, (but then we could take no more, becauſe that of 2 is included in 4; and 67 would bring in a fourth 31.) Or it might be ſupplied by one of thoſe (ſuppoſe 5,) with that of 67. And thus we ſhould have a ſupply of 31 twice, and of 7 twice, and of 3 once: But there wants another 3 (which the remaining Squares of 2 and 4 cannot ſupply) to compleat the Cube. So that this affords, not a Cube, but $\frac{1}{3}$ of a Cube. There is therefore no other Cube (but that before aſſigned) here to be had, retaining (as is hitherto ſuppoſed) the Numbers 109, 109, 109.

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
37	3, 7, 67
163	3, 7, 19, 67
29	13, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
4, 5	31
67	3, 7, 7, 31

Let us therefore now leave out 109, and conſequently the Squares of 263 and 499, where it is found; and ſee whether the remaining Squares will afford ſuch a Cube as is deſired. Now theſe are,

2	7	3	13	7	3,19	29	13,67	67	3,7,7,31	191	7,13,13,31
4	31	5	31	11	7,19	37	3,7,67	163	3,7,19,67	439	3,31,31,67

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
439	3, 31, 31, 67

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
29	13, 67
3	13

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
29	13, 67
191	7, 13, 13, 31
67	3, 7, 7, 31
4, 5	31

7	3, 19
11	7, 19
163	3, 7, 19, 67
29	13, 67
439	3, 31, 31, 67
191	7, 13, 13, 31

Of theſe, we will firſt begin with 19, which comes thrice (and but thrice) at the Squares of 7, 11, 163. Where we have 67 once, and therefore muſt have it twice more. Now if, for one of theſe, we take the Square of 37; we muſt, for the other, take either the Square of 439, or of 29. If that of 439; this brings in 3 a fourth time; which may not be, becauſe it comes not twice more to make up ſix times. Therefore (if at all) it muſt be that of 29, (or elſe 37 muſt be laid aſide;) But this brings in 13 once, for which we may have a ſecond at the Square of 3, but then we cannot have a third without a fourth, at the Square of 191. Therefore (waving that at the Square of 3) we muſt take both (if at all) at the Square of 191. Now this brings in 7 a fourth time, which calls for a fifth and ſixth: One of theſe we might have at the Square of 2; but then we cannot have a ſixth without a ſeventh. Therefore (waving that at 2) we muſt (if at all) take both at the Square of 67. But here, beſide a ſecond 31 (for which we may have a third at the Square of 4, or of 5,) we have 3 a fourth time (which will make up, not a Cube, but the Triple of a Cube,) which is not to be admitted, becauſe we cannot have a fifth and ſixth. And conſequently, the Square of 37 muſt be laid aſide, (as not to be joined either with that of 439 or 29;) but (waving that) we muſt have recourſe to the other two (at 29 and 439) for Tripling of 67. Now here we have 13 once; and therefore muſt have it twice more; not from the Square of 3, (becauſe, as before, if we take a ſecond here, we cannot have a third without a fourth;) but from that of 191. Which doth not only ſupply 13 twice; but alſo 7 and 31 which were alſo wanting: So that we have now a ſecond Cube, ſuch as was deſired; whoſe Components are, 3, 7, 13, 19, 31, 67, thrice taken. And the Square whence it ariſeth, is that of 7 × 11 × 29 × 163 × 191 × 439.

And if, from the remaining Square of 2, 4, 3, 5, 37, 67, we could form a third; this, Compounded with the laſt foregoing (as Prime to it) would form a fourth. But this cannot be, becauſe no Prime doth here thrice occur, but only 7 and 31: And neither of theſe can be thrice taken, without being incumbered with 3, which cannot be Tripled. So that, retaining 19 (as is hitherto ſuppoſed) we can have (from thence) no other Cube than what is already found.

2	7
4	31
3	13
5	31
37	3, 7, 67
67	3, 7, 7, 31

Let us now therefore lay by 19; and conſequently the Squares of 7, 11, 163, wherein it is found. And we have then theſe only left for conſideration.

2	7	3	13	29	13,67	67	3,7,7,31	439	3,31,31,67
4	31	5	31	37	3,7,67	191	7,13,13,31		

We have here 67 three times, at the Squares of 29, 37, 439. And (with theſe) we have 3 twice; which calls for a third from the Square of 67. And we have 13 once, for which we might have a ſecond at the Square of 3; but could not then have a third without a fourth; therefore (waving that) we take both from the Square of 191. And we have then 31 four times, and therefore muſt take it twice more from the Squares of 4, and of 5. But we have 7 four

29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31
191	7, 13, 13, 31
4	31
5	31

times; yet cannot find it twice more to make it up ſix times; nor indeed once more, becauſe we cannot here Compound the Square of 2, as being included in that of 4. So that, with 67, we may make up, not a Cube, but a Sextuple of a Cube.

Suppoſe we then that 67 be laid aſide; and therefore the Squares of 29, 37, 439. Thoſe that then remain are,

2	7	4	31	3	13	5	31	67	3,7,7,31	191	7,13,13,31

Of theſe, that of 67 muſt be laid aſide (becauſe 3 occurs but once,) and conſequently (becauſe 7 comes then but twice) that of 2 and 191. And for the other three (of 3, 4, 5,) the Number 13 comes but once; and 31 but twice. So that no further Cube can be hence expected.

4	31
5	31
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31
163	3, 7, 19, 67
191	7, 13, 13, 31
263	7, 7, 13, 169
439	3, 31, 31, 67
499	3, 7, 109, 109

We conclude therefore (having fully conſidered all) that (within the extent of this Table) we may have two Squares (and but two) ſuch as are deſired; whoſe Aggregate of Diviſors ſhall be a Cube. Namely, the Square of $7\times11\times29\times163\times191\times439$, whoſe Aggregate of Diviſors is the Cube of $3\times7\times13\times19\times31\times67$. And the Square of $4\times5\times7\times11\times37\times67\times163\times191\times263\times439\times499$; whoſe Aggregate of Diviſors is the Cube of $3\times3\times7\times7\times7\times13\times19\times31\times31\times67\times109$.

And, if any think it worth the pains to ſeek out more; they muſt enlarge the Table, to take in more Primes, or more Quadratick Powers of theſe Primes.

7	3, 19
11	7, 19
29	13, 67
163	3, 7, 19, 67
191	7, 13, 13, 31
439	3, 31, 31, 67

It had been eaſy to have rendered this buſineſs more ſtupendous (as ſome other would have done,) if (concealing the methods whereby I came at them) I would have performed the Multiplications here directed; and then, in thoſe great Numbers, exhibited theſe two Squares, with the two Cubes thence ariſing; affirming, that (within ſuch extent of Numbers) there is no other Square Number (beſide theſe two, vaſtly great,) which added to all its Aliquot Parts will make a Cube: Or perhaps, having aſſigned thoſe two, propoſed a Challenge to all the Mathematicians in *France*;) to find a third within thoſe limits. But this would ſerve only to amuſe a Reader, not to inſtruct him. And I chuſe rather (in what I publiſh) to inform my Reader, by what ſteps I come at thoſe diſcoveries I make, and whereby he may (if he pleaſe) attain the like; deſigning more, the benefit of others, than oſtentation.

I may here add (as is done after the former Queſtion;) that the ſame method is to be uſed, if (inſtead of a Cube) it had been demanded, that ſuch Aggregate ſhould be the Triple (or other deſigned Multiple) of a Cube: (ſuppoſing ſuch deſigned Multiple to be poſſible:) Of which I have given ſome inſtances as I paſſed along; and might have done more if it had been needful.

But we muſt not then demand the Duple, Quadruple, Sextuple of a Cube, or otherwiſe Multiple thereof by an *even* Number: For all ſuch are impoſſible. For, ſince every Quadratick power of a Prime Number (be it the firſt, ſecond, third, or further Square thereof,) hath, for its Diviſors, (beſide 1) all its Degrees or Powers ſo far; (as, for inſtance, a^6 hath for its Diviſors 1, a, aa, a^3, a^4, a^5, a^6,) and all theſe (becauſe it is a Quadratick Power) are (excluding 1) in Number even; (and every of them either odd or even according as is the Prime a whence it ariſeth;) and conſequently, the Aggregate of all except 1, an even Number; (for an even Number of *odds*, as well as an even Number of *evens*, will ſtill make an even Number;) to this even Number, if 1 be added (which is alſo an Aliquot Part, and therefore a Diviſor,) this always makes the whole Aggregate an odd Number: Which therefore cannot be Duple of Cube, or its Multiple by an even Number. And the ſame will hold as well for the Quadratick Powers of any Compound Number: For (as was ſhewed before) the Aggregate of Diviſors of ſuch Compound Square, is always Compounded of ſuch Aggregates of Diviſors of ſome Quadratick Powers of Primes; which, being (as is now ſhewed) odd Numbers, their Compound muſt be ſo too. For an odd Number, Multiplied by an odd Number (and ſo continually) will ſtill produce an odd Number; and therefore, not the Duple (or otherwiſe Multiple by an even Number) of any Number whatſoever.

In the former Queſtion, concerning Cubick Powers, whoſe Aggregate of Diviſors ſhould be equal to a Square, (or a deſigned Multiple of a Square,) this will not hold, For there the Aggregate may be either an odd or an even Number. Yet with this diverſity: If the Prime *a* be 2, then all the Degrees thereof will be even Numbers, to which when 1 is added the Aggregate will be odd. If the Prime *a* be 3 (or other odd Prime,) and the Cube thence ariſing be the firſt, third, fifth Cube, (or other in odd places) whoſe Number of dimenſions is 3, 9, 15, or other odd Number; the Number of Diviſors, without 1, will be odd alſo; and therefore, with 1, it will become even. But if ſuch Prime *a*, be odd, and the Cubick Power thereof be the ſecond, fourth, ſixth, or other in even places, whoſe Number of dimenſions will therefore be 6, 12, 18, or other even Number (which will therefore be Quadratick as well as Cubick;) here the Number of Diviſors without 1, will be even, and their Aggregate even; and therefore with 1, the Aggregate will be odd. And accordingly an eſtimate is to be made of the Compounds of ſuch Aggregates: For, if all the Compounding Aggregates be odd, the Compound will be alſo odd; but if any one of them be even, the Compound Aggregate will be even. I forbear to purſue this to any nicer determination: But any who pleaſe may purſue it further.

Of the Third Queſtion mentioned above in pages 322, 323; *which was propoſed by Dr.* WALLIS *to Monſieur* FERMAT.

III. A third Queſtion I added to thoſe two; not as a new difficulty, but as a trial whether Monſieur *Fermat* did thoroughly underſtand the myſtery of his own two Queſtions; and did not only by chance light on them: For if he thoroughly underſtood thoſe, he muſt needs be able to ſolve this with much eaſe; which it ſeems, by Epiſt. 37, he did not find ſo eaſy; and therefore, what ſolution he did find, he choſe rather to conceal than let us know it. Nor doth any where let us know, whether he were able to ſolve his own Queſtions. But Monſieur *Frenicle* gives ſolutions both of this and thoſe; but without acquainting us by what methods he came at them; which makes me think they are not better than mine.

The Queſtion is this: *To find two Square Numbers, which added to their Aliquot Parts ſhall make the ſame Number* (or, whoſe Aggregate of Diviſors ſhall be the ſame;) *As for inſtance* $16 + 8 + 4 + 2 + 1 = 31 = 25 + 5 + 1$; *Let two ſuch other be found.*

Now 'tis manifeſt (by what hath been before delivered) that any Multiple of thoſe two (16 and 25) by any other Square which is a Prime to both of them (as 9, 49, 121, &c,) will do what is deſired. For the Multiple of 31, by the Aggregate of Diviſors of any ſuch other Square, will be the Aggregate of Diviſors, both of 16, and of 25, Multiplied by ſuch Square. As for inſtance, becauſe $9 + 3 + 1 = 13$; therefore $31 \times 13 = 403$, is the Aggregate of the Diviſors, as well of $16 \times 9 = 144$, as of $25 \times 9 = 225$.

But, if we would have others than the Equimultiples of 16 and 25; we may make uſe of the former Table of Squares; wherein (becauſe we do not meet with any ſingle

Squares, (other than thoſe of 4 and of 5,) whoſe Aggregate of Diviſors is the ſame) we are ſo to Compound two or more of them in ſeveral parties, as that the Aggregates be the ſame. As, the Squares of

$$\left.\begin{matrix}4\\5\end{matrix}\right\}31. \qquad \left.\begin{matrix}29\times67\\2\times3\times5\times37\end{matrix}\right\}3\times7\times7\times13\times31\times67.$$

$$\left.\begin{matrix}2\times19\times29\\3\times8\times37\end{matrix}\right\}3\times7\times13\times67\times127.$$

$$\left.\begin{matrix}7\times8\times29\times67\\3\times4\times11\times19\times37\end{matrix}\right\}3\times3\times7\times7\times13\times19\times31\times67\times127.$$

$$\left.\begin{matrix}7\times8\times92\times76\\3\times5\times11\times19\times37\end{matrix}\right\}3\times3\times7\times7\times13\times19\times31\times67\times127.$$

All which ariſe from Compounding the Squares of the Primes leſs than 100, taking into the Number the ſecond and third Squares of 2.

And more Couples than theſe are not to be found within thoſe limits, unleſs by Multiplying both the Numbers of ſome of theſe Couples by ſome common Square which is a Prime to both of them; which may be done at pleaſure. But if we extend the limits, to other Primes, and other Powers of theſe Primes, we may have more without ſtint.

And by the ſame means we may have Three or more ſuch Squares, whoſe Aggregate of Diviſors ſhall make the ſame ſum. As (amongſt theſe) we have Three. Namely the Squares of

$$\left.\begin{matrix}7\times8\times29\times67\\3\times4\times11\times19\times37\\3\times5\times11\times19\times37\end{matrix}\right\}3\times3\times7\times7\times13\times19\times31\times67\times127.$$

But if we enlarge the bounds, we may find others (Two's, Threes, Fours, &c,) in great Multitudes, whoſe Aggregate of Diviſors ſhall be the ſame. As any man by experience, may find, who (without going farther) will give himſelf the trouble of purſuing the whole Table here given, as I have done thoſe Primes which are ſmaller than 100.

I forbear to purſue more Queſtions of this nature; but, according to the ſame method, any others of like kind may be diſpatched.

FINIS.

AN

APPENDIX

TO THE

ENGLISH TRANSLATION

OF

RHONIUS'S GERMAN TREATISE OF ALGEBRA,

MADE BY

MR. THOMAS BRANCKER, M. A.

And Publiſhed by him,

With the Advice and Aſſiſtance of Dr. JOHN PELL,

At London, in the Year 1668;

CONTAINING

A TABLE OF ODD NUMBERS LESS THAN ONE HUNDRED THOUSAND,

SHEWING,

Firſt, Which of them are INCOMPOSIT, or PRIME, NUMBERS,

And, Secondly, The FACTORS, or CO-EFFICIENTS, by the Multiplication of which the others are produced; Supputated, or Computed, by the ſame THOMAS BRANCKER,

THE

TRANSLATOR'S PREFACE.

THE *Title-Page ſays that this Book was a* Tranſlation, *but hath been much* altered. *If any man deſire to know* what *the* alterations *are, and* why *they were made; he may do well to compare it with the Original: A Printed Copy whereof may be had at* Francfort *in* Germany, *by any that inquires there for it by this Title,* Algebra Rhonii Germanicè; Tiguri [*] apud Bodmerum, 1659, in quarto. *The Copy which I have, was given me* anno 1662, *by a good Friend, who* then *told me he much deſired to read it in ſome Language that he underſtood; I then promiſed him to* Engliſh *it. As ſoon as my leiſure permitted, I corrected it according to the Printed Catalogue of Errata, and then began the Tranſlation. When it was finiſhed, I deſired to ſee it Printed, and got it Licenſed* May 18, 1665, *with the name of* An Introduction to Algebra. *And ſo* without any alteration *either in the Precepts or Examples, ſave only the correction of many Miſtakes: It was ſent to the Preſs, with order to Re-print the ſix leaves of* His Table of Incompoſits *preciſely as they ſtand there.*

A little after, I heard that there was at that time in London, *a Perſon of Note* [†] *very worthy to be made acquainted with my deſign, before I made any farther progreſs in the Impreſſion. Being admitted to ſpeak with him, I found him not only* able *to direct me, but alſo very* willing *ſo to do, ſo far as his leiſure would permit. He gave me divers cautions concerning the Work. He ſhewed me the way of making the* Table of Incompoſits, *of examining it, and of continuing it as far as I would. He encouraged me to extend it to* 100 *thouſand: Telling me that by that time that I had Calculated and Printed that Table, he*

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[*] That is, at Zurich, in Switzerland.

[†] Dr. John Pell.

hoped to be at leiſure to review ſome of Monſieur Rhonius *his Problemes, and to work them anew; and that he would ſend them to me, with leave to publiſh them or to keep them by me.*

I had finiſhed and Printed that Table, as alſo Twelve ſheets of the Book itſelf, *before he ſent me his* Alterations. *They begin with* Probl. 24, pag. 100. *All from thence to the end is his Work: As alſo* pag. 79, 80, 81, 82, *which he ſent laſt of all: So that inſtead of the firſt* 124 pages *of* Rhonius, this *hath juſt twice as many: Inſtead of thoſe* 8 *or* 9 *ſheets remaining in* Rhonius, *how much ſhall be hereafter publiſhed, I will not adventure to foretell, becauſe of the uncertainty of life, health, leiſure, and of the acceptance which* this *ſhall find amongſt the Lovers of theſe Studies, to whom this might have been more acceptable, if it had been wholly void of Preſs-faults.*

As for the Table of Incompoſits, *I was very ſenſible of the bad effects of perfunctorineſs in Supputating, Tranſcribing, or Printing of it. My care therefore was not ſmall: yet* pag. 198, *is almoſt filled with* Errata, *and I dare not warrant that none have eſcaped unſeen: But ſeeing ſo few are fit to undertake to Supputate it anew, whoſoever ſhall happen to diſcover any other fault in that Table, ſhall do well to ſignify it to the* Book-ſeller, *or to any other likely to be concerned in the next Impreſſion.*

The Errata *in* the reſt of the Book *are many, notwithſtanding my care, and the diligence of a good friend, who Corrected part of it, after my removal to an abode ſo far from* London. *Moſt of them cannot trouble the more exerciſed ſort of Readers. But fear of leaving any ſtumbling-block in the way of Beginners hath cauſed this larger Enumeration of them in the three next following* pages.

White-gate *in* Cheſhire,
April 22, 1668.

T. B.

From pages 34 *and* 35 *of* BRANCKER's *Tranſlation of* RHONIUS's *Algebra.*

BUT it is oftentimes very troubleſome to find a Square, Cube, &c, whereby this *Abbreviation* may be performed. Find therefore all the *Partes aliquotæ*, or *juſt Dividers*, and theſe will tell us whether, and how often any Cube, Square, &c, is contained in the Quantity aſſigned.

Foraſmuch then as the Diſcovery of the *Partes aliquotæ* is many waies uſeful in *Vulgar Arithmetick*, I have adjoyned a Table in the End of this Book, which diſcovers them in all uneven Numbers as far as 100,000.

In which Table [p] ſtands for a *Prime Number* throughout.

The Uſe of that Table is

To diſcover *at view* whether any given Quantity be compound or ſimple, *i. e.* be diviſible or indiviſible, and how many *Partes aliquotæ* it hath. On the left ſide you ſee, run down all the odd Numbers to 99, which muſt be ſet after the Numbers in the Head-Row, as Occaſion is, thus. Let the Number given be 21449, ſeek 49 in the ſide, and the other 214 in the head, then run downward, and ſide-waies till their Rows meet in a Square, where we find 89, which is a *Pars aliquota*, which dividing 21449, gives Quotient 241. With this 241 do as before (*i. e.* ſeek 41 on the ſide, and 2 in the head) and in its Square you find (P) which ſhews that it is an indiviſible or *Prime* Number. Wherefore the aliquot Parts of this 21449 ſtand thus.

1
89 · 241
21449.

If the even Number 21696 were given, *ſubdivide* it continually by 2 till the Quotient be an odd Number (as at the ſixth Time you will here find 339.) Seek this 339 in the Table as you are directed above. In its Square we find 3, which dividing 339 gives Quotient 113, which 113 we find to be a *Prime* Number. The *Partes aliquotæ* of the Number 21696 ſtand as follows. Out of 1, 2, 3, 113, we may find the reſt.

1
2 . 2 . 2 2 . 2 . 2
4 . 8 . 16 . 32 . 64
3 . 6 . 12 . 24 . 48 . 96 . 192
113.226.452.904.1808.3616.7232.339.678.1356.2712.5424
10848 . 21696.

How thoſe *Principal* Diviſors (1, 2, 3, 113,) are multiplied into each other, and into their Products, lies plain before the Eyes without any more words.

Mr. THOMAS BRANCKER'*s Preface to his long Table of Incompoſit, or Prime, Numbers; from pages* 193, 194, 195, 196, 197, *and* 198, *of his Tranſlation of* RHONIUS'*s Algebra, publiſhed in the Year* 1668.

This is *the Table* mentioned page 34, line 8. It fills 50 pages. Its firſt page calls it *a Table of Incompoſit numbers leſs than* 100,000; but it contains far more *compoſit* numbers, than *incompoſit*; For it doth not only give an *Orderly enumeration of all odd numbers* which are not *compoſit*: but alſo it ſhews that none of the reſt are ſo. To every other odd number there expreſſed, the Table ſets ſome *incompoſit* that will *divide it without fraction.*

Each page hath 21 columels, whereof the *firſt* is filled with 40 odd numbers ſtanding in their natural order. The *following twenty* columels are diſtinguiſhed on their Tops, by numbers in their natural order 0, 1, 2, 3, to 998, 999. Theſe *Top numbers* are *hundreds*; the 40 *marginal* numbers are Unites *adhering to the Centuries.* A line running from any marginal croſs the page, ſhews, in any column, the *place* of the number made up of the Top-number and that marginal. In every ſuch *place* of concourſe you ſhall either find the letter *p*, or ſome *incompoſit* leſs than 317. The letter *p* ſhews the number to be a *prime or incompoſit*, (See Euclid, VII. def. 11 and 13.) If any number leſs than 100,000, do end in 1, 3, 7, or 9, you may find its place in one of thoſe 50 pages, and then ſee whether it be a *prime* or no: If it be compoſit, you will there find its *leaſt Diviſor.* Thus in page 1, where the line marked with the marginal 67, croſſeth the columel whoſe Top-number is 16; there you find *p*, that is, 1667 is a prime. Where the ſame line croſſeth the next columel, you find 3; That is, 1767 is no prime, and 3 is the *leaſt Diviſor of it.* So in page 25, you ſee 49031, 49033, 49037 are primes; but 49039 is a Compoſit, and 19 is its ſmalleſt Diviſor.

It may be of great uſe ſometimes to have *a complete and orderly enumeration of all incompoſits between* 0, *and* 100,000, *without any mixture of Compoſits*; thus 1. 2. 3. 5. 7. 11. 13. &c, leaving out 9, 21 and all other compoſits. The numbers 2 and 5 are primes, though they be left out of the long Table, becauſe no other incompoſit ends ſo. Theſe two prime numbers 2 and 5 being duely placed, all the reſt of the primes are taken out of the long Table as they there ſtand *marked with p*, from 7 in the firſt page to 99991 in the end of the 50*th* page.

If to *each of theſe primes* you ſet the Briggian Logarithm, you may find the Logarithms for *all the reſt of the numbers* in the firſt 100 Chiliads, by addition of the Logarithms of their incompoſit Factors.

The Reſolving of a number into all its incompoſit Factors [as 4620 into 2. 2. 3. 5. 7. 11.] is altogether neceſſary, for the determining *how many Diviſors* that number hath, and *which they be;* As in pages 194, 195.

(29)
1.abcdef
a. bcdef
b. acdef
c. abdef
d. abcef
e. abcdf
f. abcde
ab. cdef
ac. bdef
ad. bcef
ae. bcdf
af. bcde
bc. adef
bd. acef
be. acdf
bf. acde
cd. abef
ce. abdf
cf. abde
de. abcf
df. abce
ef. abcd
abc. def
abd. cef
abe. cdf
abf. cde
acd. bef
ace. bdf
acf. bde
ade. bcf
adf. bce
aef. bcd

(28)
1.aabcde
a. abcde
b. aacde
c. aabde
d. aabce
e. aabcd
aa. bcde
ab. acde
ac. abde
ad. abce
ae. abcd
bc. aade
bd. aace
be. aacd
cd. aabe
ce. aabd
de. aabc
aab. cde
aac. bde
aad. bce
aae. bcd
abc. ade
abd. ace
abe. acd

(27)
1.aabbcd
a. abbcd
b. aabcd
c. aabbd
d. aabbc
aa. bbcd
ab. abcd
ac. abbd
ad. abbc
bb. aacd
bc. aabd
bd. aabc
cd. aabb
aab. bcd
aac. bbd
aad. bbc
abb. acd
abc. abd

(26)
1.aaabcd
a. aabcd
b. aaacd
c. aaabd
d. aaabc
aa. abcd
ab. aacd
ac. aabd
ad. aabc
bc. aaad
bd. aaac
cd. aaab
aaa. bcd
aab. acd
aac. abd
aad. abc

(25)
1.aabbcc
a. abbcc
b. aabcc
c. aabbc
aa. bbcc
ab. abcc
ac. abbc
bb. aacc
bc. aabc
cc. aabb
aab. bcc
aac. bbc
abb. acc
abc. abc

(24)
1.aaabbc
a. aabbc
b. aaabc
c. aaabb
aa. abbc
ab. aabc
ac. aabb
bb. aaac
bc. aaab
aaa. bbc
aab. abc
aac. abb

(23)
1.aaaabc
a. aaabc
b. aaaac
c. aaaab
aa. aabc
ab. aaac
ac. aaab
bc. aaaa
aaa. abc
aab. aac

(22)
1.aaabbb
a. aabbb
b. aaabb
aa. abbb
ab. aabb
bb. aabb
aaa. bbb
aab. abb

(21)
1.aaaabb
a. aaabb
b. aaaab
aa. aabb
ab. aaab
bb. aaaa
aaa. abb
aab. aab

(20)
1.aaaaab
a. aaaab
b. aaaaa
aa. aaab
ab. aaaa
aaa. aab

(19)
1.aaaaaa
a. aaaaa
aa. aaaa
aaa. aaa

(18)
1.abcde
a. bcde
b. acde
c. abde
d. abce
e. abcd
ab. cde
ac. bde
ad. bce
ae. bcd
bc. ade
bd. ace
be. acd
cd. abe
ce. abd
de. abc

(17)
1.aabcd
a. abcd
b. aacd
c. aabd
d. aabc
aa. bcd
ab. acd
ac. abd
ad. abc
bc. aad
bd. aac
cd. aab

(16)
1.aabbc
a. abbc
b. aabc
c. aabb
aa. bbc
ab. abc
ac. abb
bb. aac
bc. aab

(15)
1.aaabc
a. aabc
b. aaac
c. aaab
aa. abc
ab. aac
ac. aab
bc. aaa

(14)
1.aaabb
a. aabb
b. aaab
aa. abb
ab. aab
bb. aaa

(13)
1.aaaab
a. aaab
b. aaaa
aa. aab
ab. aaa

(12)
1.aaaaa
a. aaaa
aa. aaa

(11)
1.abcd
a. bcd
b. acd
c. abd
d. abc
ab. cd
ac. bd
ad. bc

(10)
1, aabc
a. abc
b. aac
c. aab
aa. bc
ab. ac

(9)
1.aabb
a. abb
b. aab
aa. bb
ab. ab

(8)
1.aaab
a. aab
b. aaa
aa. ab

(7)
1.aaaa
a. aaa
aa. aa

(6)
1.abc
a. bc
b. ac
c. ab

(5)
1.aab
a. ab
b. aa

(4)
1.aaa
a. aa

(3)
1.ab
a. b

(2)
1.aa
a. a

(1)
1.a

(29)
1.30030
2.15015
3.10010
5.6006
7.4290
11.2730
13.2310
6.5005
10.3003
14.2145
22.1365
26.1155
15.2002
21.1430
33. 910
39. 770
35. 858
55. 546
65. 462
77. 390
91. 330
143.210
30.1001
42.715
66.455
78.385
70.429
110.273
130.231
154.195
182.165
286.105

(28)
1.4620
2.2310
3.1540
5.924
7.660
11.420
4.1155
6.770
10.462
14.330
22.210
15.308
21.220
33.140
35.132
55. 84
77. 60
12.385
20.231
28.165
44.105
30.154
42.110
66. 70

(27)
1.1260
2.630
3.420
5.252
7.180
4.315
6.210
10.126
14. 90
9. 140
15. 84
21. 60
35. 36
12.105
20. 63
28. 45
18. 70
30. 42

(26)
1.840
2.420
3.280
5.168
7.120
4.210
6.140
10. 84
14. 60
15. 56
21. 40
35. 24
8. 105
12. 70
20. 42
28. 30

(25)
1.900
2.450
3.300
5.180
4.225
6.150
10. 90
9.100
15. 60
25. 36
12. 75
20. 45
18. 50
30. 30

(24)
1.360
2.180
3.120
5. 72
4. 90
6. 60
10. 36
9. 40
15. 24
8. 45
12. 30
20.18

(23)
1.240
2.120
3. 80
5. 48
4. 60
6.40
10.24
15.16
8.30
12.20

(22)
1.216
2.108
3.72
4.54
6.36
9.24
8.27
12.18

(21)
1.144
2.72
3 48
4 36
6.24
9.16
8.18
12.12

(20)
1.96
2.48
3.32
4.24
6.16
8.12

(19)
1.64
2.32
4. 16
8. 8

(18)
1.2310
2.1155
3. 770
5. 462
7. 330
11.210
6.385
10.231
14.165
22.105
15.154
21.110
33. 70
35. 66
55. 42
77. 30

(17)
1.420
2.210
3.140
5. 84
7. 60
4.105
6. 70
10. 42
14. 30
15. 28
21. 20
35. 12

(16)
1.180
2. 90
3. 60
5. 36
4. 45
6. 30
10. 18
9.20
15.12

(15)
1.120
2. 60
3. 40
5. 24
4. 30
6. 20
10.12
15. 8

(14)
1.72
2.36
3.24
4.18
6.12
9. 8

(13)
1.48
2.24
3.16
4.12
6. 8

(12)
1.32
2.16
4. 8

(11)
1.210
2.105
3.70
5.42
7.30
6.35
10.21
14.15

(10)
1.60
2.30
3.20
5.12
4.15
6.10

(9)
1.36
2.18
3.12
4. 9
6. 6

(8)
1.24
2.12
3. 8
4. 6

(7)
1.16
2. 8
4. 4

(6)
1.30
2.15
3.10
5. 6

(5)
1.12
2. 6
3. 4

(4)
1. 8
2. 4

(3)
1. 6
2. 3

(2)
1. 4
2. 2

(1)
1. 2

Sort	Forme	Div.
29	*abcdef*	64
28	*aabcde*	48
27	*aabbcd*	36
26	*aaabcd*	32
25	*aabbcc*	27
24	*aaabbc*	24
23	*aaaabc*	20
22	*aabbb*	16
21	*aaabb*	15
20	*aaaaab*	12
19	*aaaaaa*	7
18	*abcde*	32
17	*aabcd*	24
16	*aabbc*	18
15	*aaabc*	16
14	*aaabb*	12
13	*aaaab*	10
12	*aaaaa*	6
11	*abcd*	16
10	*aabc*	12
5	*aabb*	9
8	*aaab*	8
7	*aaaa*	5
6	*abc*	8
5	*aab*	6
4	*aaa*	4
3	*ab*	4
2	*aa*	3
1	*a*	2

That is, The 29th ſort hath 64 Diviſors; the 18th hath but 32, &c.

Uſe of the Long Table of Numbers, ending in 1, 3, 7, *or* 9.

Every *Aliquot part* of a Number is one of the juſt Diviſors of it. The greateſt Diviſor being equal to the *whole* Dividend, muſt not be called a *Part:* Wherefore, ſubſtract 1 from every number in the laſt columel of page 195, you ſhall have the number of *aliquot* parts belonging to every one of thoſe 29 ſorts.

Having the leaſt Diviſor of any Number of the long Table, to find all its other incompoſit Co-efficients.

If that Diviſor end in 1 or 9, and have a black ſtroke *under* it in the Dividend's place in the long table; or if the Diviſor end in 3 or 7, and have ſuch a ſtroke *over* it in the Dividend's place; the Dividend is the *ſquare of an incompoſit*, and the Quotient is given, for it is equal to the Diviſor.

If the leaſt Diviſor have no ſuch ſtroke by it, let it divide the propoſed number, the Quotient ſhall be the greateſt *aliquot part* of that Dividend: Seek that Quotient in the ſame long Table; if it be there *marked with p*, your inquiry is at an end; the Dividend is of the form AB. If it be *not ſo marked*, by the Prime there found, divide your *firſt* Quotient, deal with the *ſecond* Quotient as you had done with the *firſt*, repeating ſuch Diviſions, till the Quotient be incompoſit. Thus 53191 is found in page 27, with its ſmalleſt Diviſor 43. Now 53191 divided by 43 gives 1237. Page 1 ſays, this 1237 is a prime. Inquire no farther.

But, deſiring the incompoſit factors of 93611, I find it in page 47 of the long Table, with 7 for its leaſt Diviſor. The Quotient 13373 is found in page 7, with its leaſt Diviſor 43. This 43 gives a ſecond quotient 311. Page 1 ſays, this 311 is an incompoſit. So the prime Co-efficients of 93611 are 7. 43. 311. (Hence infer that 53191 is to 93611, as 1237 to 2177 = 7 * 311, or 7×311.

If you divide *any odd* number by *all the primes in order*, beginning with 3, The firſt Diviſor that finds a Quotient without fraction, is the leaſt Diviſor that the Dividend can have. Thus, 239 is the leaſt number that meaſures 1 111 111. Try 3, 7, 11, &c. No prime can divide 1 111 111 till you come to 239. If no ſuch Diviſor find an Integer Quotient, before the Quotient is leſs than the Diviſor, pronounce your Dividend to be incompoſit, and that laſt Diviſor to be greater than the Dividend's ſquare root. Frequent occaſion of Dividing by Incompoſits calls for a *Tariffa* of as many *primes* as ſhall be needful. For reſolving of numbers leſs than 100,000, it ſufficeth if it be extended to 313, as in the next page.

A Tariffa, or Table, of all Incompoſit, or Prime, Numbers, leſs than √ 100,000, *multiplied by* 2, 3, 4, 5, 6, 7, 8, 9.

1	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73
2	4	6	10	14	22	26	34	38	46	58	62	74	82	86	94	106	118	122	134	142	146
3	6	9	15	21	33	39	51	57	69	87	93	111	123	129	141	159	177	183	201	213	219
4	8	12	20	28	44	52	68	76	92	116	124	148	164	172	188	212	236	244	268	284	292
5	10	15	25	35	55	65	85	95	115	145	155	185	205	215	235	265	295	305	335	355	365
6	12	18	30	42	66	78	102	114	138	172	186	222	246	258	282	318	354	366	402	426	438
7	14	21	35	49	77	91	119	133	161	203	217	259	287	301	329	371	413	427	469	497	511
8	16	24	40	56	88	104	136	152	184	232	248	296	328	344	376	424	472	488	536	568	584
9	18	27	45	63	99	117	153	171	207	261	279	333	369	387	423	477	531	549	603	939	657

1	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157
2	158	166	178	194	202	206	214	218	226	254	262	274	278	298	302	314
3	237	249	267	291	303	309	321	327	339	381	393	411	417	447	453	471
4	316	332	356	388	404	412	428	436	452	508	524	548	556	596	604	628
5	395	415	445	485	505	515	535	545	565	635	655	685	695	745	755	785
6	474	498	534	582	606	618	642	654	678	762	786	822	834	894	906	942
7	553	581	623	679	707	721	749	763	791	889	917	959	973	1043	1057	1099
8	632	664	712	776	808	824	856	872	904	1016	1048	1096	1112	1192	1208	1256
9	711	747	801	873	909	927	963	981	1017	1143	1179	1233	1251	1341	1359	1413

1	163	167	173	179	181	191	193	197	199	211	223	227	229	233
2	326	334	346	358	362	382	386	394	398	422	446	454	458	466
3	489	501	519	537	543	573	579	591	597	633	669	681	687	699
4	652	668	692	716	724	764	772	788	796	844	892	908	916	932
5	815	835	865	895	905	955	965	985	995	1055	1115	1135	1145	1165
6	978	1002	1038	1074	1086	1146	1158	1182	1194	1266	1338	1362	1374	1398
7	1141	1169	1211	1253	1267	1337	1351	1379	1393	1477	1561	1589	1603	1631
8	1304	1336	1384	1432	1448	1528	1544	1576	1592	1688	1784	1816	1832	1864
9	1467	1503	1557	1611	1629	1719	1737	1773	1791	1899	2007	2043	2061	2097

1	239	241	251	257	263	269	271	277	281	283	293	307	311	313
2	478	482	502	514	526	538	542	554	562	566	586	614	622	626
3	717	723	753	771	789	807	813	831	843	849	879	921	933	939
4	956	964	1004	1028	1052	1076	1084	1108	1124	1132	1172	1228	1244	1252
5	1195	1205	1255	1285	1315	1345	1355	1385	1405	1415	1465	1535	1555	1565
6	1434	1446	1506	1542	1578	1614	1626	1662	1686	1698	1758	1842	1866	1878
7	1673	1687	1757	1799	1841	1883	1897	1939	1967	1981	2051	2149	2177	2191
8	1912	1928	2008	2056	2104	2152	2168	2216	2248	2264	2344	2456	2488	2504
9	2151	2169	2259	2313	2367	2421	2439	2493	2529	2547	2637	2763	2799	2817

Amendments to the following Table.

P. Guldin ſayes, 149 is diviſible by 7, and 229 by 31. *Schooten* leaves 809 out of his *Catalogue of Incompoſits*. *Rhonius* makes 1209 and 1673 incompoſits, and ſays 11833 is diviſible by 19. But *this Table* ſayes more truly, that 149.229. 809.11833 are Incompoſits: and that 1209 is diviſible by 3, and 1673 by 7. Yet truſt it not, before you have amended *theſe faults* in it.

Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.
5	9211	19	61	21	40277	13	p		60779	63	163		72381	p	3
	9799	40	41		40591	3	p		61779	p	3		72383	3	p
6	10199	p	7		40593	p	3	32	62011	3	p		72557	73	37
	10813	13	11		40597	3	p		62013	p	3		72601	97	79
7	13201	23	43	23	44659	11	17		62017	3	p		73023	p	3
9	17563	3	7		45353	p	7		62019	p	3		73051	7	11
	17981	41	p		45837	p	3		63839	71	p		73481	179	197
10	18903	7	3	24	466..	470	466		63883	191	193		73493	p	7
	18907	3	7		46089	7	3	33	641..	541	641		73913	p	7
	18909	7	3		46457	3	p		659..	569	659	40	78199	p	11
12	23203	3	p		47201	11	7		64237	61	p	41	80333	67	11
	23381	193	103		47577	7	3		64693	3	p		80663	p	11
13	24011	3	13		47579	3	7		64973	23	43	42	83123	163	101
	25093	13	23		47663	p	7		65959	17	71	43	84311	57	59
	25873	23	p	25	48601	53	7	34	66239	19	p	44	86699	281	181
14	27433	31	113	27	53361	7	3		66561	7	3	46	91189	p	7
	27517	p	7		53791	3	p		66563	3	7		91707	p	3
15	28201	3	p	28	54507	7	3		66567	p	3		91793	23	17
	28203	p	3		54509	3	7		66569	3	p	47	92701	3	7
	29599	blank	p		54589	71	79		66761	191	101		92703	7	3
17	32297	71	p	29	56323	157	151		66951	2	3		92773	163	113
	33259	97	79	30	58123	11	13	35	68809	53	13		93101	151	157
	33591	7	3		58181	71	73	36	70313	167	p		93161	52	59
	33593	3	7		58301	137	173		70981	167	p	48	95371	281	283
18	34089	7	3		58901	blank	p		71113	3	7		95797	p	13
	34209	23	3		59901	7	3		71603	p	7	49	97903	3	13
	35089	3	p		59909	137	139		71983	167	p	50	98099	26	263
20	39263	p	7	31	60079	63	73	37	72357	7	3		98551	39	139
	39589	p	11		60293	7	p		72359	3	7		99443	17	277

Mr. THOMAS BRANCKER's TABLE *of* INCOMPOSIT, *or* PRIME, NUMBERS, *leſs than* 100,000.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
01	1·P	P	3	7	P	3	P	P	3	17	7	3	P	P	3	19	P	3	P	P
03	P	P	7	3	13	P	3	19	11	3	17	P	3	P	23	3	7	13	3	11
07	P	P	3	P	11	3	P	7	3	P	19	3	17	P	3	11	P	3	13	P
09	3	P	11	3	P	P	3	P	P	3	P	P	3	7	P	3	P	P	3	23
11	P	3	P	P	3	7	13	3	P	P	3	11	7	3	17	P	3	29	P	3
13	P	P	3	P	7	3	P	23	3	11	P	3	P	13	3	17	P	3	7	P
17	P	3	7	P	3	11	P	3	19	7	3	P	P	3	13	37	3	17	23	3
19	P	7	3	11	P	3	P	P	3	P	P	3	23	P	3	7	P	3	17	19
21	3	11	13	3	P	P	3	7	P	3	P	19	3	P	7	3	P	P	3	17
23	P	3	P	17	3	P	7	3	P	13	3	P	P	3	P	P	3	P	P	3
27	3	P	P	3	7	17	3	P	P	3	13	7	3	P	P	3	P	11	3	41
29	P	3	P	7	3	23	17	3	P	P	3	P	P	3	P	11	3	7	31	3
31	P	P	3	P	P	3	P	17	3	7	P	3	P	11	3	P	7	3	P	P
33	3	7	P	3	P	13	3	P	7	3	P	11	3	31	P	3	23	P	3	P
37	P	P	3	P	19	3	7	11	3	P	17	3	P	7	3	29	P	3	11	13
39	3	P	P	3	P	7	3	P	P	3	P	17	3	13	P	3	11	37	3	7
41	P	3	P	11	3	P	P	3	29	P	3	7	17	3	11	23	3	P	7	3
43	P	11	3	7	P	3	P	P	3	23	7	3	11	17	3	P	31	3	19	29
47	P	3	13	P	3	P	P	3	7	P	3	31	29	3	P	7	3	P	P	3
49	7	P	3	P	P	3	11	7	3	13	P	3	P	19	3	P	17	3	43	P
51	3	P	P	3	11	19	3	P	23	3	P	P	3	7	P	3	13	17	3	P
53	P	3	11	P	3	7	P	3	P	P	3	P	7	3	P	P	3	P	17	3
57	3	P	P	3	P	P	3	P	P	3	7	13	3	23	31	3	P	7	3	19
59	P	3	7	P	3	13	P	3	P	7	3	19	P	3	P	P	3	P	11	3
61	P	7	3	19	P	3	P	P	3	31	P	3	13	P	3	7	11	3	P	37
63	3	P	P	3	P	P	3	7	P	3	P	P	3	29	7	3	P	41	3	13
67	P	P	3	P	P	3	23	13	3	P	11	3	7	P	3	P	P	3	P	7
69	3	13	P	3	7	P	3	P	11	3	P	7	3	37	13	3	P	29	3	11
71	P	3	P	7	3	P	11	3	13	P	3	P	31	3	P	P	3	7	P	3
73	P	P	3	P	11	3	P	P	3	7	29	3	19	P	3	11	7	3	P	P
77	7	3	P	13	3	P	P	3	P	P	3	11	P	3	7	19	3	P	P	3
79	P	P	3	P	P	3	7	19	3	11	13	3	P	7	3	P	23	3	P	P
81	3	P	P	3	13	7	3	11	P	3	23	P	3	P	P	3	41	13	3	7
83	P	3	P	P	3	11	P	3	P	P	3	7	P	3	P	P	3	P	7	3
87	3	11	7	3	P	P	3	P	P	3	P	P	3	19	P	3	7	P	3	P
89	P	3	17	P	3	19	13	3	7	23	3	29	P	3	P	7	3	P	P	3
91	7	P	3	17	P	3	P	7	3	P	P	3	P	13	3	37	19	3	31	11
93	3	P	P	3	17	P	3	13	19	3	P	P	3	7	P	3	P	11	3	P
97	P	P	3	P	7	3	17	P	3	P	P	3	P	11	3	P	P	3	7	P
99	3	P	13	3	P	P	3	17	29	3	7	11	3	P	P	3	P	7	3	P

	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
01	3	11	31	3	7	41	3	37	P	3	P	7	3	P	19	3	13	P	3	47
03	P	3	P	7	3	P	19	3	P	P	3	29	P	3	41	31	3	7	P	3
07	3	7	P	3	29	23	3	P	7	3	31	13	3	P	P	3	P	11	3	P
09	7	3	47	P	3	13	P	3	53	P	3	P	P	3	7	11	3	P	13	3
11	P	P	3	P	P	3	7	P	3	41	P	3	13	7	3	P	23	3	37	P
13	3	P	P	3	19	7	3	P	29	3	23	11	3	P	P	3	P	47	3	7
17	P	29	3	7	P	3	P	11	3	P	7	3	P	31	3	P	P	3	11	P
19	3	13	7	3	41	11	3	P	P	3	P	P	3	P	13	3	7	P	3	P
21	43	3	P	11	3	P	P	3	7	23	3	P	P	3	11	7	3	61	P	3
23	7	11	3	23	P	3	43	7	3	37	P	3	11	P	3	13	P	3	P	P
27	P	3	17	13	3	7	37	3	11	P	3	53	7	3	23	P	3	P	43	3
29	P	P	3	17	7	3	11	P	3	29	13	3	P	P	3	P	19	3	7	P
31	3	P	23	3	11	P	3	P	19	3	7	31	3	P	47	3	P	7	3	P
33	19	3	7	P	3	17	P	3	P	7	3	13	53	3	P	P	3	P	P	3
37	3	P	P	3	P	43	3	7	P	3	P	P	3	47	7	3	P	37	3	31
39	P	3	P	P	3	P	7	3	17	P	3	43	41	3	19	P	3	P	11	3
41	13	P	3	P	P	3	19	P	3	17	P	3	7	13	3	P	11	3	23	7
43	3	P	P	3	7	P	3	13	P	3	17	7	3	P	11	3	P	19	3	P
47	23	19	3	P	P	3	P	41	3	7	11	3	17	P	3	P	7	3	P	P
49	3	7	13	3	31	P	3	P	7	3	P	47	3	17	P	3	41	23	3	11
51	7	3	P	P	3	P	11	3	P	13	3	23	P	3	7	53	3	11	P	3
53	P	P	3	13	11	3	7	P	3	P	43	3	P	7	3	11	13	3	P	59
57	11	3	37	P	3	P	P	3	P	P	3	7	P	3	P	P	3	13	7	3
59	29	17	3	7	P	3	P	31	3	11	7	3	P	P	3	P	P	3	17	37
61	3	P	7	3	23	13	3	11	P	3	P	29	3	P	P	3	7	P	3	17
63	P	3	31	17	3	11	P	3	7	P	3	P	13	3	P	7	3	53	P	3
67	3	11	P	3	P	17	3	P	47	3	P	P	3	7	P	3	19	P	3	P
69	P	3	P	23	3	7	17	3	19	P	3	P	7	3	P	43	3	P	53	3
71	19	13	3	P	7	3	P	17	3	P	37	3	P	P	3	P	P	3	7	11
73	3	41	P	3	P	31	3	47	13	3	7	19	3	P	23	3	P	7	3	29
77	31	7	3	P	P	3	P	P	3	13	17	3	29	11	3	7	P	3	P	41
79	3	P	43	3	37	P	3	7	P	3	P	11	3	31	7	3	13	P	3	23
81	P	3	P	P	3	29	7	3	43	11	3	P	17	3	59	P	3	19	P	3
83	P	37	3	P	13	3	P	11	3	19	P	3	7	17	3	P	29	3	11	7
87	P	3	P	7	3	13	P	3	P	29	3	P	19	3	11	17	3	7	13	3
89	P	11	3	P	19	3	P	P	3	7	P	3	11	P	3	37	7	3	P	P
91	3	7	29	3	47	P	3	P	7	3	11	P	3	P	P	3	P	17	3	13
93	7	3	P	P	3	P	P	3	11	41	3	31	37	3	7	P	3	P	17	3
97	3	13	P	3	11	7	3	P	P	3	19	23	3	43	13	3	P	P	3	7
99	P	3	11	P	3	23	P	3	13	P	3	7	P	3	P	59	3	29	7	3

	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
01	p	3	p	11	3	7	43	3	p	13	3	p	7	3	11	p	3	p	p	3
03	p	11	3	13	7	3	p	p	3	p	p	3	11	p	3	p	13	3	7	p
07	p	3	7	59	3	p	17	3	11	7	3	p	41	3	p	p	3	13	p	3
09	19	7	3	31	p	3	11	17	3	p	p	3	p	p	3	7	71	3	37	19
11	3	p	p	3	11	13	3	7	17	3	p	19	3	47	7	3	31	p	3	23
13	p	3	11	19	3	p	7	3	p	17	3	p	13	3	p	37	3	29	p	3
17	3	23	p	3	7	p	3	53	p	3	29	7	3	13	p	3	41	p	3	61
19	p	3	p	7	3	p	31	3	61	p	3	p	17	3	p	p	3	7	11	3
21	p	13	3	29	p	3	p	p	3	7	p	3	23	17	3	p	7	3	p	31
23	3	7	41	3	p	p	3	p	7	3	p	47	3	p	11	3	p	59	3	p
27	p	p	3	p	19	3	7	29	3	13	11	3	p	7	3	p	17	3	p	p
29	3	p	p	3	43	7	3	p	11	3	47	23	3	73	61	3	13	17	3	7
31	29	3	p	61	3	23	11	3	p	p	3	7	p	3	p	p	3	11	7	3
33	37	p	3	7	11	3	41	p	3	p	7	3	p	p	3	11	43	3	19	17
37	11	3	19	p	3	13	p	3	7	p	3	11	p	3	p	7	3	p	13	3
39	7	p	3	p	23	3	p	7	3	11	p	3	13	19	3	29	p	3	p	p
41	3	41	p	3	p	19	3	11	47	3	71	53	3	7	p	3	p	p	3	13
43	13	3	p	43	3	7	p	3	29	p	3	37	7	3	p	23	3	p	p	3
47	3	11	31	3	p	p	3	47	37	3	7	p	3	p	13	3	p	7	3	19
49	p	3	7	p	3	p	p	3	13	7	3	19	29	3	p	31	3	p	p	3
51	p	7	3	19	p	3	p	p	3	p	p	3	59	p	3	7	p	3	p	11
53	3	p	p	3	61	29	3	7	23	3	31	p	3	53	7	3	p	11	3	p
57	p	p	3	p	p	3	p	67	3	p	13	3	7	11	3	p	p	3	p	7
59	3	p	p	3	7	47	3	p	43	3	p	7	3	23	53	3	p	13	3	59
61	31	3	p	7	3	p	59	3	p	11	3	13	p	3	43	67	3	7	p	3
63	17	23	3	p	p	3	p	11	3	7	61	3	19	31	3	p	7	3	11	67
67	7	3	17	11	3	p	13	3	31	p	3	p	23	3	7	19	3	73	p	3
69	13	11	3	17	41	3	7	19	3	p	37	3	11	7	3	p	p	3	p	47
71	3	43	p	3	17	7	3	13	p	3	11	p	3	41	p	3	53	29	3	7
73	p	3	p	p	3	17	p	3	11	p	3	7	p	3	13	p	3	23	7	3
77	3	p	7	3	11	23	3	17	p	3	p	31	3	19	p	3	7	53	3	43
79	p	3	11	29	3	19	p	3	7	13	3	p	p	3	p	p	3	p	p	3
81	7	37	3	13	p	3	31	7	3	17	p	3	p	p	3	p	13	3	p	p
83	3	47	p	3	p	p	3	p	19	3	13	71	3	7	p	3	p	p	3	31
87	61	53	3	41	7	3	43	p	3	p	p	3	17	p	3	37	11	3	7	p
89	3	59	p	3	67	13	3	p	p	3	7	p	3	17	11	3	p	7	3	53
91	p	3	7	p	3	p	p	3	67	7	3	29	11	3	17	p	3	p	43	3
93	p	7	3	23	p	3	13	p	3	p	11	3	67	p	3	7	p	3	71	13
97	17	3	p	p	3	p	7	3	59	19	3	p	p	3	23	29	3	11	p	3
99	p	13	3	53	11	3	37	p	3	p	p	3	7	p	3	11	41	3	17	7

	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
01	17	p	3	p	37	3	7	p	3	67	p	3	19	7	3	13	11	3	29	p
03	3	17	p	3	19	7	3	p	p	3	47	p	3	67	11	3	p	p	3	7
07	p	31	3	7	43	3	p	19	3	p	7	3	p	p	3	p	p	3	37	p
09	3	41	7	3	13	23	3	p	11	3	43	p	3	p	31	3	7	13	3	11
11	p	3	p	p	3	17	11	3	7	p	3	13	p	3	p	7	3	11	73	3
13	7	p	3	59	11	3	17	7	3	31	p	3	p	71	3	11	23	3	13	41
17	11	3	p	p	3	7	13	3	17	p	3	11	7	3	p	p	3	p	p	3
19	13	29	3	71	7	3	p	p	3	11	p	3	p	13	3	73	19	3	7	p
21	3	p	p	3	p	p	3	11	19	3	7	p	3	p	41	3	p	7	3	89
23	19	3	7	p	3	11	37	3	p	7	3	17	31	3	13	p	3	p	p	3
27	3	11	13	3	p	61	3	7	p	3	p	p	3	17	7	3	29	p	3	p
29	p	3	p	p	3	p	7	3	p	13	3	p	p	3	17	p	3	59	p	3
31	37	p	3	13	59	3	19	53	3	29	79	3	7	p	3	17	13	3	41	7
33	3	p	23	3	7	47	3	p	p	3	13	7	3	p	p	3	17	11	3	p
37	p	17	3	p	41	3	p	p	3	7	31	3	p	11	3	p	7	3	17	p
39	3	7	17	3	47	13	3	23	7	3	p	11	3	41	43	3	p	71	3	17
41	7	3	79	17	3	31	29	3	p	11	3	37	13	3	7	p	3	p	p	3
43	p	p	3	p	17	3	7	11	3	53	p	3	p	7	3	19	p	3	11	13
47	p	3	p	11	3	p	17	3	41	p	3	7	p	3	11	p	3	61	7	3
49	23	11	3	7	p	3	61	17	3	p	7	3	11	p	3	p	p	3	47	p
51	3	p	7	3	p	p	3	43	13	3	11	p	3	p	p	3	7	23	3	p
53	p	3	13	p	3	p	p	3	7	17	3	23	p	3	29	7	3	p	p	3
57	3	47	p	3	11	79	3	29	p	3	p	17	3	7	p	3	13	p	3	73
59	73	3	11	p	3	7	p	3	19	p	3	p	7	3	p	p	3	p	29	3
61	11	61	3	p	7	3	p	p	3	p	23	3	53	17	3	p	47	3	7	19
63	3	p	p	3	23	p	3	p	p	3	7	13	3	37	17	3	79	7	3	p
67	p	7	3	p	29	3	59	67	3	p	37	3	13	53	3	7	11	3	p	31
69	3	31	p	3	p	p	3	7	p	3	p	67	3	p	7	3	p	17	3	13
71	13	3	p	23	3	p	7	3	p	p	3	71	11	3	31	67	3	19	17	3
73	p	p	3	p	p	3	p	13	3	19	11	3	7	73	3	p	p	3	p	7
77	59	3	p	7	3	p	11	3	13	p	3	p	19	3	p	p	3	7	p	3
79	p	37	3	p	11	3	p	p	3	7	p	3	29	47	3	11	7	3	p	79
81	3	7	11	3	p	p	3	p	7	3	73	43	3	11	p	3	p	31	3	23
83	7	3	61	13	3	29	41	3	p	p	3	11	p	3	7	p	3	43	p	3
87	3	23	p	3	13	7	3	11	71	[illegible]	19	p	3	83	p	3	p	13	3	7
89	p	3	19	p	3	11	p	3	83	29	3	7	37	3	p	p	3	p	7	3
91	p	41	3	7	p	3	p	p	3	p	7	3	23	19	3	p	p	3	13	61
93	3	11	7	3	43	19	3	p	61	3	41	p	3	p	59	3	7	p	3	p
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09	P	3	P	7	3	67	P	3	23	59	3	P	P	3	97	37	3	7	17	3
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17	P	P	3	P	19	3	7	23	3	37	71	3	13	7	3	31	59	3	P	47
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41	11	7	3	19	23	3	P	P	3	P	P	3	P	P	3	7	31	3	13	P
43	3	17	P	3	P	P	3	7	37	3	P	41	3	P	7	3	P	P	3	61
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71	7	P	3	11	43	3	13	7	3	P	47	3	73	P	3	17	19	3	P	13
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79	3	P	17	3	61	23	3	P	13	3	7	67	3	83	P	3	P	7	3	17
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87	P	3	P	P	3	31	7	3	P	11	3	P	19	3	53	P	3	P	P	3
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03	7	p	3	p	101	3	23	7	3	p	p	3	17	89	3	p	41	3	11	p
07	p	3	59	11	3	7	p	3	101	13	3	29	7	3	11	37	3	23	p	3
09	p	11	3	13	7	3	103	p	3	p	101	3	11	43	3	17	13	3	7	p
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97	23	3	7	37	3	p	19	3	17	7	3	p	11	3	p	p	3	47	p	3
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31	17	3	p	7	3	61	p	3	p	p	3	37	p	3	p	47	3	7	11	3
33	p	13	3	p	p	3	p	29	3	7	p	3	19	p	3	89	7	3	17	79
37	7	3	13	17	3	23	127	3	113	p	3	p	11	3	7	13	3	p	p	3
39	43	p	3	p	17	3	7	19	3	13	11	3	p	7	3	p	31	3	p	p
41	3	p	109	3	41	7	3	p	11	3	p	61	3	p	107	3	13	113	3	7
43	61	3	37	59	3	71	11	3	p	p	3	7	43	3	p	53	3	11	7	3
47	3	67	7	3	p	p	3	p	17	3	p	13	3	11	73	3	7	p	3	131
49	11	3	p	p	3	13	p	3	7	17	3	11	47	3	p	7	3	p	13	3
51	7	31	3	83	p	3	p	7	3	11	17	3	13	p	3	p	19	3	p	29
53	3	29	p	3	p	p	3	11	19	3	p	17	3	7	31	3	127	41	3	13
57	p	107	3	11	7	3	p	13	3	31	37	3	p	17	3	97	p	3	7	p
59	3	11	71	3	109	29	3	p	23	3	7	p	3	p	13	3	p	7	3	p
61	p	3	7	p	3	p	p	3	13	7	3	131	41	3	19	17	3	p	53	3
63	p	7	3	p	101	3	19	p	3	p	113	3	61	97	3	7	17	3	p	11
67	p	3	p	13	3	p	7	3	101	19	3	p	31	3	p	11	3	109	17	3
69	p	19	3	p	43	3	79	41	3	71	13	3	7	11	3	p	p	3	107	7
71	3	103	53	3	7	73	3	31	p	3	43	7	3	29	p	3	41	13	3	p
73	p	3	p	7	3	p	p	3	47	11	3	13	23	3	101	p	3	7	61	3
77	3	7	41	3	p	11	3	19	7	3	p	89	3	p	p	3	11	29	3	p
79	7	3	73	11	3	59	13	3	p	p	3	41	37	3	7	p	3	23	19	3
81	13	11	3	p	p	3	7	97	3	p	19	3	11	7	3	p	p	3	p	p
83	3	p	19	3	53	7	3	13	p	3	11	p	3	p	p	3	p	p	3	7
87	p	p	3	7	p	3	11	p	3	p	7	3	59	p	3	43	23	3	31	p
89	3	p	7	3	11	53	3	103	p	3	23	p	3	p	p	3	7	p	3	p
91	p	3	11	37	3	47	p	3	7	13	3	p	p	3	p	7	3	p	p	3
93	7	p	3	13	p	3	p	7	3	p	p	3	p	p	3	73	13	3	29	19
97	p	3	43	19	3	7	59	3	61	23	3	29	7	3	p	p	3	13	11	3
99	17	97	3	23	7	3	p	107	3	89	p	3	p	127	3	p	11	3	7	41

	180	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199
01	47	23	3	p	p	3	11	p	3	41	p	3	7	p	3	p	17	3	p	7
03	3	43	109	3	7	p	3	59	p	3	31	7	3	97	p	3	p	17	3	13
07	11	19	3	p	79	3	23	13	3	7	83	3	p	43	3	p	7	3	29	17
09	3	7	131	3	41	83	3	53	7	3	p	97	3	p	13	3	p	p	3	43
11	7	3	p	p	3	107	37	3	13	p	3	29	p	3	7	109	3	23	11	3
13	p	59	3	p	p	3	7	p	3	p	p	3	p	7	3	13	11	3	p	p
17	43	3	p	13	3	p	p	3	31	p	3	7	11	3	p	29	3	p	7	3
19	37	p	3	7	113	3	43	p	3	p	7	3	p	p	3	131	23	3	p	p
21	3	p	7	3	13	p	3	97	11	3	23	p	3	139	p	3	7	13	3	11
23	67	3	p	73	3	p	11	3	7	127	3	13	47	3	p	7	3	11	43	3
27	3	p	11	3	p	97	3	61	67	3	53	31	3	7	p	3	19	p	3	p
29	11	3	p	p	3	7	13	3	19	23	3	11	7	3	p	59	3	109	79	3
31	13	p	3	23	7	3	31	p	3	11	p	3	p	13	3	p	67	3	7	19
33	3	p	p	3	p	43	3	11	37	3	7	19	3	p	p	3	29	7	3	31
37	17	7	3	11	103	3	p	41	3	29	p	3	p	61	3	7	73	3	83	p
39	3	11	13	3	p	p	3	7	p	3	79	p	3	83	7	3	41	p	3	127
41	p	3	17	p	3	p	7	3	83	13	3	p	71	3	p	p	3	19	p	3
43	p	p	3	13	p	3	103	p	3	19	137	3	7	23	3	p	13	3	p	7
47	p	3	71	7	3	17	29	3	47	p	3	41	19	3	p	11	3	7	89	3
49	p	p	3	59	19	3	17	p	3	7	43	3	p	11	3	113	7	3	23	p
51	3	7	p	3	p	13	3	17	7	3	p	11	3	37	53	3	43	p	3	71
53	7	3	p	p	3	p	23	3	17	11	3	107	13	3	7	p	3	p	p	3
57	3	67	p	3	p	7	3	p	109	3	17	p	3	13	p	3	11	23	3	7
59	p	3	19	11	3	67	47	3	p	p	3	7	p	3	11	p	3	p	7	3
61	p	11	3	7	p	3	p	73	3	67	7	3	11	19	3	31	p	3	p	p
63	3	41	7	3	37	19	3	29	13	3	11	p	3	17	p	3	7	p	3	p
67	7	37	3	p	59	3	11	7	3	13	23	3	p	107	3	17	71	3	p	41
69	3	p	p	3	11	31	3	137	p	3	p	29	3	7	p	3	13	53	3	19
71	17	3	11	p	3	7	p	3	113	61	3	19	7	3	p	p	3	17	31	3
73	11	17	3	19	7	3	71	p	3	p	p	3	p	p	3	23	103	3	7	p
77	p	3	7	17	3	13	19	3	43	7	3	127	37	3	p	p	3	p	11	3
79	101	7	3	p	17	3	p	89	3	p	p	3	13	p	3	7	11	3	103	p
81	3	p	101	3	p	17	3	7	79	3	p	p	3	p	7	3	p	131	3	13
83	13	3	47	31	3	p	7	3	23	41	3	p	11	3	p	p	3	73	59	3
87	3	13	p	3	7	p	3	p	11	3	p	7	3	p	13	3	p	47	3	11
89	p	3	p	7	3	29	11	3	13	17	3	31	p	3	p	19	3	7	p	3
91	79	p	3	53	11	3	p	19	3	7	17	3	101	p	3	11	7	3	p	p
93	3	7	11	3	p	p	3	p	7	3	61	17	3	11	101	3	47	p	3	p
97	p	31	3	p	53	3	7	p	3	11	13	3	23	7	3	p	p	3	101	p
99	3	p	29	3	13	7	3	11	p	3	71	73	3	19	17	3	p	13	3	7

	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219
01	3	P	P	3	23	13	3	127	11	3	P	P	3	7	P	3	P	P	3	11
03	83	3	89	79	3	7	11	3	71	P	3	47	7	3	17	P	3	11	P	3
07	3	P	11	3	P	P	3	P	P	3	7	P	3	11	P	3	17	7	3	19
09	11	3	7	23	3	P	37	3	P	7	3	11	127	3	79	137	3	17	113	3
11	P	7	3	19	P	3	P	139	3	11	P	3	P	101	3	7	P	3	17	P
13	3	P	17	3	137	73	3	7	13	3	P	43	3	P	7	3	P	P	3	17
17	37	P	3	11	17	3	53	P	3	13	P	3	7	P	3	P	P	3	P	7
19	3	11	P	3	7	17	3	P	109	3	P	7	3	P	P	3	13	37	3	23
21	P	3	73	7	3	P	17	3	47	P	3	P	P	3	31	P	3	7	P	3
23	P	P	3	P	13	3	41	17	3	7	P	3	19	P	3	P	7	3	139	11
27	7	3	113	P	3	13	P	3	59	17	3	37	P	3	7	11	3	P	13	3
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31	3	41	P	3	P	7	3	P	37	3	P	11	3	83	29	3	97	31	3	7
33	13	3	P	P	3	P	47	3	83	11	3	7	17	3	P	61	3	103	7	3
37	3	13	7	3	107	11	3	89	67	3	109	23	3	19	13	3	7	P	3	P
39	29	3	37	11	3	19	P	3	7	P	3	P	67	3	11	7	3	P	P	3
41	7	11	3	P	P	3	P	7	3	43	53	3	11	P	3	13	17	3	P	37
43	3	P	31	3	P	P	3	P	19	3	11	P	3	7	41	3	23	17	3	P
47	P	P	3	P	7	3	11	P	3	P	13	3	P	P	3	29	P	3	7	17
49	3	P	P	3	11	P	3	P	P	3	7	P	3	37	89	3	P	7	3	47
51	P	3	7	47	3	P	107	3	29	7	3	13	79	3	19	23	3	P	P	3
53	11	7	3	P	113	3	19	P	3	23	37	3	53	131	3	7	59	3	13	29
57	31	3	47	P	3	61	7	3	P	19	3	P	29	3	43	P	3	P	11	3
59	13	19	3	P	41	3	73	P	3	P	P	3	7	13	3	P	11	3	P	7
61	3	P	P	3	7	29	3	13	23	3	P	7	3	41	11	3	P	47	3	P
63	P	3	23	7	3	P	P	3	31	P	3	P	11	3	13	P	3	7	P	3
67	3	7	13	3	97	131	3	19	7	3	P	61	3	23	P	3	47	P	3	11
69	7	3	P	P	3	67	11	3	41	13	3	P	P	3	7	P	3	11	19	3
71	P	23	3	13	11	3	7	P	3	67	19	3	89	7	3	11	13	3	P	127
73	3	P	11	3	59	7	3	P	P	3	13	31	3	11	109	3	P	P	3	7
77	17	P	3	7	P	3	23	79	3	11	7	3	P	P	3	P	53	3	131	P
79	3	17	7	3	P	13	3	11	P	3	107	P	3	P	47	3	7	29	3	31
81	43	3	17	89	3	11	P	3	7	P	3	59	13	3	P	7	3	23	P	3
83	7	P	3	11	P	3	13	7	3	P	29	3	P	P	3	113	P	3	79	13
87	53	3	P	19	3	7	137	3	P	31	3	P	7	3	P	P	3	P	43	3
89	P	13	3	P	7	3	17	P	3	139	P	3	61	73	3	P	23	3	7	11
91	3	61	103	3	31	59	3	17	13	3	7	P	3	P	P	3	109	7	3	P
93	71	3	7	P	3	P	P	3	17	7	3	P	107	3	P	11	3	19	P	3
97	3	19	P	3	103	43	3	7	P	3	17	11	3	P	7	3	13	71	3	P
99	101	3	53	P	3	P	7	3	P	11	3	17	19	3	P	P	3	P	61	3

	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239
01	7	3	149	29	3	P	97	3	151	P	3	13	P	3	7	71	3	137	P	3
03	P	23	3	P	43	3	7	73	3	37	P	3	P	7	3	19	P	3	13	11
07	59	3	53	P	3	71	13	3	P	P	3	7	23	3	89	11	3	151	7	3
09	13	P	3	7	P	3	23	P	3	31	7	3	P	11	3	P	P	3	29	P
11	3	P	7	3	73	P	3	13	P	3	P	11	3	P	41	3	7	131	3	P
13	P	3	97	53	3	47	P	3	7	11	3	29	139	3	13	7	3	23	P	3
17	3	17	13	3	29	11	3	P	P	3	P	P	3	7	P	3	11	37	3	P
19	97	3	17	11	3	7	P	3	19	13	3	61	7	3	11	29	3	P	P	3
21	19	11	3	13	7	3	P	P	3	P	P	3	11	P	3	43	13	3	7	19
23	3	P	71	3	17	101	3	31	29	3	7	19	3	83	59	3	P	7	3	47
27	P	7	3	83	41	3	11	P	3	101	P	3	P	P	3	7	P	3	P	71
29	3	P	P	3	11	13	3	7	37	3	P	101	3	41	7	3	P	61	3	P
31	P	3	11	137	3	P	7	3	17	23	3	P	13	3	P	P	3	19	P	3
33	11	P	3	23	P	3	13	127	3	17	31	3	7	P	3	101	P	3	P	7
37	P	3	37	7	3	31	P	3	41	P	3	17	19	3	23	P	3	7	11	3
39	P	13	3	89	19	3	P	P	3	7	P	3	17	P	3	P	7	3	31	37
41	3	7	23	3	P	P	3	P	7	3	P	73	3	17	11	3	47	P	3	89
43	7	3	13	P	3	P	P	3	53	P	3	P	11	3	7	13	3	P	113	3
47	3	P	P	3	P	7	3	23	11	3	19	79	3	37	P	3	13	P	3	7
49	17	3	19	P	3	P	11	3	73	53	3	7	67	3	131	P	3	11	7	3
51	P	17	3	7	11	3	P	P	3	59	7	3	P	19	3	11	67	3	17	43
53	3	P	7	3	P	19	3	61	P	3	P	13	3	11	47	3	7	P	3	17
57	7	P	3	79	17	3	139	7	3	11	P	3	13	P	3	P	41	3	P	P
59	3	P	P	3	37	17	3	11	P	3	P	P	3	7	P	3	59	23	3	13
61	13	3	113	59	3	7	17	3	P	P	3	19	7	3	29	P	3	P	107	3
63	P	37	3	11	7	3	131	13	3	P	P	3	43	61	3	P	P	3	7	31
67	P	3	7	P	3	P	19	3	13	7	3	P	53	3	31	P	3	P	29	3
69	29	7	3	P	P	3	P	P	3	103	17	3	P	P	3	7	P	3	P	11
71	3	P	P	3	23	P	3	7	P	3	P	17	3	P	7	3	P	11	3	P
73	P	3	P	13	3	P	7	3	89	P	3	P	17	3	P	11	3	P	P	3
77	3	67	P	3	7	107	3	P	P	3	47	7	3	97	17	3	P	13	3	P
79	P	3	P	7	3	67	P	3	137	11	3	13	P	3	53	17	3	7	P	3
81	71	41	3	P	P	3	37	11	3	7	P	3	31	103	3	P	7	3	11	P
83	3	7	P	3	P	11	3	P	7	3	41	97	3	67	23	3	11	17	3	29
87	13	11	3	61	113	3	7	P	3	127	P	3	11	7	3	103	P	3	P	17
89	3	P	31	3	43	7	3	13	47	3	11	P	3	19	83	3	P	P	3	7
91	P	3	P	P	3	19	P	3	11	83	3	7	P	3	13	31	3	37	7	3
93	P	P	3	7	83	3	11	23	3	P	7	3	P	149	3	P	19	3	P	P
97	19	3	11	P	3	59	P	3	7	13	3	P	P	3	P	7	3	53	23	3
99	7	79	3	13	149	3	P	7	3	109	P	3	23	P	3	P	13	3	P	103

	240	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259
01	p	7	3	19	13	3	73	17	3	37	23	3	11	p	3	7	p	3	p	59
03	3	p	p	3	23	107	3	7	17	3	11	13	3	p	7	3	p	p	3	p
07	p	p	3	109	p	3	11	31	3	p	17	3	7	p	3	23	29	3	131	7
09	3	p	43	3	7	p	3	p	p	3	89	7	3	p	p	3	p	47	3	13
11	13	3	11	7	3	127	p	3	43	29	3	p	17	3	p	97	3	7	53	3
13	11	p	3	41	p	3	151	13	3	7	p	3	19	17	3	31	7	3	83	p
17	7	3	61	p	3	p	103	3	13	p	3	p	151	3	7	17	3	p	11	3
19	p	89	3	83	p	3	7	19	3	p	127	3	p	7	3	13	11	3	p	p
21	3	p	53	3	p	7	3	59	p	3	131	p	3	p	11	3	p	17	3	7
23	p	3	p	13	3	137	p	3	103	p	3	7	11	3	p	p	3	29	7	3
27	3	23	7	3	13	p	3	79	11	3	29	p	3	19	47	3	7	13	3	11
29	p	3	p	p	3	19	11	3	7	97	3	13	p	3	59	7	3	11	23	3
31	7	59	3	29	11	3	p	7	3	107	p	3	23	73	3	11	19	3	13	p
33	3	p	11	3	53	p	3	p	19	3	p	41	3	7	29	3	p	p	3	p
37	13	p	3	p	7	3	71	29	3	11	p	3	p	13	3	p	31	3	7	37
39	3	101	p	3	p	53	3	11	59	3	7	23	3	p	p	3	p	7	3	p
41	29	3	7	101	3	11	41	3	p	7	3	31	43	3	13	p	3	p	p	3
43	p	7	3	11	p	3	19	109	3	p	79	3	p	p	3	7	p	3	43	p
47	139	3	p	97	3	p	7	3	p	13	3	p	p	3	p	59	3	p	p	3
49	p	19	3	13	23	3	157	p	3	61	37	3	7	p	3	29	13	3	p	7
51	3	p	p	3	7	p	3	53	p	3	13	7	3	101	31	3	113	11	3	p
53	67	3	79	7	3	43	89	3	29	p	3	p	p	3	p	11	3	7	103	3
57	3	7	127	3	37	13	3	19	7	3	p	11	3	p	p	3	p	43	3	101
59	7	3	17	p	3	41	p	3	p	11	3	139	13	3	7	61	3	p	19	3
61	p	37	3	17	61	3	7	11	3	109	19	3	p	7	3	p	67	3	11	13
63	3	73	19	3	17	7	3	p	23	3	71	p	3	13	p	3	11	p	3	7
67	41	11	3	7	43	3	17	p	3	p	7	3	11	p	3	37	p	3	p	23
69	3	p	7	3	p	79	3	17	13	3	11	p	3	23	p	3	7	73	3	p
71	p	3	13	p	3	p	p	3	7	p	3	p	37	3	p	7	3	p	41	3
73	7	23	3	p	p	3	11	7	3	13	p	3	127	p	3	107	p	3	p	19
77	p	3	11	19	3	7	p	3	p	p	3	17	7	3	73	p	3	149	113	3
79	11	p	3	p	7	3	23	71	3	p	31	3	17	41	3	p	p	3	7	83
81	3	p	p	3	p	47	3	p	139	3	7	13	3	17	83	3	61	7	3	p
83	p	3	7	37	3	13	p	3	149	7	3	p	131	3	17	p	3	19	11	3
87	3	19	149	3	47	23	3	7	41	3	p	89	3	53	7	3	17	107	3	13
89	13	3	107	29	3	67	7	3	p	p	3	p	11	3	71	p	3	17	p	3
91	p	17	3	p	19	3	p	13	3	67	11	3	7	p	3	157	23	3	17	7
93	3	13	17	3	7	p	3	p	11	3	23	7	3	67	13	3	p	p	3	11
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37	101	3	7	P	3	P	19	3	11	7	3	41	167	3	P	P	3	13	P	3
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49	79	3	29	7	3	P	P	3	P	P	3	P	161	3	P	19	3	7	11	3
51	17	13	3	P	47	3	P	19	3	7	P	3	P	23	3	73	7	3	P	P
53	3	7	P	3	131	109	3	23	7	3	P	P	3	P	11	3	101	P	3	157
57	P	P	3	17	P	3	7	P	3	13	11	3	P	7	3	31	181	3	23	41
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61	P	3	P	P	3	17	11	3	71	P	3	7	37	3	P	43	3	11	7	3
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67	11	3	P	P	3	13	P	3	7	73	3	11	P	3	29	7	3	47	13	3
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71	3	P	43	3	P	181	3	11	P	3	17	P	3	7	79	3	P	P	3	13
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87	89	17	3	137	P	3	P	43	3	59	13	3	7	11	3	19	127	3	17	7
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91	73	3	53	7	3	P	113	3	23	11	3	13	P	3	P	P	3	7	19	3
93	103	31	3	163	17	3	P	11	3	7	19	3	29	P	3	P	7	3	11	P
97	7	3	P	11	3	29	13	3	P	79	3	61	47	3	7	P	3	P	P	3
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03	3	79	41	3	59	173	3	17	13	3	p	11	3	7	113	3	31	37	3	29
07	p	p	3	p	7	3	p	11	3	13	23	3	29	p	3	p	p	3	7	p
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47	19	3	83	17	3	157	41	3	7	67	3	13	59	3	23	7	3	11	163	3
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21	p	3	p	23	3	211	p	3	7	29	3	p	11	3	53	7	3	13	p	3
23	7	p	3	127	31	3	p	7	3	167	11	3	41	61	3	p	43	3	p	19
27	p	3	47	19	3	7	11	3	23	p	3	p	7	3	p	53	3	11	p	3
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41	p	37	3	11	19	3	p	p	3	13	73	3	7	p	3	p	p	3	p	7
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47	17	131	3	61	13	3	p	29	3	7	107	3	p	137	3	37	7	3	19	11
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53	p	67	3	17	p	3	7	p	3	p	p	3	13	7	3	p	71	3	p	p
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69	127	3	p	13	3	7	19	3	11	193	3	17	7	3	41	p	3	37	p	3
71	p	p	3	p	7	3	11	p	3	p	13	3	17	59	3	199	109	3	7	p
73	3	163	p	3	11	29	3	p	23	3	7	199	3	17	37	3	p	7	3	31
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79	3	p	p	3	19	p	3	7	p	3	61	p	3	23	7	3	17	p	3	p
81	17	3	p	p	3	109	7	3	37	31	3	p	p	3	p	19	3	17	11	3
83	13	17	3	p	p	3	p	19	3	p	p	3	7	13	3	79	11	3	17	7
8[illegible]	p	3	67	7	3	p	p	3	p	p	3	73	11	3	13	p	3	7	p	3
89	p	p	3	p	17	3	23	p	3	7	11	3	p	p	3	p	7	3	109	p
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97	31	3	67	13	3	17	7	3	23	p	3	109	p	3	p	11	3	p	211	3
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03	3	11	19	3	97	7	3	113	37	3	p	p	3	47	127	3	p	23	3	7
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03	7	p	3	193	13	3	41	7	3	p	p	3	83	151	3	p	11	3	173	19
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67	p	3	p	7	3	p	p	3	29	p	3	79	p	3	127	17	3	7	11	3
69	p	13	3	p	71	3	31	p	3	7	p	3	p	83	3	p	7	3	103	29
71	3	7	167	3	137	p	3	113	7	3	73	p	3	19	11	3	191	17	3	31
73	7	3	13	83	3	19	p	3	37	p	3	p	11	3	7	13	3	p	17	3
77	3	p	61	3	97	7	3	89	11	3	p	41	3	p	53	3	13	p	3	7
79	19	3	23	p	3	p	11	3	p	31	3	7	p	3	p	131	3	11	7	3
81	p	p	3	7	11	3	139	47	3	p	7	3	p	p	3	11	p	3	p	23
83	3	p	7	3	31	p	3	p	p	3	109	13	3	11	79	3	7	p	3	37
87	7	23	3	p	73	3	19	7	3	11	p	3	13	197	3	41	37	3	p	p
89	3	p	p	3	p	43	3	11	p	3	p	p	3	7	89	3	53	19	3	13
91	13	3	p	p	3	7	p	3	227	19	3	43	7	3	149	p	3	p	p	3
93	113	19	3	11	7	3	23	13	3	197	p	3	137	107	3	p	p	3	7	p
97	59	3	7	151	3	149	p	3	13	7	3	p	223	3	61	p	3	23	p	3
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01	p	p	3	13	p	3	p	19	3	7	p	3	p	17	3	p	7	3	41	p
03	3	7	67	3	p	p	3	11	7	3	13	p	3	29	17	3	p	53	3	p
07	53	61	3	11	41	7	7	227	3	p	67	3	p	7	3	47	17	3	p	37
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17	19	3	p	29	3	p	p	3	7	p	3	p	p	3	151	7	3	p	p	3
19	7	13	3	p	p	3	193	7	3	p	37	3	p	11	3	59	p	3	p	199
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23	89	3	13	p	3	7	p	3	73	11	3	199	7	3	19	13	3	103	p	3
27	3	113	211	3	37	11	3	p	109	3	7	p	3	61	43	3	11	7	3	p
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31	71	7	3	p	13	3	p	229	3	163	113	3	11	p	3	7	p	3	31	p
33	3	p	193	3	29	23	3	7	p	3	11	13	3	p	7	3	p	p	3	p
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39	3	p	73	3	7	p	3	19	29	3	23	7	3	p	p	3	p	139	3	13
41	13	3	11	7	3	p	101	3	173	p	3	67	37	3	p	p	3	7	19	3
43	11	29	3	31	p	3	53	13	3	7	19	3	p	p	3	67	7	3	p	43
47	7	3	17	p	3	p	p	3	13	23	3	p	101	3	7	p	3	107	11	3
49	p	173	3	17	p	3	7	53	3	p	p	3	p	7	3	13	11	3	p	p
51	3	p	p	3	17	7	3	p	p	3	p	131	3	p	11	3	19	197	3	7
53	191	3	227	13	3	17	31	3	19	179	3	7	11	3	23	73	3	127	7	3
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59	p	3	29	19	3	p	11	3	7	p	3	13	p	3	31	7	3	11	83	3
61	7	41	3	p	11	3	47	7	3	17	p	3	73	23	3	11	p	3	13	107
63	3	p	11	3	107	p	3	23	83	3	17	p	3	7	37	3	p	p	3	191
67	13	p	3	p	7	3	p	p	3	11	53	3	17	13	3	181	p	3	7	p
69	3	19	p	3	p	197	3	11	p	3	7	43	3	17	p	3	179	7	3	97
71	139	3	7	p	3	11	23	3	37	7	3	p	19	3	13	61	3	43	p	3
73	23	7	3	11	19	3	p	p	3	p	p	3	31	p	3	7	p	3	59	223
77	17	3	p	p	3	p	7	3	p	13	3	23	167	3	29	149	3	17	71	3
79	41	17	3	13	157	3	p	p	3	p	p	3	7	79	3	p	13	3	17	7
81	3	p	17	3	7	p	3	29	p	3	13	7	3	p	109	3	p	11	3	17
83	p	3	19	7	3	p	149	3	71	p	3	139	59	3	113	11	3	7	29	3
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97	47	11	3	7	p	3	83	37	3	43	7	3	11	31	3	53	p	3	p	p
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	560	561	562	563	564	565	566	567	568	569	570	571	572	573	574	575	576	577	578	579
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13	3	p	67	3	7	31	3	p	p	3	11	7	3	37	p	3	17	p	3	29
17	13	17	3	199	p	3	11	43	3	7	23	3	29	13	3	113	7	3	17	p
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23	11	p	3	157	17	3	7	131	3	p	127	3	p	7	3	23	29	3	53	p
27	179	3	59	23	3	p	17	3	p	13	3	7	89	3	p	p	3	p	7	3
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33	137	3	53	p	3	p	p	3	7	17	3	19	11	3	79	7	3	13	151	3
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39	p	3	p	53	3	7	11	3	113	97	3	p	7	3	71	163	3	11	p	3
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47	41	7	3	29	47	3	37	p	3	11	p	3	19	p	3	7	17	3	p	p
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53	p	233	3	11	p	3	181	19	3	13	59	3	7	83	3	67	p	3	p	7
57	29	3	101	7	3	23	53	3	p	p	3	61	31	3	p	p	3	7	47	3
59	61	89	3	p	13	3	p	211	3	7	p	3	p	41	3	p	7	3	p	11
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71	47	p	3	7	149	3	p	11	3	23	7	3	p	103	3	p	101	3	11	29
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81	p	3	23	13	3	7	p	3	11	19	3	211	7	3	47	71	3	p	p	3
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	580	581	582	583	584	585	586	587	588	589	590	591	592	593	594	595	596	597	598	599
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03	11	97	3	7	p	3	p	47	3	13	7	3	73	31	3	157	19	3	79	37
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43	p	3	p	41	3	p	13	3	19	p	3	7	p	3	p	11	3	p	7	3
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49	p	3	31	19	3	p	223	3	7	11	3	p	179	3	13	7	3	149	97	3
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	600	601	602	603	604	605	606	607	608	609	610	611	612	613	614	615	616	617	618	619
01	29	p	3	47	11	3	p	101	3	p	p	3	7	59	3	11	229	3	23	7
03	3	p	11	3	7	17	3	p	41	3	53	7	3	11	p	3	p	p	3	103
07	23	p	3	13	29	3	p	17	3	7	p	3	97	101	3	p	7	3	19	31
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37	p	7	3	p	13	3	p	p	3	p	67	3	11	83	3	7	p	3	p	241
39	3	p	59	3	19	p	3	7	83	3	11	13	3	p	7	3	53	107	3	23
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79	73	7	3	11	197	3	p	p	3	17	103	3	233	p	3	7	37	p	p	p
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93	3	7	7	3	p	13	3	p	7	3	199	11	3	29	p	3	17	61	3	47
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	620	621	622	623	624	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639
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03	p	3	17	p	3	7	p	3	13	p	3	p	7	3	19	11	3	p	p	3
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23	13	23	3	p	p	3	11	p	3	7	19	3	17	13	3	139	7	3	p	97
27	7	3	11	p	3	31	p	3	p	p	3	p	23	3	7	p	3	p	83	3
29	11	p	3	157	163	3	7	149	3	p	p	3	53	7	3	17	p	3	29	p
31	3	p	13	3	149	7	3	p	83	3	p	p	3	p	137	3	17	101	3	7
33	17	3	p	83	3	p	p	3	19	13	3	7	37	3	229	p	3	17	7	3
37	3	p	7	3	29	23	3	43	31	3	13	19	3	p	11	3	7	p	3	17
39	p	3	109	17	3	p	p	3	7	p	3	103	11	3	p	7	3	13	71	3
41	7	p	3	31	17	3	37	7	3	113	11	3	p	97	3	p	23	3	p	43
43	3	p	67	3	41	13	3	p	11	3	23	233	3	7	p	3	31	p	3	11
47	p	29	3	p	7	3	13	17	3	19	67	3	p	p	3	11	p	3	7	13
49	3	19	11	3	197	p	3	131	17	3	7	p	3	11	67	3	p	7	3	p
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53	p	7	3	23	19	3	p	p	3	11	17	3	43	p	3	7	53	3	p	31
57	p	3	13	127	3	11	7	3	239	157	3	137	17	3	23	13	3	103	p	3
59	229	61	3	11	p	3	p	97	3	13	p	3	7	17	3	p	p	3	19	7
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63	53	3	19	7	3	p	223	3	37	79	3	83	41	3	p	17	3	7	p	3
67	3	7	71	3	p	19	3	23	7	3	p	13	3	p	p	3	p	11	3	47
69	7	3	73	47	3	13	29	3	p	p	3	181	151	3	7	11	3	43	13	3
71	p	p	3	97	179	3	7	41	3	p	59	3	13	7	3	151	p	3	23	17
73	3	79	p	3	p	7	3	p	p	3	p	11	3	127	p	3	41	p	3	7
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79	3	13	7	3	43	11	3	67	227	3	p	p	3	61	13	3	7	23	3	137
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83	7	11	3	p	p	3	p	7	3	p	199	3	11	241	3	13	43	3	191	109
87	47	3	199	13	3	7	p	3	11	p	3	179	7	3	p	p	3	227	29	3
89	29	p	3	89	7	3	11	37	3	p	13	3	19	p	3	p	p	3	7	61
91	3	p	167	3	11	p	3	p	61	3	7	29	3	p	173	3	p	7	3	89
93	31	3	7	43	3	53	71	3	109	7	3	13	167	3	p	19	3	p	181	3
97	p	37	p	3	p	p	3	7	p	3	p	p	3	p	7	3	p	131	3	p
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01	7	3	19	p	3	53	p	3	11	p	3	p	113	3	7	17	3	p	29	3
03	29	13	3	p	p	3	7	89	3	41	p	3	p	7	3	31	17	3	23	59
07	p	3	11	107	3	251	23	3	229	47	3	7	197	3	p	13	3	p	7	3
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13	p	3	157	73	3	p	p	3	7	139	3	19	p	3	p	7	3	p	11	3
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67	p	3	7	191	3	p	p	3	11	7	3	p	p	3	17	173	3	13	p	3
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79	139	3	p	7	3	p	p	3	p	181	3	p	29	3	p	p	3	7	11	3
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87	19	p	3	31	59	3	7	17	3	13	11	3	p	7	3	p	p	3	41	19
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97	11	3	113	71	3	13	31	3	7	p	3	11	17	3	p	7	3	19	13	3
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01	13	7	3	p	23	3	p	p	3	149	11	3	17	13	3	7	p	3	p	p
03	3	p	239	3	p	73	3	7	11	3	p	p	3	17	7	3	67	79	3	11
07	149	p	3	61	11	3	43	41	3	23	37	3	7	p	3	11	p	3	p	7
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93	3	37	p	3	7	p	3	17	151	3	13	7	3	19	p	3	139	11	3	p
97	157	53	3	67	29	3	p	p	3	7	229	3	173	11	3	23	7	3	43	97
99	3	7	167	3	p	13	3	67	7	3	17	11	3	p	p	3	p	151	3	53

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41	P	P	3	7	89	3	83	53	3	71	7	3	17	P	3	197	11	3	211	P
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87	P	3	23	11	3	107	P	3	7	149	3	43	193	3	11	7	3	19	17	3
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03	p	11	3	229	23	3	13	17	3	7	19	3	11	113	3	p	p	3	59	13
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83	3	p	67	3	7	p	3	p	73	3	31	7	3	13	p	3	97	23	3	167
87	109	13	3	59	p	3	p	71	3	7	67	3	p	p	3	17	7	3	p	p
89	3	7	p	3	p	p	3	29	7	3	p	257	3	p	11	3	17	p	3	193
91	7	3	13	43	3	73	223	3	p	p	3	p	11	3	7	13	3	17	29	3
93	29	17	3	p	157	3	7	p	3	13	11	3	p	7	3	p	p	3	17	p
97	191	3	p	17	3	227	11	3	31	p	3	7	83	3	19	p	3	11	7	3
99	p	p	3	7	11	3	19	83	3	p	7	3	37	p	3	11	p	3	p	p

	720	721	722	723	724	725	726	727	728	729	730	731	732	733	734	735	736	737	738	739
01	89	p	3	17	7	3	97	p	3	p	37	3	71	23	3	31	11	3	7	67
03	3	p	103	3	17	p	3	23	47	3	7	41	3	p	11	3	89	7	3	263
07	13	7	3	p	61	3	17	p	3	p	11	3	19	13	3	7	p	3	23	p
09	3	p	163	3	19	31	3	7	11	3	p	29	3	p	7	3	p	p	3	11
11	107	3	p	167	3	59	7	3	17	p	3	113	179	3	13	19	3	11	31	3
13	23	37	3	p	11	3	p	19	3	17	p	3	7	167	3	11	p	3	223	p
17	11	3	257	7	3	127	p	3	p	13	3	11	211	3	p	p	3	7	97	3
19	p	41	3	13	139	3	101	p	3	7	p	3	17	157	3	37	7	3	p	193
21	3	7	p	3	p	47	3	11	7	3	13	p	3	17	p	3	83	p	3	29
23	7	3	p	31	3	11	p	3	p	p	p	83	37	3	7	p	3	13	p	3
27	3	11	p	3	23	7	3	p	19	3	103	p	3	p	101	3	17	p	3	7
29	17	3	p	151	3	29	59	3	67	233	3	7	13	3	97	p	3	17	7	3
31	p	17	3	7	p	3	13	257	3	p	7	3	67	p	3	23	29	3	17	11
33	3	53	7	3	113	p	3	p	173	3	199	p	3	13	p	3	7	11	3	17
37	7	13	3	p	17	3	19	7	3	p	p	3	p	11	3	151	p	3	47	107
39	3	p	29	3	107	17	3	p	13	3	p	11	3	7	23	3	211	19	3	p
41	61	3	13	p	3	7	17	3	23	11	3	p	7	3	271	13	3	37	41	3
43	p	19	3	73	7	3	p	11	3	13	p	3	p	71	3	251	p	3	7	p
47	p	3	7	11	3	p	p	3	97	7	3	193	89	3	11	p	3	29	p	3
49	109	7	3	71	13	3	p	23	3	p	17	3	11	41	3	7	47	3	p	73
51	3	23	p	3	53	p	3	7	263	3	7	13	3	p	7	3	p	p	3	p
53	p	3	p	p	3	13	7	3	11	p	3	191	17	3	p	p	3	131	13	3
57	3	59	19	7	7	73	3	31	41	3	43	7	3	109	17	3	73	p	3	13
59	13	3	11	3	3	p	113	3	p	p	3	149	p	3	p	17	3	7	p	3
61	11	p	3	269	p	3	p	13	3	7	p	3	61	p	3	p	7	3	233	p
63	3	7	127	3	233	149	3	p	7	3	p	23	3	p	13	3	19	17	3	37
67	19	p	3	p	p	3	7	p	3	131	31	3	41	7	3	13	11	3	p	17
69	3	p	p	3	p	7	3	53	p	3	89	19	3	p	11	3	23	71	3	7
71	97	3	p	13	3	31	p	3	p	43	3	7	11	3	p	p	3	p	7	3
73	p	p	3	7	23	3	p	61	3	p	7	3	47	239	3	29	p	3	31	p
77	p	3	p	157	3	p	11	3	7	p	3	13	p	3	p	7	3	11	p	3
79	7	89	3	p	11	3	p	7	3	19	p	3	127	p	3	11	p	3	13	29
81	3	19	11	p	p	181	3	73	31	3	107	p	3	7	179	3	p	89	3	167
83	11	3	41	3	3	7	13	3	p	59	3	11	7	3	p	p	3	p	p	3
87	3	37	p	3	173	29	3	11	23	3	7	163	3	p	43	3	31	7	3	241
89	p	3	7	191	3	11	p	3	p	7	3	p	83	3	13	p	3	113	37	3
91	p	7	3	11	71	3	157	83	3	47	p	3	p	79	3	7	59	3	19	23
93	3	11	13	3	p	229	3	7	p	3	19	53	3	23	p	3	p	109	3	61
97	17	23	3	13	p	3	139	p	3	p	67	3	7	19	3	p	13	3	p	7
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	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756	757	758	759
01	3	p	p	3	47	7	3	11	131	3	179	13	3	257	p	3	19	17	3	7
03	43	3	p	67	3	11	61	3	19	p	3	7	157	3	p	p	3	p	7	3
07	3	11	7	3	37	p	3	p	239	3	107	19	3	p	p	3	7	p	3	13
09	13	3	p	19	3	p	p	3	7	173	3	p	p	3	73	7	3	p	41	3
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17	p	137	3	p	7	3	29	p	3	19	p	3	p	11	3	13	p	3	7	89
19	3	19	p	3	p	43	3	p	23	3	7	11	3	109	53	3	p	7	3	31
21	p	3	7	13	3	p	71	3	p	7	3	43	19	3	199	p	3	p	p	3
23	79	7	3	p	19	3	p	11	3	p	13	3	p	p	3	7	47	3	11	23
27	p	3	199	11	3	p	7	3	p	31	3	13	p	3	11	p	3	41	191	3
29	181	11	3	239	263	3	37	p	3	p	p	3	7	p	3	47	p	3	13	7
31	3	p	p	3	7	p	3	p	p	3	11	7	3	71	p	3	53	p	3	p
33	101	3	19	7	3	73	13	3	11	p	3	p	23	3	241	p	3	7	p	3
37	3	7	61	3	11	19	3	13	7	3	p	227	3	p	p	3	43	53	3	p
39	7	3	11	79	3	131	101	3	67	137	3	29	p	3	7	p	3	23	181	3
41	11	151	3	17	p	3	7	31	3	p	p	3	67	7	3	p	p	3	149	p
43	3	p	13	3	17	7	3	41	p	3	101	163	3	59	37	3	67	p	3	7
47	p	53	3	7	109	3	17	p	3	149	7	3	47	p	3	31	11	3	73	173
49	3	p	7	3	p	127	3	17	29	3	13	p	3	151	11	3	7	211	3	53
51	p	3	41	149	3	p	19	3	7	241	3	223	11	3	197	7	3	13	101	3
53	7	29	3	p	p	3	p	7	3	17	11	3	p	p	3	p	p	3	p	151
57	103	3	p	p	3	7	11	3	p	23	3	17	7	3	61	p	3	11	31	3
59	31	p	3	23	7	3	13	p	3	p	47	3	17	179	3	11	p	3	7	13
61	3	p	11	3	19	p	3	p	p	3	7	p	3	11	59	3	29	7	3	37
63	11	3	7	p	3	173	197	3	43	7	3	11	73	3	17	19	3	239	107	3
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71	p	17	3	11	p	3	89	p	3	13	41	3	7	23	3	p	31	3	17	7
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77	p	p	3	p	13	3	53	37	3	7	193	3	p	p	3	p	7	3	23	11
79	3	7	p	3	71	17	3	p	7	3	p	13	3	43	p	3	p	11	3	p
81	7	3	59	p	3	13	17	3	103	97	3	p	83	3	7	11	3	p	13	3
83	23	31	3	p	211	3	7	17	3	167	p	3	13	7	3	p	p	3	p	p
87	13	3	p	73	3	p	p	3	p	11	3	7	79	3	19	131	3	p	7	3
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	760	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779
01	p	3	181	41	3	113	7	3	p	11	3	p	p	3	17	19	3	13	p	3
03	p	p	3	p	p	3	p	11	3	53	p	3	7	23	3	17	71	3	11	7
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33	139	19	3	p	7	3	197	p	3	107	11	3	13	17	3	23	29	3	7	p
37	13	3	7	23	3	p	11	3	p	7	3	p	p	3	211	17	3	11	277	3
39	p	7	3	97	11	3	173	13	3	47	41	3	p	p	3	7	17	3	p	59
41	3	13	11	3	p	p	3	7	43	3	p	p	3	11	7	3	p	17	3	41
43	11	3	p	p	3	p	7	3	13	p	3	11	p	3	43	p	3	p	17	3
47	3	p	19	3	7	41	3	11	p	3	p	7	3	p	p	3	p	p	3	23
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51	59	271	3	11	89	3	p	23	3	7	13	3	67	p	3	p	7	3	127	p
53	3	7	p	3	13	37	3	p	7	3	29	p	3	103	73	3	19	13	3	137
57	19	p	3	29	101	3	7	p	3	41	251	3	23	7	3	p	79	3	13	11
59	3	p	p	3	157	7	3	59	151	3	263	19	3	p	29	3	p	11	3	7
61	23	3	p	19	3	p	13	3	101	p	3	7	p	3	71	11	3	p	7	3
63	13	p	3	7	p	3	31	29	3	p	7	3	p	11	3	p	37	3	p	53
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73	127	3	89	11	3	7	p	3	p	13	3	229	7	3	11	p	3	p	43	3
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79	p	3	7	p	3	p	p	3	11	7	3	113	p	3	p	23	3	13	47	3
81	p	7	3	17	p	3	11	p	3	23	p	3	109	223	3	7	p	3	19	29
83	3	29	p	3	11	13	3	7	p	3	19	79	3	p	7	3	131	p	3	p
87	11	47	3	p	p	3	13	31	3	167	157	3	7	19	3	p	p	3	71	7
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93	47	13	3	79	p	3	271	41	3	7	p	3	37	193	3	31	7	3	p	23
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	780	781	782	783	784	785	786	787	788	789	790	791	792	793	794	795	796	797	798	799
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07	p	37	3	p	7	3	p	p	3	19	41	3	103	71	3	43	11	3	7	p
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17	p	3	17	p	3	p	7	3	269	53	3	61	37	3	13	131	3	11	p	3
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41	p	3	p	p	3	p	19	3	7	p	3	29	p	3	17	7	3	23	p	3
43	7	13	3	157	47	3	p	7	3	89	p	3	109	11	3	17	73	3	p	p
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73	101	p	3	181	97	3	7	37	3	151	107	3	p	7	3	13	11	3	p	p
77	163	3	p	13	3	p	29	3	p	p	3	7	11	3	19	17	3	p	7	3
79	p	p	3	7	p	3	19	p	3	p	7	3	p	p	3	p	17	3	23	p
81	3	37	7	3	13	179	3	p	11	3	31	p	3	163	p	3	7	13	3	11
83	113	3	p	103	3	p	11	3	7	19	3	13	p	3	61	7	3	11	17	3
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91	13	p	3	277	7	3	p	p	3	11	139	3	37	13	3	19	p	3	7	41
93	3	p	59	3	53	p	3	11	p	3	7	p	3	p	p	3	p	7	3	167
97	29	7	3	11	p	3	p	p	3	197	19	3	179	p	3	7	p	3	109	p
99	3	p	13	3	23	53	3	7	257	3	83	29	3	p	7	3	p	199	3	p

	800	801	802	803	804	805	806	807	808	809	810	811	812	813	814	815	816	817	818	819
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03	7	3	139	131	3	19	p	3	p	17	3	11	p	3	7	149	3	p	179	3
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29	191	7	3	p	p	3	p	11	3	p	13	3	29	167	3	7	p	3	11	p
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33	163	3	p	67	3	29	7	3	p	p	3	13	p	3	11	p	3	37	19	3
37	3	127	19	3	7	p	3	p	229	3	11	7	3	163	31	3	p	p	3	p
39	p	3	p	7	3	43	13	3	11	29	3	41	p	3	p	67	3	7	p	3
41	13	p	3	p	257	3	11	263	3	7	p	3	137	13	3	73	7	3	223	67
43	3	7	29	3	11	239	3	13	7	3	p	53	3	p	23	3	19	43	3	p
47	11	p	3	p	p	3	7	p	3	61	p	3	113	7	3	p	p	3	p	19
49	3	p	13	3	p	7	3	p	p	3	p	19	3	p	79	3	p	p	3	7
51	p	3	p	19	3	109	p	3	233	13	3	7	31	3	47	p	3	29	7	3
53	17	p	3	7	43	3	59	23	3	p	7	3	193	p	3	p	11	3	p	p
57	223	3	17	107	3	p	p	3	7	73	3	p	11	3	p	7	3	13	23	3
59	7	71	3	17	61	3	79	7	3	19	11	3	23	p	3	p	37	3	109	41
61	3	19	83	3	17	13	3	p	11	3	103	277	3	7	29	3	127	p	3	11
63	23	3	p	p	3	7	p	3	p	p	3	p	7	3	p	p	3	11	71	3
67	3	p	11	3	67	p	3	17	193	3	7	23	3	11	41	3	p	7	3	p
69	11	3	7	p	3	23	p	3	17	7	3	11	181	3	257	p	3	p	p	3
71	p	7	3	179	p	3	p	37	3	11	p	3	67	p	3	7	p	3	19	p
73	3	p	p	3	p	197	3	7	13	3	17	p	3	p	7	3	23	p	3	p
77	p	p	3	11	23	3	p	p	3	13	p	3	7	19	3	29	p	3	41	7
79	3	11	p	3	7	19	3	p	31	3	89	7	3	17	59	3	13	53	3	73
81	73	3	43	7	3	61	p	3	29	47	3	p	p	3	17	23	3	7	37	3
83	53	181	3	31	13	3	p	p	3	7	p	3	p	97	3	17	7	3	p	11
87	7	3	p	p	3	13	p	3	47	109	3	19	29	3	7	11	3	17	13	3
89	283	17	3	19	p	3	7	p	3	p	131	3	13	7	3	83	p	3	17	163
91	3	p	17	3	p	7	3	173	23	3	83	11	3	199	19	3	151	89	3	7
93	13	3	23	17	3	83	19	3	41	11	3	7	p	3	227	139	3	263	7	3
97	3	13	7	3	101	11	3	43	p	3	p	p	3	23	13	3	7	157	3	167
99	173	3	59	11	3	p	17	3	7	107	3	p	p	3	11	7	3	p	p	3

	820	821	822	823	824	825	826	827	828	829	830	831	832	833	834	835	836	837	838	839
01	43	3	7	P	3	17	P	3	31	7	3	P	19	3	P	11	3	P	47	3
03	P	7	3	13	19	3	17	191	3	P	P	3	P	11	3	7	13	3	181	P
07	P	3	P	P	3	P	7	3	17	11	3	41	P	3	P	113	3	13	43	3
09	P	47	3	53	23	3	P	11	3	17	P	3	7	227	3	37	P	3	11	7
11	3	157	229	3	7	11	3	107	P	3	17	7	3	P	239	3	11	97	3	P
13	P	3	19	7	3	109	P	3	P	P	3	17	13	3	11	23	3	7	P	3
17	3	7	P	3	73	19	3	181	7	3	11	P	3	13	P	3	P	P	3	31
19	7	3	P	263	3	179	P	3	11	283	3	43	P	3	7	47	3	P	79	3
21	P	13	3	191	P	3	7	P	3	101	61	3	P	7	3	17	P	3	109	P
23	3	41	P	3	11	7	3	P	13	3	P	103	3	97	P	3	17	29	3	7
27	11	17	3	7	139	3	53	P	3	13	7	3	P	103	3	101	241	3	17	23
29	3	P	7	3	31	P	3	P	113	3	79	97	3	23	19	3	7	101	3	17
31	P	3	P	17	3	P	19	3	7	127	3	59	P	3	P	7	3	31	11	3
33	7	23	3	281	13	3	P	7	3	239	43	3	P	167	3	103	11	3	P	P
37	P	3	P	137	3	7	17	3	P	197	3	P	7	3	P	P	3	P	13	3
39	P	P	3	P	7	3	23	17	3	P	11	3	13	P	3	139	P	3	7	P
41	3	P	P	3	19	59	3	97	11	3	7	71	3	P	181	3	P	7	3	11
43	13	3	7	67	3	197	11	3	37	7	3	29	P	3	P	19	3	11	P	3
47	3	13	11	3	29	23	3	7	P	3	P	17	3	11	7	3	233	83	3	127
49	11	3	233	P	3	P	7	3	13	109	3	11	17	3	P	29	3	89	191	3
51	P	113	3	P	41	3	P	83	3	11	53	3	7	17	3	13	23	3	71	7
53	3	P	83	3	7	31	3	11	29	3	23	7	3	19	17	3	P	61	3	37
57	31	29	3	11	P	3	P	P	3	7	13	3	P	P	3	P	7	3	P	59
59	3	7	43	3	13	P	3	P	7	3	P	137	3	31	P	3	269	13	3	113
61	7	3	P	P	3	P	131	3	41	23	3	13	139	3	7	P	3	P	17	3
63	137	P	3	23	P	3	7	P	3	P	P	3	53	7	3	P	P	3	13	11
67	P	3	P	31	3	P	13	3	173	163	3	7	P	3	19	11	3	211	7	3
69	13	127	3	7	P	3	19	37	3	29	7	3	P	11	3	193	31	3	P	P
71	3	P	7	3	P	P	3	13	79	3	P	11	3	263	P	3	7	19	3	131
73	P	3	29	P	3	71	47	3	7	11	3	31	P	3	13	7	3	P	P	3
77	3	37	13	3	67	11	3	23	179	3	P	P	3	7	P	3	11	P	3	79
79	211	3	P	11	3	7	29	3	67	13	3	223	7	3	11	P	3	199	37	3
81	79	11	3	13	7	3	89	P	3	P	251	3	11	199	3	19	13	3	7	137
83	3	P	107	3	P	269	3	19	P	3	7	193	3	P	31	3	67	7	3	P
87	23	7	3	P	P	3	11	P	3	31	19	3	37	61	3	7	53	3	149	P
89	3	P	19	3	11	13	3	7	P	3	P	41	3	P	7	3	P	23	3	47
91	103	3	11	47	3	P	7	3	P	37	3	23	13	3	29	P	3	P	P	3
93	11	P	3	P	P	3	13	P	3	149	P	3	7	89	3	179	127	3	43	7
97	53	3	17	7	3	151	41	3	19	P	3	271	31	3	P	P	3	7	11	3
99	19	13	3	17	P	3	P	P	3	7	23	3	P	P	3	41	7	3	53	19

	840	841	842	843	844	845	846	847	848	849	850	851	852	853	854	855	856	857	858	859
01	167	37	3	7	p	3	11	p	3	59	7	3	p	197	3	13	p	3	239	17
03	3	31	7	3	11	p	3	71	137	3	167	p	3	p	41	3	7	p	3	p
07	7	151	3	p	p	3	19	7	3	197	13	3	139	23	3	37	p	3	53	271
09	3	241	107	3	13	p	3	23	p	3	p	p	3	7	223	3	59	13	3	p
11	p	3	p	57	3	7	211	3	p	19	3	13	7	3	p	233	3	p	11	3
13	29	19	3	p	7	3	191	p	3	p	151	3	p	p	3	p	11	3	7	53
17	p	3	7	p	3	223	13	3	89	7	3	47	11	3	229	p	3	p	p	3
19	13	7	3	p	29	3	37	p	3	p	11	3	31	13	3	7	p	3	p	151
21	3	p	p	3	p	p	3	7	11	3	p	p	3	41	7	3	p	23	3	11
23	73	3	p	37	3	p	7	3	271	163	3	23	p	3	13	p	3	11	19	3
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31	17	p	3	13	p	3	p	p	3	7	23	3	29	p	3	p	7	3	p	p
33	3	7	131	3	23	p	3	11	7	3	13	p	3	p	37	3	19	p	3	p
37	19	p	3	11	p	3	7	p	3	157	p	3	p	7	3	23	29	3	p	19
39	3	11	p	3	17	7	3	101	43	3	277	19	3	61	p	3	p	83	3	7
41	31	3	61	19	3	17	53	3	37	29	3	7	13	3	43	113	3	179	7	3
43	229	p	3	7	p	3	13	83	3	173	7	3	p	31	3	131	p	3	p	11
47	p	3	p	p	3	59	47	3	7	p	3	p	p	3	p	7	3	19	p	3
49	7	13	3	p	p	3	p	7	3	17	p	3	163	11	3	p	41	3	293	61
51	3	19	173	3	79	p	3	p	13	3	17	11	3	7	p	3	97	p	3	23
53	p	3	13	67	3	7	p	3	53	11	3	17	7	3	p	13	3	29	p	3
57	3	23	109	3	p	11	3	131	p	3	7	31	3	17	97	3	11	7	3	43
59	p	3	7	11	3	p	p	3	p	7	3	p	p	3	11	67	3	191	23	3
61	p	7	3	29	13	3	31	p	3	p	p	3	11	p	3	7	p	3	19	67
63	3	p	p	3	p	103	3	7	113	3	11	13	3	p	7	3	17	139	3	31
67	p	17	3	239	p	3	11	29	3	p	257	3	7	19	3	41	p	3	17	7
69	3	73	17	3	7	19	3	103	p	3	97	7	3	p	p	3	p	199	3	13
71	13	3	11	7	3	23	227	3	p	31	3	53	71	3	127	p	3	7	43	3
73	11	41	3	139	17	3	p	13	3	7	241	3	269	59	3	83	7	3	79	149
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79	83	p	3	19	23	3	7	17	3	p	149	3	107	7	3	13	11	3	157	127
81	3	p	271	3	p	7	3	149	17	3	p	103	3	p	11	3	47	p	3	7
83	47	3	89	13	3	41	19	3	29	17	3	7	11	3	73	23	3	109	7	3
87	3	29	7	3	13	251	3	p	11	3	p	17	3	103	p	3	7	13	3	11
89	p	3	31	p	3	p	11	3	7	37	3	13	17	3	53	7	3	11	p	3
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93	3	59	11	3	19	29	3	p	23	3	p	p	3	7	17	3	67	p	3	113
97	13	269	3	37	7	3	p	19	3	11	43	3	p	13	3	p	17	3	7	23
99	3	p	p	3	p	31	3	11	73	3	7	p	3	23	193	3	43	7	3	p

.	860	861	862	863	864	865	866	867	868	869	870	871	872	873	874	875	876	877	878	879
01	3	29	p	3	7	p	3	277	11	3	19	7	3	67	71	3	17	p	3	11
03	17	3	13	7	3	23	11	3	61	43	3	p	29	3	p	13	3	7	p	3
07	3	7	11	3	71	19	3	31	7	3	167	p	3	11	p	3	13	229	3	17
09	7	3	p	17	3	p	257	3	47	233	3	11	37	3	7	p	3	139	277	3
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13	3	p	73	3	p	7	3	11	p	3	p	13	3	p	61	3	p	239	3	7
17	p	p	3	7	103	3	37	17	3	23	7	3	13	p	3	p	41	3	137	p
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21	13	3	151	37	3	31	19	3	7	17	3	p	p	3	p	7	3	p	53	3
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33	227	3	7	13	3	p	41	3	71	7	3	p	83	3	p	17	3	59	p	3
37	3	p	83	3	13	11	3	7	p	3	p	79	3	p	7	3	11	13	3	47
39	97	3	p	11	3	p	7	3	37	p	3	13	23	3	11	p	3	p	17	3
41	139	11	3	p	p	3	23	127	3	227	p	3	7	167	3	p	p	3	13	7
43	3	p	p	3	7	37	3	p	p	3	11	7	3	19	p	3	p	p	3	p
47	13	277	3	79	137	3	11	223	3	7	61	3	43	13	3	p	7	3	107	31
49	3	7	p	3	11	23	3	13	7	3	p	p	3	113	157	3	p	47	3	37
51	7	3	11	p	3	41	73	3	p	p	3	p	p	3	7	29	3	p	59	3
53	11	101	3	p	p	3	7	p	3	89	263	3	p	7	3	p	23	3	p	281
57	47	3	p	p	3	101	193	3	p	13	3	7	p	3	19	p	3	127	7	3
59	41	29	3	7	31	3	19	101	3	p	7	3	71	p	3	p	11	3	103	p
61	3	p	7	3	p	p	3	53	p	3	13	43	3	199	11	3	7	19	3	p
63	89	3	p	67	3	107	79	3	7	19	3	101	11	3	149	7	3	13	41	3
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71	17	p	3	p	7	3	13	p	3	29	p	3	197	41	3	11	p	3	7	13
73	3	17	11	3	43	p	3	19	109	3	7	179	3	11	p	3	73	7	3	p
77	p	7	3	17	p	3	p	107	3	11	19	3	p	23	3	7	43	3	p	p
79	3	p	19	3	17	p	3	7	13	3	31	p	3	59	7	3	p	61	3	97
81	59	3	13	p	3	11	7	3	283	p	3	p	p	3	p	13	3	41	p	3
83	p	p	3	11	197	3	17	p	3	13	p	3	7	p	3	p	p	3	23	7
87	31	3	p	7	3	p	23	3	17	37	3	p	191	3	89	p	3	7	p	3
89	19	79	3	p	13	3	p	59	3	7	73	3	41	31	3	p	7	3	179	11
91	3	7	p	3	p	131	3	229	7	3	17	13	3	281	p	3	p	11	3	p
93	7	3	p	19	3	13	p	3	31	p	3	17	p	3	7	11	3	p	13	3
97	3	p	p	3	67	7	3	29	113	3	251	11	3	17	59	3	p	p	3	7
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	880	881	882	883	884	885	886	887	888	889	890	891	892	893	894	895	896	897	898	899
01	p	3	193	p	3	7	41	3	p	19	3	p	7	3	13	p	3	271	89	3
03	p	19	3	227	7	3	251	107	3	p	p	3	p	p	3	37	p	3	7	11
07	p	3	7	233	3	67	p	3	p	7	3	p	37	3	29	11	3	109	31	3
09	17	7	3	13	211	3	p	43	3	67	p	3	p	11	3	7	13	3	p	p
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37	p	3	p	p	3	29	151	3	7	p	3	p	p	3	17	7	3	19	11	3
39	7	53	3	p	13	3	137	7	3	19	269	3	233	41	3	17	11	3	p	p
41	3	19	p	3	59	37	3	p	73	3	p	13	3	7	11	3	17	43	3	53
43	17	3	79	23	3	7	p	3	p	29	3	97	7	3	p	151	3	17	13	3
47	3	181	17	3	241	p	3	p	11	3	7	239	3	47	23	3	157	7	3	11
49	13	3	7	17	3	73	11	3	23	7	3	59	31	3	p	149	3	11	p	3
51	191	7	3	53	11	3	p	13	3	p	p	3	149	199	3	7	37	3	19	293
53	3	13	11	3	197	17	3	7	p	3	19	p	3	11	7	3	p	p	3	23
57	173	199	3	149	53	3	p	17	3	11	p	3	7	19	3	13	p	3	59	7
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61	107	3	p	7	3	11	p	3	p	17	3	163	p	3	137	p	3	7	23	3
63	83	131	3	11	p	3	p	37	3	7	13	3	23	p	3	p	7	3	73	p
67	7	3	61	97	3	31	p	3	p	43	3	13	17	3	7	p	3	p	p	3
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71	3	37	103	3	p	7	3	p	181	3	p	23	3	p	17	3	p	11	3	7
73	29	3	41	67	3	23	13	3	p	193	3	7	p	3	131	11	3	107	7	3
77	3	p	7	3	103	101	3	13	31	3	281	11	3	139	p	3	7	17	3	p
79	p	3	43	p	3	283	71	3	7	11	3	257	73	3	13	7	3	p	17	3
81	7	109	3	31	23	3	p	7	3	101	229	3	19	p	3	29	p	3	11	17
83	3	163	13	3	19	11	3	47	p	3	p	101	3	7	43	3	11	p	3	p
87	59	11	3	13	7	3	131	19	3	23	p	3	11	p	3	101	13	3	7	29
89	3	29	p	3	107	p	3	p	103	3	7	p	3	71	109	3	p	7	3	p
91	137	3	7	157	3	p	31	3	11	7	3	79	29	3	p	p	3	13	p	3
93	p	7	3	37	p	3	11	p	3	p	41	3	p	p	3	7	257	3	241	31
97	37	3	11	p	3	19	7	3	p	p	3	191	13	3	31	p	3	p	p	3
99	11	89	3	109	p	3	13	p	3	61	139	3	7	p	3	p	19	3	p	7

	900	901	902	903	904	905	906	907	908	909	910	911	912	913	914	915	916	917	918	919
01	p	11	3	73	p	3	7	13	3	p	17	3	11	7	3	37	139	3	p	29
03	3	13	p	3	p	7	3	p	p	3	11	17	3	p	13	3	47	p	3	7
07	p	p	3	7	p	3	11	61	3	p	7	3	223	17	3	13	101	p	p	73
09	3	251	7	3	11	29	3	p	71	3	p	31	3	p	17	3	7	293	3	p
11	p	3	11	13	3	p	19	3	7	p	3	179	197	3	p	7	3	p	p	3
13	7	97	3	p	23	3	31	7	3	229	13	3	53	127	3	p	17	3	p	107
17	p	3	p	37	3	7	p	3	197	p	3	13	7	3	113	23	3	41	11	3
19	p	227	3	181	7	3	p	83	3	23	p	3	19	53	3	71	11	3	7	17
21	3	p	83	3	19	131	3	257	p	3	7	p	3	29	11	3	p	7	3	p
23	p	3	7	41	3	p	13	3	p	7	3	293	11	3	p	19	3	37	p	3
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33	3	173	11	3	7	p	3	41	p	3	p	7	3	11	p	3	43	p	3	149
37	179	23	3	13	p	3	233	31	3	7	59	3	p	149	3	239	7	3	p	89
39	3	7	p	3	p	37	3	11	7	3	13	p	3	241	61	3	p	199	3	p
41	7	3	31	61	3	11	p	3	p	211	3	p	23	3	7	p	3	13	p	3
43	127	109	3	11	149	3	7	103	3	199	181	3	p	7	3	31	113	3	29	p
47	53	3	p	167	3	p	p	3	p	p	3	7	13	3	19	43	3	23	7	3
49	17	p	3	7	151	3	13	p	3	163	7	3	p	167	3	83	37	3	53	11
51	3	17	7	3	29	23	3	151	47	3	83	p	3	13	109	3	7	11	3	p
53	p	3	17	p	3	83	269	3	7	19	3	p	p	3	p	7	3	p	31	3
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59	p	3	13	p	3	7	p	3	43	11	3	p	7	3	p	13	3	89	97	3
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67	p	7	3	23	13	3	71	139	3	17	19	3	11	p	3	7	31	3	p	p
69	3	37	19	3	p	41	3	7	89	3	11	13	3	p	7	3	29	163	3	p
71	p	3	p	p	3	13	7	3	11	p	3	17	107	3	23	p	3	p	13	3
73	p	p	3	p	p	3	11	43	3	29	61	3	7	p	3	p	p	3	p	7
77	13	3	11	7	3	53	p	3	19	p	3	73	97	3	17	p	3	7	79	3
79	11	31	3	p	173	3	p	13	3	7	p	3	37	23	3	17	7	3	139	19
81	3	7	p	3	p	239	3	23	7	3	p	19	3	p	13	3	17	p	3	59
83	7	3	137	19	3	p	29	3	13	37	3	p	p	3	7	p	3	17	11	3
87	3	p	17	3	41	7	3	p	p	3	79	67	3	p	11	3	277	263	3	7
89	p	3	p	13	3	157	23	3	97	p	3	p	11	3	191	67	3	19	7	3
91	23	p	3	7	17	3	89	163	3	19	7	3	p	59	3	p	p	3	43	67
93	3	19	7	3	13	17	3	p	11	3	71	p	3	p	p	3	7	23	3	11
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	920	921	922	923	924	925	926	927	928	929	930	931	932	933	934	935	936	937	938	939
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03	p	3	p	241	3	p	7	7	17	61	3	p	11	3	23	p	3	p	19	3
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17	19	251	3	p	13	3	7	p	3	11	191	3	31	7	3	17	179	3	23	19
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39	31	3	7	13	3	29	p	3	263	7	3	p	p	3	41	89	3	p	107	3
41	p	7	3	107	97	3	p	11	3	p	13	3	p	31	3	7	29	3	11	p
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47	83	11	3	p	193	3	p	163	3	41	p	3	7	17	3	139	37	3	13	7
49	3	43	29	3	7	19	3	137	p	3	11	7	3	277	17	3	71	241	3	p
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53	13	p	3	p	59	3	11	p	3	7	p	3	p	13	3	p	7	3	127	47
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61	3	23	13	3	p	7	3	p	p	3	29	52	3	89	19	3	229	p	3	7
63	43	3	257	p	3	151	19	3	p	13	3	7	p	3	p	p	3	p	7	3
67	3	37	7	3	p	p	3	p	p	3	13	151	3	73	11	3	7	41	3	p
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71	7	61	3	71	89	3	p	7	3	239	11	3	19	p	3	137	47	3	p	p
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79	3	p	11	3	p	43	3	p	131	3	7	p	3	11	p	3	23	7	3	p
81	11	3	7	p	3	p	p	3	293	7	3	11	p	3	p	p	3	191	269	3
83	p	7	3	p	23	3	p	31	3	11	p	3	p	p	3	7	p	3	223	p
87	71	3	13	p	3	11	7	3	29	p	3	p	p	3	p	13	3	p	p	3
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93	19	3	17	7	3	p	p	3	p	p	3	41	29	3	p	173	3	7	p	3
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	940	941	942	943	944	945	946	947	948	949	950	951	952	953	954	955	956	957	958	959
01	23	3	p	181	3	11	13	3	7	43	3	p	31	3	p	7	3	p	p	3
03	7	139	3	11	67	3	p	7	3	p	p	3	p	13	3	43	p	3	p	29
07	p	3	p	p	3	7	89	3	113	p	3	p	7	3	13	p	3	p	149	3
09	p	p	3	p	7	3	37	p	3	107	p	3	19	191	3	149	67	3	7	11
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13	41	3	7	37	3	p	p	3	59	7	3	227	p	3	p	11	3	p	p	3
17	3	p	71	3	263	47	3	7	53	3	13	11	3	p	7	3	p	p	3	p
19	149	3	p	257	3	31	7	3	p	11	3	73	p	3	p	23	3	13	p	3
21	167	p	3	p	p	3	p	11	3	23	p	3	7	199	3	59	p	3	11	7
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33	p	13	3	17	p	3	7	61	3	p	29	3	p	7	3	83	p	3	47	23
37	271	3	11	29	3	17	101	3	p	139	3	7	131	3	19	13	3	p	7	3
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41	3	47	7	3	p	p	3	17	p	3	101	89	3	67	p	3	7	19	3	37
43	157	3	73	p	3	p	31	3	7	19	3	p	23	3	p	7	3	67	11	3
47	3	31	79	3	p	p	3	p	p	3	17	13	3	7	11	3	101	p	3	p
49	p	3	307	p	3	7	p	3	p	p	3	17	7	3	31	p	3	23	13	3
51	163	p	3	p	7	3	p	41	3	p	11	3	13	97	3	19	p	3	7	229
53	3	p	p	3	29	23	3	19	11	3	7	p	3	17	53	3	41	7	3	11
57	p	7	3	157	11	3	103	13	3	269	19	3	p	167	3	7	23	3	p	p
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63	p	17	3	197	p	3	181	193	3	11	p	3	7	47	3	13	271	3	17	7
67	109	3	107	7	3	11	137	3	19	23	3	59	p	3	p	227	3	7	37	3
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79	p	3	29	p	3	271	13	3	79	17	3	7	p	3	p	11	3	19	7	3
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83	3	19	7	3	p	p	3	13	239	3	p	11	3	p	p	3	7	p	3	53
87	7	97	3	37	19	3	p	7	3	43	p	3	p	17	3	61	103	3	11	p
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93	23	11	3	13	7	3	p	p	3	p	p	3	11	p	3	109	13	3	7	59
97	73	3	7	p	3	p	281	3	11	7	3	23	233	3	29	p	3	p	17	3
99	p	7	3	p	53	3	11	47	3	p	61	3	157	19	3	7	83	3	41	17

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01	p	17	3	23	p	3	p	11	3	7	p	3	13	p	3	p	7	3	11	47
03	3	7	17	3	149	11	3	p	7	3	p	p	3	p	257	3	11	41	3	3
07	19	11	3	193	17	3	7	13	3	p	p	3	11	7	3	281	p	3	47	19
09	3	13	23	3	229	7	3	97	131	3	11	19	3	31	13	3	p	199	3	7
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13	p	223	3	7	67	3	11	17	3	199	7	3	p	23	3	13	p	3	p	179
17	p	3	11	13	3	p	79	3	7	17	3	p	67	3	61	7	3	19	29	3
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21	3	19	p	3	13	263	3	311	p	3	p	17	3	7	37	3	41	13	3	181
23	131	3	p	p	3	7	23	3	p	103	3	13	7	3	p	p	3	79	11	3
27	3	97	41	3	211	p	3	197	p	3	7	p	3	p	11	3	233	7	3	p
29	109	3	7	p	3	83	13	3	37	7	3	23	11	3	p	17	3	p	p	3
31	13	7	3	p	p	3	71	p	3	p	11	3	p	13	3	7	17	3	19	p
33	3	251	p	3	73	37	3	7	11	3	19	137	3	131	7	3	89	17	3	11
37	137	p	3	p	11	3	41	p	3	31	23	3	7	19	3	11	163	3	227	7
39	3	127	11	3	7	19	3	p	179	3	p	7	3	11	139	3	251	43	3	37
41	11	3	157	7	3	29	241	3	113	13	3	11	p	3	p	103	3	7	p	3
43	p	79	3	13	p	3	p	89	3	7	53	3	47	311	3	23	7	3	p	p
47	7	3	109	23	3	11	127	3	p	29	3	19	31	3	7	p	3	13	p	3
49	139	p	3	11	43	3	7	p	3	67	107	3	79	7	3	p	p	3	p	41
51	3	11	29	3	p	7	3	31	p	3	37	p	3	67	19	3	p	239	3	7
53	p	3	101	p	3	p	19	3	23	p	3	7	13	3	p	p	3	67	7	3
57	3	p	7	3	p	p	3	p	p	3	71	p	3	13	41	3	7	11	3	23
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71	23	3	7	11	3	269	p	3	73	7	3	p	211	3	11	p	3	p	p	3
73	191	7	3	17	13	3	277	29	3	p	p	3	11	p	3	7	p	3	97	p
77	29	3	43	p	3	13	7	3	11	37	3	p	89	3	107	p	3	p	13	3
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97	p	19	3	7	p	3	p	p	3	p	7	3	149	p	3	17	151	3	223	43
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	980	981	982	983	984	985	986	987	988	989	990	991	992	993	994	995	996	997	998	999
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03	23	3	7	197	3	137	151	3	29	7	3	P	13	3	107	19	3	179	11	3
07	3	17	P	3	P	P	3	7	P	3	181	23	3	13	7	3	P	P	3	P
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17	P	59	3	P	11	3	17	P	3	7	P	3	47	P	3	11	7	3	P	41
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23	83	P	3	P	13	3	7	269	3	11	P	3	P	7	3	P	P	3	P	P
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29	167	P	3	7	P	3	19	P	3	P	7	3	13	71	3	P	67	3	P	P
31	3	11	7	3	257	37	3	P	23	3	167	P	3	17	P	3	7	19	3	13
33	13	3	23	107	3	P	53	3	7	19	3	P	P	3	17	7	3	P	P	3
37	3	13	193	3	173	211	3	P	P	3	97	P	3	7	13	3	17	11	3	37
39	17	3	31	29	3	7	P	3	13	P	3	P	7	3	P	11	3	17	P	3
41	P	17	3	43	7	3	P	293	3	163	P	3	P	11	3	13	37	3	7	139
43	3	P	17	3	P	P	3	19	97	3	7	11	3	41	17	3	P	7	3	17
47	P	7	3	P	17	3	23	11	3	P	13	3	61	P	3	7	251	3	11	89
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51	71	3	P	11	3	39	7	3	41	53	3	13	P	3	11	P	3	23	31	3
53	31	11	3	59	P	3	47	17	3	P	P	3	7	73	3	113	227	3	13	7
57	P	3	P	7	3	67	13	3	11	17	3	229	P	3	271	29	3	7	61	3
59	13	103	3	41	P	3	11	61	3	7	17	3	P	13	3	P	7	3	P	19
61	3	7	97	3	11	P	3	13	7	3	23	17	3	67	79	3	P	P	3	P
63	7	3	11	19	3	P	P	3	109	P	3	53	17	3	7	P	3	67	37	3
67	3	89	13	3	P	7	3	283	P	3	157	131	3	P	17	3	P	P	3	7
69	281	3	P	P	3	241	P	3	P	13	3	7	53	3	P	17	3	19	7	3
71	101	127	3	7	59	3	79	43	3	19	7	3	37	P	3	P	11	3	P	P
73	3	19	7	3	P	P	3	P	P	3	13	P	3	43	11	3	7	17	3	257
77	7	31	3	P	19	3	101	7	3	29	11	3	P	P	3	P	263	3	P	17
79	3	P	23	3	P	13	3	P	11	3	P	41	3	7	31	3	P	113	3	11
81	P	3	29	131	3	7	11	3	61	P	3	P	7	3	53	P	3	11	P	3
83	43	47	3	37	7	3	13	173	3	31	P	3	101	23	3	11	83	3	7	13
87	11	3	7	P	3	311	29	3	P	7	3	11	43	3	P	53	3	P	59	3
89	47		3	P	149	3	P	223	3	11	P	3	P	19	3	7	P	3	23	P
91	3	149	227	3	P	19	3	7	13	3	197	P	3	P	7	3	131	73	3	P
93	233	3	13	61	3	11	7	3	P	P	3	281	31	3	37	13	3	P	191	3
97	3	11	P	3	7	P	3	31	P	3	41	7	3	P	P	3	13	23	3	19
99	263	3	P	7	3	43	229	3	P	P	3	19	109	3	29	137	3	7	283	3

Of Rational Numbers that expreſs the Sides of Right-angled Triangles.

A PROBLEM.

Article 1. To find as many right-angled triangles as we pleaſe, of which the three ſides ſhall be expreſſible in rational numbers.

SOLUTION.

Let the numbers that expreſs the lengths of the two ſides that contain the right angle, be denoted by the letters m and n. Then will the number that denotes the hypotenuſe of the triangle, or the line that ſubtends the right angle, be greater than either of the two numbers m and n, and its exceſs above either of the ſaid numbers will be a rational number: for, if it were not, the number itſelf which expreſſes the ſaid hypotenuſe would not be a rational number. Let the exceſs of this number, which expreſſes the hypotenuſe, above the number m, which expreſſes one of the ſides containing the right angle, be called e. Then will the number which expreſſes the hypotenuſe be $m + e$, and its ſquare will be $mm + 2me + ee$. But (by El. 1, 47,) the ſquare of the hypotenuſe of a right-angled triangle is equal to the ſum of the ſquares of the two ſides of it. Therefore

$mm + 2me + ee$ will be equal to $mm + nn$; and, conſequently (ſubtracting mm from both ſides), $2me + ee$ will be equal to nn; and (ſubtracting ee from both ſides, which is evidently leſs than $2me + ee$, and conſequently muſt be leſs alſo than nn, or the right-hand ſide of the equation $2me + ee = nn$,) $2me$ will be equal to $nn - ee$; and (dividing both ſides of the equation by $2e$,) m will be equal to $\frac{nn-ee}{2e}$, and conſequently $m + e$ will be equal to $\frac{nn-ee}{2e} + e$, or to $\frac{nn-ee}{2e} + \frac{2e \times e}{2e}$, or to $\frac{nn-ee}{2e} + \frac{2ee}{2e}$, or to $\frac{nn+ee}{2e}$. And conſequently the three numbers m, n, and $m + e$, that will expreſs the three ſides of a right-angled triangle, will be equal to $\frac{nn-ee}{2e}$, n, and $\frac{nn+ee}{2e}$, or $\frac{nn-ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn+ee}{2e}$; or, if we take any number whatſoever, and call it e, and take any other number whatſoever that is greater than e, and call it n, the three numbers $\frac{nn-ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn+ee}{2e}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. Q. E. I.

Examples of this Method of finding ſuch Rational Numbers.

Art. 2. Thus, for example, if e is $= 1$, and n is $= 2$, (which are the ſimpleſt numbers we can chuſe,) we ſhall have $ee = 1$, and $nn = 4$, and conſequently $\frac{nn-ee}{2e}$ $(= \frac{4-1}{2 \times 1} = \frac{4-1}{2}) = \frac{3}{2}$, and $\frac{2en}{2e}$ $(= \frac{2 \times 1 \times 2}{2 \times 1}) = \frac{4}{2}$, and $\frac{nn+ee}{2e}$ $(= \frac{4+1}{2 \times 1} = \frac{4+1}{2}) = \frac{5}{2}$; and conſequently

$\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$, will be three rational numbers that will expreſs the lengths of the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of the laſt of theſe numbers, to wit, $\frac{5}{2}$, which expreſſes the hypotenuſe of the triangle, is equal to the ſum of the ſquares of the two former numbers, to wit, $\frac{3}{2}$ and $\frac{4}{2}$, which expreſs the two ſides that contain the right angle. For the ſquare of $\frac{5}{2}$ is $\frac{25}{4}$, and the ſquares of $\frac{3}{2}$ and $\frac{4}{2}$ are $\frac{9}{4}$ and $\frac{16}{4}$; and $\frac{25}{4}$ is $= \frac{9}{4} + \frac{16}{4}$.

Secondly, let e be $= 2$, and n be $= 3$.

Then we ſhall have $ee = 4$, and $nn = 9$, and $2e = 4$; and conſequently $\frac{nn - ee}{2e}$ $(= \frac{9 - 4}{4}) = \frac{5}{4}$, and $\frac{2en}{2e}$ $(= \frac{2 \times 2 \times 3}{2 \times 2}) = \frac{12}{4}$, and $\frac{nn + ee}{2e}$ $(= \frac{9 + 4}{4}) = \frac{13}{4}$. Therefore $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{13}{4}$ will be equal to the ſum of the ſquares of $\frac{5}{4}$ and $\frac{12}{4}$. For the ſquare of $\frac{13}{4}$ is $= \frac{169}{16}$, and the ſquares of $\frac{5}{4}$ and $\frac{12}{4}$ are $\frac{25}{16}$ and $\frac{144}{16}$; and $\frac{169}{16}$ is $= \frac{25}{16} + \frac{144}{16}$.

Thirdly, let e be $= 3$, and n be $= 5$.

Then we ſhall have $ee = 9$, and $nn = 25$, and $2e = 6$, and conſequently $\frac{nn - ee}{2e}$ $(= \frac{25 - 9}{6}) = \frac{16}{6}$, and $\frac{2en}{2e}$ $(= \frac{2 \times 3 \times 5}{6}) = \frac{30}{6}$, and $\frac{nn + ee}{2e}$ $(= \frac{25 + 9}{6}) = \frac{34}{6}$.

Therefore $\frac{16}{6}$, $\frac{30}{6}$, and $\frac{34}{6}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{34}{6}$ will be equal to the ſum of the ſquares of $\frac{16}{6}$ and $\frac{30}{6}$. For the ſquare of $\frac{34}{6}$ is $\frac{1156}{36}$, and the ſquares of $\frac{16}{6}$ and $\frac{30}{6}$ are $\frac{256}{36}$ and $\frac{900}{36}$; and $\frac{1156}{36}$ is $= \frac{256}{36} + \frac{900}{36}$.

Theſe three numbers $\frac{16}{6}$, $\frac{30}{6}$, and $\frac{34}{6}$, might have been reduced to ſmaller numbers, by dividing both their numerators and denominators by 2. For they would then have been $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$. Therefore theſe three numbers $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$, will expreſs the three ſides of a right-angled triangle.

Fourthly, let e be $= 3$, and $n = 7$.

Then we ſhall have $ee = 9$, and $nn = 49$, and $2e = 6$, and conſequently $\frac{nn - ee}{2e}$ $(= \frac{49 - 9}{6}) = \frac{40}{6}$, and $\frac{2en}{2e}$ $(= \frac{2 \times 3 \times 7}{6}) = \frac{42}{6}$, and $\frac{nn + ee}{2e}$ $(= \frac{49 + 9}{6}) = \frac{58}{6}$. Therefore $\frac{40}{6}$, $\frac{42}{6}$, and $\frac{58}{6}$, or $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, will be three rational numbers that will expreſs the ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{29}{3}$ will be equal to the ſum of the ſquares of $\frac{20}{3}$ and $\frac{21}{3}$. For the ſquare of $\frac{29}{3}$ is $\frac{841}{9}$, and the ſquares of $\frac{20}{3}$ and $\frac{21}{3}$ are $\frac{400}{9}$ and $\frac{441}{9}$; and $\frac{841}{9}$ is $= \frac{400}{9} + \frac{441}{9}$.

Fifthly, let e be $= 3$, and $n = 11$.

Then we ſhall have $ee = 9$, and $nn = 121$, and $2e = 6$, and conſequently $\frac{nn - ee}{2e}$ $(= \frac{121 - 9}{6}) = \frac{112}{6}$, and $\frac{2en}{2e}$ $(= \frac{6 \times 11}{6}) = \frac{66}{6}$, and $\frac{nn + ee}{2e}$ $(= \frac{121 + 9}{6}) = \frac{130}{6}$. Therefore $\frac{112}{6}$, $\frac{66}{6}$, and $\frac{130}{6}$, or $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{65}{3}$ will be equal to the ſum of the ſquares of $\frac{56}{3}$ and $\frac{33}{3}$. For the ſquare of $\frac{65}{3}$ is $\frac{4225}{9}$, and the ſquares of $\frac{56}{3}$ and $\frac{33}{3}$ are $\frac{3136}{9}$ and $\frac{1089}{9}$; and $\frac{4225}{9}$ is $= \frac{3136}{9} + \frac{1089}{9}$.

Sixthly, let e be $= 5$, and $n = 13$.

Then we ſhall have $ee = 25$, and $nn = 169$, and $2e = 10$, and conſequently $\frac{nn - ee}{2e}$ $(= \frac{169 - 25}{10}) = \frac{144}{10}$, and $\frac{2en}{2e}$ $(= \frac{10 \times 13}{10}) = \frac{130}{10}$, and $\frac{nn + ee}{2e}$ $(= \frac{169 + 25}{10}) = \frac{194}{10}$. Therefore $\frac{144}{10}$, $\frac{130}{10}$, and $\frac{194}{10}$, or $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{97}{5}$ will be equal to the ſum of the ſquares of $\frac{72}{5}$ and $\frac{65}{5}$. For the ſquare of $\frac{97}{5}$ is $\frac{9409}{25}$, and the ſquares of $\frac{72}{5}$ and $\frac{65}{5}$ are $\frac{5184}{25}$ and $\frac{4225}{25}$; and $\frac{9409}{25}$ is $= \frac{5184}{25} + \frac{4225}{25}$.

Thus we have obtained ſix different ſets of rational numbers, which expreſs the lengths of the ſides of as many different right-angled triangles; to wit, 1ſt, $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$; and, 2dly, $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$; and, 3dly, $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$; and, 4thly, $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$; and, 5thly, $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$; and, 6thly, $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$. And, by changing either both the numbers denoted by e and n, or only one of thoſe numbers, and computing the values of the three fractions $\frac{nn-ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn+ee}{2e}$, we may obtain as many more ſuch ſets of numbers as we pleaſe.

Art. 3. All theſe numbers are fractions, becauſe they are derived from the general fractional expreſſions $\frac{nn-ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn+ee}{2e}$. But, if we multiply the three fractions of each of theſe ſix ſets of fractions by their common denominator, the products will be whole numbers expreſſing the ſides of greater right-angled triangles ſimilar to the former triangles, of which the ſides were expreſſed by the foregoing fractions. Thus, if we multiply the three fractions $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$, by their common denominator 2, we ſhall have the whole numbers 3, 4, and 5, for the ſides of a greater right-angled triangle ſimilar to the former triangle, of which the ſides were expreſſed by the fractions $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$. And, if we multiply the three fractions $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$, by their common denominator 4, we ſhall have the whole numbers 5, 12, and 13, for the ſides of a greater right-angled triangle ſimilar to the former triangle, of which the ſides were $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$. And, if we multiply the three fractions

$\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$, by their common denominator 3, we ſhall have the whole numbers 8, 15, and 17, for the ſides of a greater right-angled triangle ſimilar to the former triangle, of which the ſides were $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$. And, in like manner, from the fractions $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, we may derive the whole numbers 20, 21, and 29; and from the fractions $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, we may derive the whole numbers 56, 33, and 65; and from the fractions $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$, we may derive the whole numbers 72, 65, and 97; all which ſets of whole numbers will expreſs the ſides of right-angled triangles ſimilar to the three former triangles, of which the ſides were expreſſed by the fractions $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, and the fractions $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, and the fractions $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$.

Art. 4. And theſe whole numbers might have been obtained at once by computing only the numerators of the three general fractions $\frac{nn-ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn+ee}{2e}$, to wit, the expreſſions $nn - ee$, $2en$, and $nn + ee$, which are the products of the multiplication of the ſaid three fractions into their common denominator $2e$. For then we ſhould have found, in the firſt example, in which e is $= 1$ and n is $= 2$, that the ſaid expreſſions $nn - ee$, $2en$, and $nn + ee$, would have been equal to ($4 - 1$, $2 \times 1 \times 2$, and $4 + 1$, or) 3, 4, and 5; and, in the ſecond example, in which e was $= 2$, and $n = 3$, we ſhould have had $nn - ee$, $2en$, and $nn + ee$, equal to ($9 - 4$, $2 \times 2 \times 3$, and $9 + 4$, or)

5, 12,

5, 12, and 13; and, in the 3d example, in which e was $= 3$, and $n = 5$, we ſhould have had $nn - ee$, $2en$, and $nn + ee$, equal to ($25 - 9$, $2 \times 3 \times 5$, and $25 + 9$, or) 16, 30, and 34, which, when divided by 2, become 8, 15, and 17; and in the 4th example, in which e was $= 3$, and $n = 7$, we ſhould have had $nn - ee$, $2en$, and $nn + ee$, equal to ($49 - 9$, $2 \times 3 \times 7$, and $49 + 9$, or) 40, 42, and 58, which, when divided by 2, become 20, 21, and 29; and, in the 5th example, in which e is $= 3$, and n is $= 11$, we ſhould have had $nn - ee$, $2en$, and $nn + ee$, equal to ($121 - 9$, $2 \times 3 \times 11$, and $121 + 9$, or) 112, 66, and 130, which, when divided by 2, become 56, 33, and 65; and in the 6th and laſt example, in which e was $= 5$, and n was $= 13$, we ſhould have had $nn - ee$, $2en$, and $nn + ee$, equal to ($169 - 25$, $2 \times 5 \times 13$, and $169 + 25$, or) 144, 130, and 194, which, when divided by 2, become 72, 65, and 97. And thus we ſhould have obtained the ſix foregoing ſets of whole numbers to expreſs the ſides of different right-angled triangles, to wit, 1ſt, the numbers 3, 4, and 5; 2dly, the numbers 5, 12, and 13; 3dly, the numbers 8, 15, and 17; 4thly, the numbers 20, 21, and 29; 5thly, the numbers 56, 33, and 65; and, 6thly, the numbers 72, 65, and 97.

A SCHOLIUM.

Art. 5. It may be obſerved, that in the four firſt of the foregoing ſix ſets of numbers, which expreſs the ſides of right-angled triangles, to wit, in the numbers 3, 4, and 5, and in the numbers 5, 12, and 13, and in the numbers 8, 15, and 17, and in the numbers 20, 21, and 29, the firſt number of each ſet is leſs than the ſecond; but in the fifth and ſixth ſets of thoſe numbers, to wit, in the numbers 56, 33, and 65, and in the numbers 72, 65, and 97, the firſt number of each ſet is greater than the ſecond. Now this depends upon the proportion in which the number n (which is always greater than e,) exceeds the number e. For, if n were not a number, but a quantity incommenſurable to 1, which bore the ſame proportion to the number e as 1 bears to $\sqrt{2} - 1$, or to the exceſs of $\sqrt{2}$ above 1, or as the ſide of a ſquare bears to the exceſs of its diagonal above its ſide, the general expreſſion $nn - ee$, from which the firſt terms of all theſe ſets of numbers are derived, would be exactly equal to the general expreſſion $2en$, from which the ſecond terms of the ſaid ſets of numbers are derived: and, when the proportion of n to e is leſs than that of 1 to $\sqrt{2} - 1$, or n is leſs than $e \times \frac{1}{\sqrt{2-1}}$, the general expreſſion $nn - ee$ will be leſs than $2en$: and, when the proportion of n to e is greater than the ſaid proportion of 1 to $\sqrt{2} - 1$, or n is greater than $e \times \frac{1}{\sqrt{2-1}}$, the general expreſſion $nn - ee$ will be greater than $2en$. Theſe things may be demonſtrated in the manner following.

Art. 6. In the 1ſt place, if n is $= e \times \frac{1}{\sqrt{2-1}}$, we ſhall have $nn = ee \times \frac{1}{\overline{\sqrt{2-1}}|^2} = \frac{ee}{2 - 2\sqrt{2} + 1}$, and $nn - ee$ (= $\frac{ee}{2-2\sqrt{2}+1} - ee = \frac{ee}{2-2\sqrt{2}+1} - \frac{\overline{2-2\sqrt{2}+1} \times ee}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} - \frac{\overline{2ee - 2\sqrt{2} \times ee + ee}}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} - \frac{\overline{3ee - 2\sqrt{2} \times ee}}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} \quad \frac{-3ee + 2\sqrt{2} \times ee}{2-2\sqrt{2}+1} = + \frac{2\sqrt{2} \times ee - 2ee}{2-2\sqrt{2}+1} = \frac{2ee \times \overline{\sqrt{2}-1}}{2-2\sqrt{2}+1}$) $= \frac{2ee}{\sqrt{2}-1}$; and we ſhall alſo have $2en$ ($= 2e \times e \times \frac{1}{\sqrt{2}-1}$) $= \frac{2ee}{\sqrt{2}-1}$. And conſequently $nn - ee$ will in this caſe be equal to $2en$. Q. E. D.

2dly, If n is leſs than $e \times \frac{1}{\sqrt{2}-1}$, the compound quantity $nn - ee$ will be leſs than $2en$.

For, if we ſuppoſe n, from being equal to $e \times \frac{1}{\sqrt{2}-1}$, to become leſs than that quantity, but ſtill to be greater than e, and the decrement of n, or its difference from its firſt value, to be denoted by the letter d, it is evident that while n is decreaſing from n to $n - d$, the compound quantity $nn - ee$ will decreaſe from $nn - ee$ to $\overline{n-d}|^2 - ee$, that is, to $nn - 2nd + dd - ee$, or to $nn - ee - 2nd + dd$, or to $nn - ee - \overline{2nd - dd}$, or will be leſs than it was before by the quantity $2nd - dd$; and in the ſame time the quantity $2en$ will decreaſe from its firſt value, $2en$, (which was equal to $nn - ee$) to $2e \times \overline{n-d}$, or $2en - 2ed$, or will be leſs than it was before by the quantity $2ed$. Now, becauſe $n - d$ is greater than e, it follows that n muſt be greater than $e + d$, and conſequently that $n - e$ muſt be greater than d. Therefore $2d \times \overline{n-e}$ will be greater than $2d \times d$, or $2nd - 2ed$ will be greater than $2dd$, and conſequently $2nd$ will be greater than $2dd + 2ed$, and $2nd - dd$ will be greater than $dd + 2ed$. Therefore, *à fortiori*, $2nd - dd$ will be greater than $2ed$; that is, the decrement of the quantity $nn - ee$ while n decreaſes from n to $n - d$, will be greater than the decrement of the quantity $2en$ in the ſame time: and conſequently the quantity $nn - ee - \overline{2nd - dd}$, to which $nn - ee$ will have decreaſed, while n was decreaſing from n to $n - d$, will be leſs than the quantity $2en - 2ed$, to which the quantity $2en$ (which was at firſt equal to $nn - ee$,) will have decreaſed in the ſame time; or, if n is of any magnitude leſs than $e \times \frac{1}{\sqrt{2}-1}$, but greater than e, the quantity $nn - ee$ will be leſs than the quantity $2en$. Q. E. D.

And, 3dly, if n is greater than $e \times \frac{1}{\sqrt{2}-1}$, the compound quantity $nn - ee$ will be greater than $2en$.

For, if we ſuppoſe n, from being equal to $e \times \frac{1}{\sqrt{2}-1}$, to become greater than that quantity, and the increment of n, or its difference from its former value, to be denoted by the letter d, it is evident that, while n is increaſing from n to $n + d$, the compound quantity $nn - ee$ will increaſe from $nn - ee$ to $\overline{n+d}|^2 - ee$, that is, to $nn + 2nd + dd - ee$, or to $nn - ee + 2nd + dd$, or it will be greater than it was before by the quantity $2nd + dd$; and in the ſame time the quantity $2en$ will increaſe from its firſt value, $2en$, (which was equal to $nn - ee$,) to $2e \times \overline{n+d}$, or $2en + 2ed$, or will be greater than it was before by the quantity $2ed$. Now, becauſe n is greater than e, it follows that $2nd$ muſt be greater than $2ed$; and conſequently, *à fortiori*, $2nd + dd$ will be greater than $2ed$; that is, the increment of $nn - ee$ during the increaſe of n from n to $n + d$ will be greater than the contemporary increment of $2en$. Therefore the quantity $nn - ee + 2nd + dd$, to which $nn - ee$ will have increaſed while n increaſed from n to $n + d$, will be greater than the quantity $2en + 2ed$, to which the quantity $2en$ (which was at firſt equal to $nn - ee$,) will have increaſed in the ſame time; or, if n is of any magnitude greater than $e \times \frac{1}{\sqrt{2}-1}$, the quantity $nn - ee$ will be greater than the quantity $2en$. Q. E. D.

Art. 7. If we take two numbers for e and n that are nearly in the proportion of $\sqrt{2} - 1$ and 1, we ſhall find that $nn - ee$ will be very nearly equal to $2en$. Now $\sqrt{2}$ is $= 1.414$ &c. Therefore $\sqrt{2} - 1$ is $= 1.414$ &c $- 1 = 0.414$ &c, and $\sqrt{2} - 1$ is to 1 pretty nearly in the proportion of 0.414 to 1, or of 414 to 1000, or of 207 to 500. Therefore, if we ſuppoſe e to be $= 207$, and n to be $= 500$, the value of $nn - ee$ ought to be nearly equal to that of $2en$. And ſo we ſhall find them to be. For, upon theſe ſuppoſitions, we ſhall have ee $(= \overline{207}|^2) = 42{,}849$, and nn $(= \overline{500}|^2) = 250{,}000$, and $nn - ee$ $(= 250{,}000 - 42{,}849) = 207{,}151$. And $2en$ will be (=

2 × 207 × 500 = 207 × 1000) = 207,000; which is very nearly equal to 207,151, or $nn - ee$.

Art. 8. And, upon theſe ſuppoſitions, $nn + ee$ will be (= 250,000 + 42,849) = 292,849; which gives us a ſeventh ſet of numbers that expreſs the ſides of a right-angled triangle, to wit, the numbers 207,151, 207,000, and 292,849. And accordingly we ſhall find that the ſquare of the number 292,849, which repreſents the hypotenuſe of the triangle, will be equal to the ſum of the ſquares of the other two numbers 207,151 and 207,000, which repreſent the ſides that contain the right angle. For the ſquare of 292,849 is 85,760,536,801, and the ſquares of 207,151 and 207,000, are 42,911,536,801 and 42,849,000,000; and 85,760,536,801 is = 42,911,536,801 + 42,849,000,000.

Art. 9. If we ſuppoſe e to be = 2, and n to be = 5, we ſhall have $nn - ee$ (= 25 − 4) = 21, and $2en$ (= 2 × 2 × 5) = 20, and $nn + ee$ (= 25 + 4) = 29. Therefore 21, 20, and 29, will be three numbers that will expreſs the lengths of the three ſides of a right-angled triangle. And theſe numbers, we may obſerve, are the ſame with the three numbers 20, 21, and 29, obtained above in art. 4, by ſuppoſing e to be equal to 3, and n to be equal to 7, excepting that the order of the two firſt numbers 20 and 21 is different in the two ſets, 20 being the firſt number in the firſt ſet, 20, 21, and 29, and being the ſecond number in the ſecond ſet, 21, 20, and 29; the reaſon of which is, that 20 is derived from the firſt general expreſſion $nn - ee$ in the firſt ſet of numbers, 20, 21, and 29, and it is derived from the ſecond general quantity $2en$ in the ſecond ſet of numbers, 21, 20, and 29. This, however, has an odd appearance, that, when the original numbers n and e, from which the general expreſſions $nn - ee$, $2en$, and $nn + ee$ are derived, are made to bear different proportions to each other (for the proportion of 5 to 2 is greater than the proportion of 7 to 3, being equal to that of 7 to 2.8, or of 70 to 28,) the three numbers obtained by means of thoſe general expreſſions ſhould ſtill be the ſame, though placed in a different order: and therefore it may not be amiſs to inquire a little further into it.

Art. 10. In order therefore that the reaſon of this ſeeming irregularity may appear the more clearly, we will recur to the obſervation made above in art. 5, concerning the change in the proportion of the general expreſſion $nn - ee$ to the general expreſſion $2en$, when the proportion of n to e, from being at firſt a leſs proportion of majority than that of 1 to $\surd 2 - 1$, or of 1 to 0.414, &c, becomes equal to, and greater than, the ſaid ratio; to wit, that, when the ratio of n to e is leſs than the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is leſs than the quantity $2en$; and that, when the ratio of n to e is equal to the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is equal to the quantity $2en$; and that, when the ratio of n to e is greater than the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is greater than the quantity $2en$. And to this obſervation we muſt add that, if the ratio of n to e, after having been equal to the ratio of 1 to 0.414, &c, is ſuppoſed to increaſe gradually *ad infinitum*, the ratio of $nn - ee$ to $2en$ will increaſe gradually at the ſame time *ad infinitum*, or ſo as to become greater than any aſſigned ratio whatſoever. For the ratio of $nn - ee$ to $2en$ is equal to the ratio of $\frac{nn - ee}{2en}$ to ($\frac{2en}{2en}$, or) 1, or to the ratio of $\frac{nn}{2en} - \frac{ee}{2en}$ to 1, or to the ratio of $\frac{n}{2e} - \frac{e}{2n}$ to 1, which evidently increaſes *ad infinitum*, while the ratio of n to e increaſes *ad infinitum*. Therefore while the ratio of n to e increaſes, from being equal to the ratio of 1 to 0.414, &c, *ad infinitum*, the ratio of $nn - ee$ to $2en$ will increaſe gradually from a ratio of equality *ad infinitum*, and conſequently will become ſucceſſively equal to all ratios of majority whatſoever. Therefore, if the ratio of n to e is at one time taken equal to the ratio of 7 to 3, (which is leſs than the ratio of 7 to 2.898, &c, or of 7 to 7 × 0.414, &c, or of 1 to 0.414, &c,) and is afterwards ſuppoſed to increaſe gradually till it becomes equal to the ſaid ratio of 1 to 0.414, &c, and then to increaſe further *ad infinitum*, the ratio of the compound quantity *nn* — *ee* to the quantity 2*en* (which is equal to the ratio of 20 to 21, when the ratio of *n* to *e* is equal to the ratio of 7 to 3,) will firſt become a ratio of equality, to wit, when the ratio of *n* to *e* becomes equal to the ratio of 1 to 0.414, &c, and afterwards will increaſe continually from being a ratio of equality, (which may be conſidered as an infinitely ſmall ratio of majority, and is uſually ſo conſidered by writers who treat of the magnitudes and meaſures of ratios,) till it becomes ſucceſſively equal to every ratio of majority whatſoever. It therefore muſt at one point of time during its ſaid increaſe become equal to the ratio of 21 to 20; or, there will be a certain ratio of majority, greater than that of 1 to 0.414, &c, to which when the ratio of *n* to *e* ſhall have become equal, the ratio of *nn* — *ee* to 2*en* will be equal to the ratio of 21 to 20. And this ratio of majority is that of 5 to 2, as has been ſhewn in the foregoing, or 9th, article.

Art. 11. I will juſt add one more example of the foregoing method of finding three whole numbers that ſhall expreſs the lengths of the ſides of a right-angled triangle.

Let *e* be = 5, and *n* be = 17. And we ſhall have *ee* = 25, and *nn* = 289, and *nn* — *ee* (= 289 — 25) = 264, and 2*en* (= 2 × 5 × 17 = 10 × 17) = 170, and *nn* + *ee* (= 289 + 25) = 314. Therefore 264, 170, and 314, or (dividing all theſe numbers by 2,) 132, 85, and 157, will be three whole numbers that will expreſs the three ſides of a right-angled triangle.

And accordingly we ſhall find that the ſquare of the number 157, which repreſents the hypotenuſe, or line ſubtending the right angle, will be equal to the ſum of the ſquares of the two numbers 132 and 85, which repreſent the ſides that contain the right angle. For the ſquare of 157 is 24,649, and the ſquares of 132 and 85 are 17,424 and 7225; and 24,649 is = 17,424 + 7225.

We have therefore now found the nine following ſets of whole numbers for expreſſing the three ſides of different right-angled triangles; to wit,

1ſt, The whole numbers 3, 4, and 5;
2dly, The whole numbers 5, 12, and 13;
3dly, The whole numbers 8, 15, and 17;
4thly, The whole numbers 20, 21, and 29;
5thly, The whole numbers 56, 33, and 65;
6thly, The whole numbers 72, 65, and 97;
7thly, The whole numbers 207,151, 207,000, and 292,849;
8thly, The whole numbers 21, 20, and 29; and
9thly, The whole numbers 132, 85, and 157.

And we may eaſily find as many more ſets of ſuch numbers as we pleaſe, by ſubſtituting different numbers for e and n, or for either of them, in the three general expreſſions $nn - ee$, $2en$, and $nn + ee$.

Art. 12. The foregoing ſubject may alſo be treated in a ſomewhat different manner, by ſolving the following Problem.

A PROBLEM.

To divide a given ſquare number into two other ſquare numbers, either whole numbers, or fractions, or mixt numbers.

SOLUTION.

SOLUTION.

Let the given ſquare number that is to be ſo divided, be denoted by the letters aa, and let xx be one of the two ſquare numbers that are ſought, and yy be the other.

Then, ſince the two numbers ſought are together to be equal to the given number, we ſhall, in the firſt place, have $xx + yy = aa$.

Now, ſince $xx + yy$ is $= aa$, it follows that yy will be $= aa - xx$. But $aa - xx$ is $= \overline{a + x} \times \overline{a - x}$. Therefore yy will be $= \overline{a + x} \times \overline{a - x}$. Therefore y will be a geometrical mean proportional between $a + x$ and $a - x$, and conſequently will be leſs than $a + x$. Let the proportion of $a + x$ to y be that of the two numbers m and n, of which m is the greater. Then will y be $= \frac{n}{m} \times \overline{a + x}$, and yy will be $= \frac{nn}{mm} \times \overline{a + x}|^2$ ($= \frac{nn}{mm} \times \overline{aa + 2ax + xx}$) $= \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx$. Therefore $xx + yy$ will be $= xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx$ ($= \frac{mm}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx$) $= \frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax$. But $xx + yy$ is $= aa$. Therefore $\frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax$ will alſo be $= aa$, and conſequently $= \frac{mm}{mm} \times aa$. Therefore $\frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times 2ax$ will be $= \frac{mm}{mm} \times aa - \frac{nn}{mm} \times aa = \frac{mm - nn}{mm} \times aa$, and conſequently (multiplying all the terms by mm,) $\overline{mm+nn}| \times xx + nn \times 2ax$ will be $= \overline{mm-nn}| \times aa$, and (dividing all the terms by $mm+nn$,) $xx + \frac{nn}{mm+nn} \times 2ax$ will be $= \frac{mm-nn}{mm+nn} \times aa$. Therefore (adding $\frac{n^4}{\overline{mm+nn}|^2} \times aa$ to both ſides of the equation,) we ſhall have $xx + \frac{nn}{mm+nn} \times 2ax + \frac{n^4}{\overline{mm+nn}|^2} \times aa = \frac{mm-nn}{mm+nn} \times aa + \frac{n^4}{\overline{mm+nn}|^2} \times aa = \frac{\overline{mm-nn} \times \overline{mm+nn}}{\overline{mm+nn} \times \overline{mm+nn}} \times aa + \frac{n^4}{\overline{mm+nn}|^2} \times aa = \frac{m^4-n^4}{\overline{mm+nn}|^2} \times aa + \frac{n^4}{\overline{mm+nn}|^2} \times aa = \frac{m^4}{\overline{mm+nn}|^2} \times aa$. Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $x + \frac{nn}{mm+nn} \times a = \frac{mm}{mm+nn} \times a$, and conſequently $x = \frac{mm}{mm+nn} \times a - \frac{nn}{mm+nn} \times a = \frac{mm-nn}{mm+nn} \times a$. Therefore $a + x$ will be $= a + \frac{mm-nn}{mm+nn} \times a \left(= \frac{mm+nn}{mm+nn} \times a + \frac{mm-nn}{mm+nn} \times a\right) = \frac{2mm}{mm+nn} \times a$; and y, or $\frac{n}{m} \times a + x$, will be $= \frac{n}{m} \times \frac{2mm}{mm+nn} \times a = \frac{2mn}{mm+nn} \times a$, that is, x, or the root, or ſide, of the firſt of the two ſquares ſought, to wit, xx and yy, will be equal to $\frac{mm-nn}{mm+nn} \times a$, and y, or the root, or ſide, of the latter of the ſaid two ſquares, will be equal to $\frac{2mn}{mm+nn} \times a$; and conſequently xx, or the firſt of the ſaid two ſquares itſelf, will be equal to $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$, and yy, or the latter of the ſaid two ſquares itſelf, will be equal to $\frac{\overline{2mn}|^2}{\overline{mm+nn}|^2} \times aa$, or $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$; or, if m and n be any two numbers whatſoever, of which m is the greater, $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$ and $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$ will be two ſquare numbers that will together be equal to the original ſquare number aa. Q. E. I.

Art. 13. That theſe two ſquare numbers $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$ and $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$, will together be equal to the original ſquare number aa, will be evident by adding them together. For $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa + \frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$ are $= \frac{m^4 - 2m^2n^2 + n^4}{m^4 + 2m^2n^2 + n^4} \times aa + \frac{4m^2n^2}{m^4 + 2m^2n^2 + n^4} \times aa = \frac{m^4 + 2m^2n^2 + n^4}{m^4 + 2m^2n^2 + n^4} \times aa = aa$.

Q. E. D.

This ſolution of the foregoing Problem ſeems to me more eaſy and natural than that of Dr. Saunderſon in his Algebra, Vol. 2, page 366, *et ſequentibus*.

Examples of the foregoing Solution.

Art. 14. Let the given ſquare number aa be 25, and let n be 1, and m be 2.

Then will nn be $= 1$, and mm be $= 4$, and $mm - nn$ will be $(= 4 - 1) = 3$, and $mm + nn$ will be $(= 4 + 1) = 5$, and conſequently $\overline{mm - nn}|^2$ will be $= 9$, and $\overline{mm + nn}|^2$ will be $= 25$, and $4m^2n^2$ will be $(= 4 \times 4 \times 1) = 16$,

$= 16$, and $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{9}{25} \times 25 = 9$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{16}{25} \times 25 = 16$. Therefore 9 and 16 will be two ſquare numbers that will together be equal to the given ſquare number 25. And it is evident that 9 and 16 are equal to 25.

Secondly, let aa be $= 25$, and $n = 1$, and $m = 3$.

Then will nn be $= 1$, and mm be $= 9$, and conſequently $mm - nn$ will be $(= 9 - 1) = 8$, and $mm + nn$ will be $= 9 + 1 = 10$, and $4m^2n^2$ will be $(= 4 \times 9 \times 1) = 36$. Therefore $\overline{mm - nn}|^2$ will be $= 64$, and $\overline{mm + nn}|^2$ will be $= 100$, and $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{64}{100} \times aa = \frac{64}{100} \times 25 = \frac{16}{25} \times 25 = 16$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{36}{100} \times aa = \frac{9}{25} \times aa = \frac{9}{25} \times 25 = 9$. Therefore 16 and 9 will be two ſquare numbers that will together be equal to the given ſquare number 25. And it is evident that they are equal to 25.

Theſe two ſquare numbers 16 and 9 are the ſame with the two former ſquare numbers 9 and 16, derived from the ſuppoſition that m was $= 3$, except in the order of their poſition.

Thirdly, let aa be $= 25$, and n be $= 1$, and m be $= 4$.

Then we ſhall have $nn = 1$, and $mm = 16$, and conſequently $mm - nn$ $(= 16 - 1) = 15$, and $mm + nn$ $(= 16 + 1) = 17$, and $\overline{mm - nn}|^2$ $(= \overline{15}|^2) = 225$, and $\overline{mm + mm}|^2$ $(= \overline{17}|^2) = 289$, and $4m^2n^2$ $(= 4 \times 16 \times 1)$ $= 64$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{225}{289} \times aa)$ $=$

$= \frac{225}{289} \times 25$, and $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$ will be $(= \frac{64}{289} \times aa)$ $= \frac{64}{289} \times 25$. Therefore $\frac{225}{289} \times 25$ and $\frac{64}{289} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And it is evident that theſe two ſquare numbers are equal to 25: for $\frac{225}{289} \times 25 + \frac{64}{289} \times 25$ are $= \frac{225+64}{289} \times 25 = \frac{289}{289} \times 25 = 25$.

If we multiply theſe three numbers $\frac{225}{289} \times 25$, $\frac{64}{289} \times 25$, and 25, by 289, we ſhall thereby obtain the three following whole numbers, to wit, 225×25, 64×25, and 289×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three laſt numbers by 25, we ſhall obtain the three following leſſer whole numbers, to wit, 225, 64, and 289, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter.

Fourthly, let aa be $= 25$, and n be $= 1$, and m be $= 5$.

Then we ſhall have $nn = 1$, and $mm = 25$, and conſequently $mm - nn = 24$, and $mm + nn = 26$, and $\overline{mm-nn}|^2$ $(= \overline{24}|^2) = 576$, and $\overline{mm+nn}|^2$ $(= \overline{26}|^2) = 676$, and $4m^2n^2$ $(= 4 \times 25 \times 1) = 100$. Therefore $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$ will be $(= \frac{576}{676} \times aa) = \frac{576}{676} \times 25$, and $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$ will be $(= \frac{100}{676} \times aa) = \frac{100}{676} \times 25$. Therefore $\frac{576}{676} \times 25$ and $\frac{100}{676} \times 25$ will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accordingly it is evident that theſe two ſquare numbers are equal to 25: for $\frac{576}{676} \times 25 + \frac{100}{676} \times 25$ are $= \frac{576 + 100}{676} \times 25 = \frac{676}{676} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{576}{676} \times 25$, $\frac{100}{676} \times 25$, and 25, by 676, we ſhall thereby obtain the three following whole numbers, to wit, 576×25, 100×25, and 676×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three laſt numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 576, 100, and 676, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe laſt numbers, 576, 100, and 676, by 4, we ſhall thereby obtain the three following ſtill leſſer whole numbers, to wit, 144, 25, and 169, which will likewiſe be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter.

Art. 15. In the 5th place, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 3$.

Then we ſhall have $nn = 4$, and $mm = 9$, and conſequently $mm - nn$ $(= 9 - 4) = 5$, and $mm + nn$ $(= 9 + 4) = 13$, and $\overline{mm - nn}|^2 = 25$, and $\overline{mm + nn}|^2 = 169$, and $4m^2n^2$ $(= 4 \times 9 \times 4) = 144$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{25}{169} \times aa) = \frac{25}{169} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{144}{169} \times aa) = \frac{144}{169} \times 25$. Therefore $\frac{25}{169} \times 25$ and $\frac{144}{169} \times 25$ will be two ſquare numbers that will, together, be equal to 25. And accordingly it is evident that theſe two ſquare numbers $\frac{25}{169} \times 25$ and $\frac{144}{169} \times 25$

$\times 25$ are equal to 25. For they are equal to $\frac{25+144}{169} \times 25$ $= \frac{169}{169} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{25}{169} \times 25$, $\frac{144}{169} \times 25$, and 25, by 169, we ſhall thereby obtain the three following whole numbers, to wit, 25×25, 144×25, and 169×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three laſt numbers 25×25, 144×25, and 169×25, by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 25, 144, and 169, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter.

Sixthly, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 5$.

Then we ſhall have $nn = 4$, and $mm = 25$, and conſequently $mm - nn$ $(= 25 - 4) = 21$, and $mm + nn$ $(= 25 + 4) = 29$, and $\overline{mm - nn}|^2$ $(= \overline{21}|^2) = 441$, and $\overline{mm + nn}|^2$ $(= \overline{29}|^2) = 841$, and $4m^2n^2$ $(= 4 \times 25 \times 4)$ $= 400$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{441}{841} \times aa)$ $= \frac{441}{841} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{400}{841} \times 25$. Therefore $\frac{441}{841} \times 25$ and $\frac{400}{841} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And ſo we ſhall find them to be; for $\frac{441}{841} \times 25 + \frac{400}{841} \times 25$ are $= \frac{441 + 400}{841} \times 25 = \frac{841}{841} \times 25$ $= 25$.

If we multiply theſe three numbers, $\frac{441}{841} \times 25$, $\frac{400}{841} \times 25$, and 25, by 841; we ſhall thereby obtain the three following whole numbers, to wit, 441×25, 400×25, and 841×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three whole numbers by 25, we ſhall obtain the three following leſſer whole numbers, to wit, 441, 400, and 841, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter.

Seventhly, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 7$.

Then we ſhall have $nn = 4$, and $mm = 49$, and conſequently $mm - nn$ $(= 49 - 4) = 45$, and $mm + nn$ $(= 49 + 4) = 53$, and $\overline{mm - nn}|^2$ $(= \overline{45}|^2) = 2025$, and $\overline{mm + nn}|^2$ $(= \overline{53}|^2) = 2809$, and $4m^2n^2$ $(= 4 \times 49 \times 4) = 784$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{2025}{2809} \times aa)$ $= \frac{2025}{2809} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{784}{2809} \times 25$. Therefore $\frac{2025}{2809} \times 25$ and $\frac{784}{2809} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accordingly it is evident that theſe two ſquare numbers $\frac{2025}{2809} \times 25$, and $\frac{784}{2809} \times 25$, are equal to 25. For they are equal to $\frac{2025 + 784}{2809} \times 25 = \frac{2809}{2809} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{2025}{2809} \times 25$, $\frac{784}{2809} \times 25$, and 25 by 2809, we ſhall thereby obtain the three following whole numbers, to wit, 2025×25, 784×25, and 2809×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three whole numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 2025, 784, and 2809, which will, all of them, be ſquare numbers, and of which the two former will, together, be equal to the latter.

Art. 16. In the 8th place, let aa be, as before, equal to 25, and let n be $= 3$, and m be $= 5$.

Then we ſhall have $nn = 9$, and $mm = 25$, and conſequently $mm - nn$ $(= 25 - 9) = 16$, and $mm + nn$ $(= 25 + 9) = 34$, and $\overline{mm - nn}|^2$ $(= \overline{16}|^2) = 256$, and $\overline{mm + nn}|^2$ $(= \overline{34}|^2) = 1156$, and $4m^2n^2$ $(= 4 \times 25 \times 9 = 100 \times 9) = 900$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{256}{1156} \times aa) = \frac{256}{1156} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{900}{1156} \times aa) = \frac{900}{1156} \times 25$. Therefore $\frac{256}{1156} \times 25$, and $\frac{900}{1156} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accordingly it is evident that theſe two ſquare numbers $\frac{256}{1156} \times 25$, and $\frac{900}{1156} \times 25$, are equal to 25. For they are $= \frac{256 + 900}{1156} \times 25 = \frac{1156}{1156} \times 25 = 25$.

If we multiply theſe three ſquare numbers, $\frac{256}{1156} \times 25$, $\frac{900}{1156} \times 25$, and 25, by 1156, we ſhall thereby obtain the three following whole numbers, to wit, 256×25, 900×25, and 1156×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three whole numbers by 25, we ſhall thereby obtain the three following leſſer,

leſſer whole numbers, to wit, 256, 900, and 1156, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter.

Ninthly, let aa be, as before, $= 25$, and let n be $= 3$, and m be $= 7$.

Then we ſhall have $nn = 9$, and $mm = 49$, and conſequently $mm - nn$ $(= 49 - 9)$ $= 40$, and $mm + nn$ $(= 49 + 9)$ $= 58$, and $\overline{mm - nn}|^2$ $(= 40^2)$ $= 1600$, and $\overline{mm + nn}|^2$ $(= 58^2)$ $= 3364$, and $4m^2n^2$ $(= 4 \times 49 \times 9)$ $= 1764$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{1600}{3364} \times aa)$ $= \frac{1600}{3364} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{1764}{3364} \times aa)$ $= \frac{1764}{3364} \times 25$. Therefore $\frac{1600}{3364} \times 25$, and $\frac{1764}{3364} \times 25$, will be two ſquare numbers that, together, will be equal to the given ſquare number 25. And accordingly it is evident that theſe two numbers are equal to 25. For $\frac{1600}{3364} \times 25 + \frac{1764}{3364} \times 25$, are $= \frac{1600 + 1764}{3364} \times 25 = \frac{3364}{3364} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{1600}{3364} \times 25$, $\frac{1764}{3364} \times 25$, and 25, by 3364, we ſhall thereby obtain the three whole numbers 1600×25, 1764×25, and 3364×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three whole numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 1600, 1764, and 3364, which are, all of them, ſquare numbers, and of which the two former are equal to the latter.

Tenthly, let aa be, as before, $= 25$, and let n be $= 3$, and m be $= 11$.

Then we ſhall have $nn = 9$, and $mm = 121$, and conſequently $mm - nn$ ($= 121 - 9$) $= 112$, and $mm + nn$ ($= 121 + 9$) $= 130$, and $\overline{mm - nn}|^2$ ($= \overline{112}|^2$) $= 12{,}544$, and $\overline{mm + nn}|^2$ ($= \overline{130}|^2$) $= 16{,}900$, and $4m^2n^2$ ($= 4 \times 121 \times 9 = 484 \times 9$) $= 4356$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be ($= \frac{12{,}544}{16{,}900} \times aa$) $= \frac{12{,}544}{16{,}900} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be ($= \frac{4356}{16{,}900} \times aa$) $= \frac{4356}{16{,}900} \times 25$. Therefore $\frac{12{,}544}{16{,}900} \times 25$, and $\frac{4356}{16{,}900} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accordingly it will eaſily appear that theſe two numbers are equal to 25. For $\frac{12{,}544}{16{,}900} \times 25 + \frac{4356}{16{,}900} \times 25$ are $= \frac{12{,}544 + 4356}{16{,}900} \times 25 = \frac{16{,}900}{16{,}900} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{12{,}544}{16{,}900} \times 25$, $\frac{4356}{16{,}900} \times 25$, and 25, by 16,900, we ſhall thereby obtain the three following whole numbers, to wit, $12{,}544 \times 25$, 4356×25, and $16{,}900 \times 25$, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter. And, if we divide theſe whole numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 12,544, 4356, and 16,900, which are, all of them, ſquare numbers, (to wit, the ſquares of the numbers 112, 66, and 130,) and of which the two former are, together, equal to the latter.

Art. 17. In the 11th place, let aa be equal, as before, to 25, and let n be $= 5$, and m be $= 7$.

Then we ſhall have $nn = 25$, and $mm = 49$, and conſequently $mm - nn$ $(= 49 - 25) = 24$, and $mm + nn$ $(= 49 + 25) = 74$, and $\overline{mm - nn}|^2$ $(= \overline{24}|^2) = 576$, and $\overline{mm + nn}|^2$ $(= \overline{74}|^2) = 5476$, and $4m^2n^2$ $(= 4 \times 49 \times 25 = 100 \times 49) = 4900$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{576}{5476} \times aa) = \frac{576}{5476} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{4900}{5476} \times aa) = \frac{4900}{5476} \times 25$. Therefore $\frac{576}{5476} \times 25$, and $\frac{4900}{5476} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accordingly, if we add theſe numbers together, we ſhall find them to be $(= \frac{576 + 4900}{5476} \times 25 = \frac{5476}{5476} \times 25)$ $= 25$.

If we multiply theſe three numbers, $\frac{576}{5476} \times 25$, $\frac{4900}{5476} \times 25$, and 25, by 5476, we ſhall thereby obtain the three following whole numbers, to wit, 576×25, 4900×25, and 5476×25, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter. And, if we divide theſe whole numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 576, 4900, and 5476, which are alſo, all of them, ſquare numbers, and of which the two former are, together, equal to the latter.

Twelfthly, let aa be, as before, $= 25$, and let n be $= 5$, and m be $= 11$.

Then we ſhall have $nn = 25$, and $mm = 121$, and conſequently $mm - nn$ $(= 121 - 25) = 96$, and $mm + nn$ $(= 121 + 25) = 146$, and $\overline{mm - nn}|^2$ $(= \overline{96}|^2) = 9216$;

and $\overline{mm + nn}|^2$ ($= \overline{146}|^2$) $= 21,316$, and $4m^2n^2$ ($= 4 \times 121 \times 25 = 121 \times 100$) $= 12,100$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be ($= \frac{9216}{21,316} \times aa$) $= \frac{9216}{21,316} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be ($= \frac{12,100}{21,316} \times aa$) $= \frac{12,100}{21,316} \times 25$. Therefore $\frac{9216}{21,316} \times 25$, and $\frac{12,100}{21,316} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And ſo we ſhall eaſily find them to be. For $\frac{9216}{21,316} \times 25 + \frac{12,100}{21,316} \times 25$, are equal to $\frac{9216 + 12,100}{21,316} \times 25 = \frac{21,316}{21,316} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{9216}{21,316} \times 25$, $\frac{12,100}{21,316} \times 25$, and 25, by 21,316, we ſhall thereby obtain the three following whole numbers, to wit, 9216×25, $12,100 \times 25$, and $21,316 \times 25$, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter. And, if we divide theſe three whole numbers by 25, we ſhall thereby obtain the three leſſer whole numbers 9216, 12,100, and 21,316, which are, all of them, ſquare numbers, (to wit, the ſquares of the numbers 96, 110, and 146,) and of which the two former are, together, equal to the latter.

Art. 18. We have now obtained, by means of the three general expreſſions $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$, $\frac{4m^2n^2}{\overline{mm + nn}|^2}$, and aa, the twelve following ſets of three whole numbers each, that are, all of them, ſquare numbers, and of which the two firſt numbers in every ſet are, together, equal to the third number; to wit, 1ſt, The numbers 9, 16, and 25;

2dly, The numbers 16, 9, and 25, which differ from the three former numbers 9, 16, and 25, only in the order in which the two firſt numbers 9 and 16 are placed;

3dly, The numbers 225, 64, and 289;

4thly, The numbers 144, 25, and 169;

5thly, The numbers 25, 144, and 169, which differ from the three foregoing numbers only in the order in which the two firſt numbers 25 and 144 are placed;

6thly, The numbers 441, 400, and 841;

7thly, The numbers 2025, 784, and 2809;

8thly, The numbers 256, 900, and 1156;

9thly, The numbers 1600, 1764, and 3364;

10thly, The numbers 12,544, 4356, and 16,900;

11thly, The numbers 576, 4900, and 5476; and,

12thly, The numbers 9216, 12,100 and 21,316.

Art. 19. The ſquare-roots of the foregoing twelve ſets of numbers are as follows; to wit,

1ſt, The numbers 3, 4, and 5;
2dly, The numbers 4, 3, and 5;
3dly, The numbers 15, 8, and 17;
4thly, The numbers 12, 5, and 13;
5thly, The numbers 5, 12, and 13;
6thly, The numbers 21, 20, and 29;
7thly, The numbers 45, 28, and 53;
8thly, The numbers 16, 30, and 34;
9thly, The numbers 40, 42, and 58;
10thly, The numbers 112, 66, and 130;
11thly, The numbers 24, 70, and 74; and
12thly, The numbers 96, 110, and 146.

Art. 20. If we divide the numbers of ſome of the foregoing

going ſets of numbers by 2, (which will not alter the proportion of ſuch numbers to each other,) the ſaid twelve ſets of numbers will be as follows; to wit,

1ſt, 3, 4, and 5;
2dly, 4, 3, and 5;
3dly, 15, 8, and 17;
4thly, 12, 5, and 13;
5thly, 5, 12, and 13;
6thly, 21, 20, and 29;
7thly, 45, 28, and 53;
8thly, 8, 15, and 17;
9thly, 20, 21, and 29;
10thly, 56, 33, and 65;
11thly, 12, 35, and 37; and
12thly, 48, 55, and 73.

And all theſe twelve ſets of numbers will expreſs the lengths of the ſides of different right-angled triangles.

Art. 21. In the three foregoing general expreſſions $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$, $\frac{4m^2n^3}{\overline{mm+nn}|^2} \times aa$, and aa, or $\frac{\overline{mm+nn}|^2}{\overline{mm+nn}|^2} \times aa$, obtained in art. 12, the letter n anſwers to the letter e in the three former general expreſſions $nn-ee$, $2ne$, and $nn+ee$, obtained in art. 4; and the letter m in the three expreſſions obtained in art. 12, anſwers to the letter n in the three former expreſſions obtained in art. 4. And accordingly we find that, where the ſame two numbers have been ſubſtituted inſtead of the letters m and n in the general expreſſions $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$, $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$, and $\frac{\overline{mm+nn}|^2}{\overline{mm+nn}|^2} \times aa$, as were ſubſtituted in art. 4, inſtead of the letters n and e reſpectively, in the general expreſſions $nn-ee$, $2ne$, and $nn+ee$, they have produced the ſame three numbers to repreſent the lengths of the three ſides of a right-angled triangle. The only difference between theſe two ſets of general expreſſions is, that the three expreſſions $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$, $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$, and $\frac{\overline{mm+nn}|^2}{\overline{mm+nn}|^2} \times aa$, give us the ſquares of ſuch numbers as will expreſs the ſides of right-angled triangles, and the three expreſſions $nn - ee$, $2ne$, and $nn + ee$, give us the ſaid numbers themſelves.

Art. 22. The whole numbers that expreſs the lengths of the ſides of a right-angled triangle, cannot, when they are reduced to the loweſt numbers poſſible by dividing them by their common diviſors, be, all of them, even numbers.

For, if they were all even numbers, they might all be divided by 2, either once, or more than once, till at laſt ſome of the quotients would be odd numbers. Thus, for example, the three even numbers 16, 30, and 34, which have been found above to expreſs the three ſides of a right-angled triangle, are all diviſible by 2, and are by ſuch diviſion reduced to the three leſſer numbers 8, 15, and 17, of which the two latter are odd numbers.

Art. 23. And further, the ſaid numbers that expreſs the lengths of the ſides of a right-angled triangle, cannot be, all of them, odd numbers.

For, if the two numbers expreſſing the lengths of the two ſides of the triangle that contain the right angle, were, both of them, odd numbers, their ſquares would alſo be odd numbers; becauſe the ſquare of every odd number is an odd number taken an odd number of times, and conſequently muſt be an odd number: and conſequently the ſum of the ſaid two ſquares muſt be an even number, becauſe two odd numbers added together always make an even number. Therefore the ſquare of the number repreſenting the hypotenuſe of the triangle, being equal to the ſaid ſum of the two other ſquares, muſt be an even number.

And conſequently the ſquare-root of the ſaid number, that is, the number repreſenting the hypotenuſe of the triangle, muſt be an even number likewiſe, which is contrary to the ſuppoſition. Therefore it is impoſſible that all the three numbers which repreſent the lengths of the ſides of a right-angled triangle, ſhould be odd numbers.

Art. 24. There is alſo another way of finding ſeveral whole numbers that ſhall repreſent the lengths of the ſides of different right-angled triangles; which conſiſts in forming a liſt, or table, of the ſquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, ſet down in their proper order; and a liſt of the differences of the ſaid ſquares, (which, it is well known, are the ſeveral odd numbers 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, &c, taken in their natural order,) and adding together the ſaid differences that follow any given ſquare number in the liſt of ſquares till their ſum amounts to another ſquare number. Such a table, carried as far as the ſquare of 100, will be as follows.

A TABLE of the Squares of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, *&c, as far as* 100, *and of their ſeveral Differences from each other, and likewiſe of the Differences of thoſe Differences, or of the Second Differences of the ſaid Squares.*

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the ſaid Squares.	Their 2d Differences, or the Differences of their Differences.
1	1	3	2
2	4	5	2
3	9	7	2
4	16	9	2
5	25	11	2
6	36	13	2
7	49	15	2
8	64	17	2
9	81	19	2
10	100	21	2
11	121	23	2
12	144	25	2
13	169	27	2
14	196	29	2
15	225	31	2
16	256	33	2
17	289	35	2
18	324	37	2
19	361	39	2
20	400	41	2
21	441	43	2
22	484	45	2
23	529	47	2
24	576	49	2
25	625	51	2
26	676	53	2
27	729	55	2
28	784	57	2

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the ſaid Squares.	Their 2d Differences, or the Differences of their Differences.
29	841	59	2
30	900	61	2
31	961	63	2
32	1024	65	2
33	1089	67	2
34	1156	69	2
*35	1225	71	2
36	1296	73	2
37	1369	75	2
38	1444	77	2
39	1521	79	2
40	1600	81	2
41	1681	83	2
42	1764	85	2
43	1841	87	2
44	1936	89	2
45	2025	91	2
46	2116	93	2
47	2209	95	2
48	2304	97	2
49	2401	99	2
50	2500	101	2
51	2601	103	2
52	2704	105	2
53	2809	107	2
54	2916	109	2
55	3025	111	2
56	3136	113	2
57	3249	115	2
58	3364	117	2
59	3481	119	2
60	3600	121	2
61	3721	123	2
62	3844	125	2
63	3969	127	2
64	4096	129	2

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the ſaid Squares.	Their 2d Differences, or the Differences of their Differences.
65	4225	131	
66	4356	133	2
67	4489	135	2
68	4624	137	2
69	4761	139	2
70	4900	141	2
71	5041	143	2
72	5184	145	2
73	5329	147	2
74	5476	149	2
75	5625	151	2
76	5776	153	2
77	5929	155	2
78	6084	157	2
79	6241	159	2
80	6400	161	2
81	6561	163	2
82	6724	165	2
83	6889	167	2
84	7056	169	2
85	7225	171	2
86	7396	173	2
87	7569	175	2
88	7744	177	2
89	7921	179	2
90	8100	181	2
91	8281	183	2
92	8464	185	2
93	8649	187	2
94	8836	189	2
95	9025	191	2
96	9216	193	2
97	9409	195	2
98	9604	197	2
99	9801	199	2
100	10,000		

Art. 25. In the foregoing table the firſt column contains all the natural numbers 1, 2, 3, 4, 5, &c, as far as 100; and the ſecond column contains the ſquares of the ſaid numbers ſet down even with the ſaid numbers themſelves, or ſo that every ſquare number ſhall be in the ſame horizontal line with the natural number of which it is the ſquare; and the third column contains the differences of the ſquare numbers in the ſecond column, ſet down in lines between the lines in which the ſquares themſelves, of which they are the differences, are ſet down; and the fourth column contains the differences of the foregoing differences that are ſet down in the third column, or the ſecond differences of the ſquare numbers that are ſet down in the ſecond column; and each of theſe ſecond differences is ſet down in a line that lies between the two lines in which the two firſt differences, of which it is the difference, are ſet down. And we may obſerve, that the differences ſet down in the third column are the odd numbers 3, 5, 7, 9, 11, 13, 15, &c, taken in their natural order; and the ſecond differences ſet down in the fourth column, being the differences of the ſaid odd numbers, are all equal to each other, and to the number 2.

Art. 26. From the conſtruction of this table, it is obvious that, if we take any number whatſoever in the firſt column, and look out its ſquare in the ſecond column, and then add together the ſeveral differences in the third column that follow the ſaid ſquare number in the ſecond column, till the ſum of the ſaid differences ſhall amount to a ſquare number, the ſquare-root of the ſaid ſquare number, or the number in the firſt column that is placed even with it, and the number firſt taken, will expreſs the lengths of the two ſides of a right-angled triangle that contain the right angle, and the number in the firſt column that immediately follows the laſt of the ſaid differences in the third column, ſo added together, will expreſs the length of the hypotenuſe of the ſame triangle. Thus, for example, if we take the number 3 in the firſt column, and find its ſquare, to wit, 9, in the ſecond column, and then add up the differences 7, 9, 11, 13, &c, in the third column, which come after the ſquare number 9, till their ſum becomes equal to a ſquare number, (for which purpoſe we need only add together the two differences 7 and 9, becauſe their ſum is 16, which is a ſquare number,) the ſquare root of the ſaid ſquare number 16, or the number in the firſt column which is placed even with it, to wit, the number 4, and the number 3, which was taken at firſt, will expreſs the lengths of the two ſides of a right-angled triangle that contain the right angle, and the number 5 in the firſt column, which immediately follows the laſt of the ſaid two differences, ſo added together, to wit, 9, will expreſs the length of the hypotenuſe of the ſame triangle. So that we we ſhall hereby obtain the whole numbers 3, 4, and 5; to expreſs the lengths of the three ſides of a right-angled triangle: which numbers we had before obtained by both the former methods of inveſtigation.

Art. 27. If we look in the 3d column, or column of differences, for thoſe differences that are themſelves ſquare numbers, without being added to any of the foregoing, or following, differences to make them ſo, (which differences are but few in number, being only the following ſix numbers, to wit, 9, 25, 49, 81, 121, and 169, in the whole table,) we may at once obtain from each of theſe differences a ſet of numbers that will expreſs the lengths of the three ſides of a right-angled triangle.

For, ſince 9 is a ſquare number, and is likewiſe the difference between the two contiguous ſquare numbers 16 and 25, and conſequently 9 + 16 are = 25, it follows that the ſquare-roots of theſe three numbers 9, 16, and 25, that is, the three numbers 3, 4, and 5, will expreſs the three ſides of a right-angled triangle.

And, in like manner, ſince 25 is a ſquare number, and is likewiſe the difference between the two contiguous ſquare numbers 144 and 169, and conſequently 25 + 144 are = 169, it follows that the ſquare-roots of the three numbers

25, 144, and 169, that is, the three numbers 5, 12, and 13, will expreſs the three ſides of a right-angled triangle.

And, ſince 49 is a ſquare number, and is likewiſe the difference of the two contiguous ſquare numbers 576 and 625, and conſequently 49 + 576 are equal to 625, it follows that the ſquare-roots of the three numbers 49, 576, and 625, that is, the three numbers 7, 24, and 25, will expreſs the three ſides of a right-angled triangle.

And, ſince 81 is a ſquare number, and is likewiſe the difference of the two contiguous ſquare numbers 1600 and 1681, and conſequently 81 + 1600 are = 1681, it follows that the ſquare-roots of the three numbers 81, 1600, and 1681, that is, the three numbers 9, 40, and 41, will expreſs the three ſides of a right-angled triangle.

And, ſince 121 is a ſquare number, and is likewiſe the difference of the two contiguous ſquare numbers 3600 and 3721, and conſequently 121 + 3600 are = 3721, it follows that the ſquare-roots of the three numbers 121, 3600, and 3721, that is, the three numbers 11, 60, and 61, will expreſs the three ſides of a right-angled triangle.

And, laſtly, ſince 169 is a ſquare number, and likewiſe is the difference of the two contiguous ſquare numbers 7056 and 7225, and conſequently 169 + 7056 are = 7225, it follows that the ſquare-roots of the three numbers 169, 7056, and 7225, that is, the three numbers 13, 84, and 85, will expreſs the three ſides of a right-angled triangle.

Art. 28. In this way of obtaining three numbers that ſhall expreſs the three ſides of a right-angled triangle, namely, by chuſing ſuch numbers in the 3d column, or column of differences, as are themſelves ſquare numbers, it is evident that the number expreſſing the hypotenuſe of the triangle will always exceed the greater of the other two numbers, that expreſs its ſides, by an unit. But, when we take ſeveral ſucceſſive differences, of which the ſum is equal to a ſquare number, the number that expreſſes the hypotenuſe of the triangle, will exceed the number taken at firſt, and between the ſquare of which, and the ſquare of the number repreſenting the hypotenuſe, the ſeveral differences that are added together lie, by as many units as there are differences that have been ſo added together in order to make a ſquare number. Of this it will not be amiſs to give a few examples.

Art. 29. Let us take 8 for the firſt number. Then, ſince the ſquare of 8 is 64, we muſt begin with the difference 17, which comes immediately after 64, and we muſt add together this difference 17, and the following differences 19, 21, 23, 25, 27, &c, till their ſum amounts to a ſquare number. For this purpoſe we need add together only two of theſe differences, to wit, 17 and 19. For 17 + 19 are = 36, which is a ſquare number, to wit, the ſquare of 6. Therefore the firſt number 8, and the number 6, (or the ſquare-root of the ſum of theſe two differences), and the number 10, (or the ſquare-root of the ſquare number 100, which comes immediately after the laſt difference 19), will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{8}|^2 + \overline{6}|^2$ will be $= \overline{10}|^2$, or 64 + 36 will be = 100. And the number 10, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 8, (which repreſents the greater of its two ſides,) by 2, or two units, or the ſame number of units as there were differences added together, in order to produce the ſquare number 36.

And, if, inſtead of taking only two of the differences 17, 19, 21, 23, 25, 27, 29, &c, we take nine of them, we ſhall find their ſum to be equal to another ſquare number, to wit, 225, which is the ſquare of 15. For 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 + 33 are = 225. Therefore the firſt number 8, and the number 15, (or the ſquare-root of the ſum of theſe nine differences,) and the number 17, (or the ſquare-root of the ſquare number 289, which comes immediately after the laſt difference 33,) will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{8}|^2 + \overline{15}|^2$ will be $= \overline{17}|^2$, or 64 +

225 will be = 289. And the number 17, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 8, (which repreſents one of the ſides that incloſe the right angle,) by 9, or nine units, or the ſame number of units as there were differences added together, in order to produce the ſquare number 225.

Art. 30. Now let the firſt number be 20. Then, ſince the ſquare of 20 is 400, we muſt begin with the difference 41, which comes immediately after 400, and muſt add together this difference 41, and the following differences 43, 45, 47, 49, 51, &c, till their ſum amounts to a ſquare number. And for this purpoſe it will be neceſſary to add together nine of theſe differences. For 41 + 43 + 45 + 47 + 49 + 51 + 53 + 55 + 57 are = 441, which is the ſquare of 21. Therefore the firſt number 20, and the number 21, (or the ſquare-root of the ſum of theſe nine differences,) and the number 29, (or the ſquare-root of the ſquare number 841, which comes immediately after the laſt difference 57,) will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{20}|^2 + \overline{21}|^2$ will be = $\overline{29}|^2$, or 400 + 441 will be = 841. And the number 29, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 20, (which repreſents one of the ſides that incloſe the right-angle,) by 9, or nine units, or the ſame number of units as there were differences added together in order to produce the ſquare number 441.

Art. 31. Now let the firſt number be 28. Then, ſince the ſquare of 28 is 784, we muſt begin with the difference 57, which comes immediately after 784, and we muſt add together this difference 57, and the following differences 59, 61, 63, 65, &c, till their ſum amounts to a ſquare number. And for this purpoſe it will be neceſſary to add together ſeven of theſe differences. For 57 + 59 + 61 + 63 + 65 + 67 + 69 are = 441, which is the ſquare of 21. Therefore the firſt number 28, and the number 21, (or the ſquare-root of the ſum of theſe ſeven differences,) and the number 35, (or the ſquare-root of the ſquare number 1225, which comes immediately after the laſt difference 69,) will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{28|}^{2} + \overline{21|}^{2}$ will be $= \overline{35|}^{2}$, or $784 + 441$ will be $= 1225$. And the number 35, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 28, (which repreſents one of the ſides that incloſe the right angle,) by 7, or ſeven units, or the ſame number of units as there were differences added together, in order to produce the ſquare number 441.

Theſe examples, I apprehend, are ſufficient to explain this method of obtaining different ſets of whole numbers that ſhall expreſs the lengths of the ſides of different right-angled triangles. And with them I ſhall conclude this little tract.

End of the Diſcourſe concerning the Methods of finding Rational Numbers that expreſs the Sides of Right-angled Triangles.

OF THE

DIFFERENCES

OF THE

CUBES

OF THE

NATURAL NUMBERS 1, 2, 3, 4, 5, 6, 7, &c.

Article 1. We have ſeen in the table of the ſquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, ſet down in the foregoing Tract, that the firſt differences of the ſquares of thoſe numbers are the odd numbers 3, 5, 7, 9, 11, 13, 15, 17, &c, in their natural order, and their ſecond differences, or the differences of their firſt differences, are all equal to each other, and to the number 2. We will now ſet down a table of the cubes of the ſeveral natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, and of their differences, and the differences of thoſe firſt differences, and the differences of thoſe ſecond differences; by which it will appear that the cubes of the ſaid numbers have three orders of differences, and that their differences of the third order are all equal to each other and to the number 6. This Table will be as follows:

A TABLE of the Cubes of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, *&c, as far as* 100; *together with their First, Second, and Third, Differences.*

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Differences of the ſaid Cubes.
1	1	7	12	6
2	8	19	28	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	6
9	729	271	60	6
10	1000	331	66	6
11	1331	397	72	6
12	1728	469	78	6
13	2197	547	84	6
14	2744	631	90	6
15	3375	721	96	6
16	4096	817	102	6
17	4913	919	108	6
18	5832	1027	114	6
19	6859	1141	120	6
20	8000	1261	126	6
21	9261	1387	132	6
22	10,648	1519	138	6
23	12,167	1657	144	6
24	13,824	1801	150	6
25	15,625	1951	156	6
26	17,576	2107	162	6
27	19,683	2269	168	6
28	21,952	2437	174	6
29	24,389	2611	180	

The

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Differences of the ſaid Cubes.
30	27,000	2791	186	6
31	29,791	2977	192	6
32	32,768	3169	198	6
33	35,937	3367	204	6
34	39,304	3571	210	6
35	42,875	3781	216	6
36	46,656	3997	222	6
37	50,653	4219	228	6
38	54,872	4447	234	6
39	59,319	4681	240	6
40	64,000	4921	246	6
41	68,921	5167	252	6
42	74,088	5419	258	6
43	79,507	5677	264	6
44	85,184	5941	270	6
45	91,125	6211	276	6
46	97,336	6487	282	6
47	103,823	6769	288	6
48	110,592	7057	294	6
49	117,649	7351	300	6
50	125,000	7651	306	6
51	132,651	7957	312	6
52	140,608	8269	318	6
53	148,877	8587	324	6
54	157,464	8911	330	6
55	166,375	9241	336	6
56	175,616	9577	342	6
57	185,193	9919	348	6
58	195,112	10267	354	6
59	205,379	10621	360	6
60	216,000	10981	366	6
61	226,981	11347	372	6
62	238,328	11719	378	6
63	250,047	12097	384	6
64	262,144	12481	390	

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Differences of the ſaid Cubes.
65	274,625	12871		
66	287,496	13267	396	6
67	300,763	13669	402	6
68	314,432	14077	408	6
69	328,509	14491	414	6
70	343,000	14911	420	6
71	357,911	15337	426	6
72	373,248	15769	432	6
73	389,017	16207	438	6
74	405,224	16651	444	6
75	421,875	17101	450	6
76	438,976	17557	456	6
77	456,533	18019	462	6
78	474,552	18487	468	6
79	493,039	18961	474	6
80	512,000	19441	480	6
81	531,441	19927	486	6
82	551,368	20419	492	6
83	571,787	20917	498	6
84	592,704	21421	504	6
85	614,125	21931	510	6
86	636,056	22447	516	9
87	658,503	22969	522	6
88	681,472	23497	528	6
89	704,969	24031	534	6
90	729,000	24571	540	6
91	753,571	25117	546	6
92	778,688	25669	552	6
93	804,357	26227	558	6
94	830,584	26791	564	6
95	857,375	27361	570	6
96	884,736	27937	576	6
97	912,673	28519	582	6
98	941,192	29107	588	6
99	970,299	29701	594	
100	1,000,000			

Art. 2. And in like manner it will be found that the fourth powers of the natural numbers 1, 2, 3, 4, 5, &c, will have four orders of differences, and that their fifth powers will have five orders of differences; and, in general, that their *n*th powers, *n* being any whole number whatſoever, will have *n* orders of differences. This is a curious property of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, and has been long known to Mathematicians. The celebrated Mr. Leibnitz, of Hanover, had taken notice of it before the month of February, 1673; and it had been obſerved before him by a French Mathematician, named *Mouton*, (who was a Canon of the Church of Lyons,) in a book on the apparent diameters of the Sun and Moon; but which Mr. Leibnitz declared he had not ſeen at the time he made the ſame diſcovery. Mr. Leibnitz's manner of conſidering the ſubject is explained pretty fully in a large extract from a letter of his to Mr. Henry Oldenburgh, the Secretary of the Royal Society of London, dated at London on the 3d of February, 167$\frac{2}{3}$, which has been publiſhed in the *Commercium Epiſtolicum* of Mr. John Collins and other Mathematicians of that time. This extract, as it contains ſeveral intereſting particulars relating to theſe numbers, I ſhall here inſert at length from the ſaid *Commercium Epiſtolicum*, pages 108, 109, 110, - - - - 114. It is as follows.

Art. 3. *Excerpta ex Epiſtolâ Domini* Gothofredi Gulielmi Leibnitzii *ad Dominum* Oldenburgh, *Londini, Anno* 167$\frac{2}{3}$, 3^{tio} *die Februarii, datâ.*

Hujus Autographon in ſcriniis Regiæ Societatis extat, et exemplar ejus in Libro Epiſtolarum dictæ Societatis, N°. 6, pag. 35, deſcriptum legitur.

CUM heri apud illuſtriſſimum *Boylium* incidiſſem in clariſſimum *Pellium* Mathematicum inſignem, ac de Numeris

incidiſſet mentio, commemoravi ego, ductus occaſione Sermonum, eſſe mihi methodum ex quodam differentiarum genere, quas voco *generatrices*, colligendi terminos Seriei cujuſcunque continuè creſcentis vel decreſcentis. Differentias autem generatrices voco, ſi datæ Seriei inveniantur differentiæ, & differentiæ differentiarum, & ipſarum ex differentiis differentiarum differentiæ, *&c.* & ſeries conſtituatur ex termino primo & primâ differentiâ, & primâ differentiâ differentiarum, & primâ differentiâ ex differentiis differentiarum, *&c.* ea Series erit differentiarum generatricium, ut ſi Series continuè creſcens vel decreſcens fuerit a, b, c, d.

Poſitâ $\backsim$ *differentiæ Notâ,*] differentiæ generatrices erunt:

$$1\,a\ .\ 2\,a\backsim b\ .\ 3\,\overline{a\backsim b}\backsim\overline{b\backsim c}\ .\ 4\,\overline{\overline{a\backsim b}\backsim\overline{b\backsim c}}\backsim\overline{\overline{b\backsim c}\backsim\overline{c\backsim d}}.$$

$$\begin{array}{lccccccc}
4 & & & \overline{\overline{a\backsim b}\backsim\overline{b\backsim c}}\backsim\overline{\overline{b\backsim c}\backsim\overline{c\backsim d}} & & & \\
3 & & a\backsim b\backsim b\backsim c & & b\backsim c\backsim c\backsim d & & \\
2 & a\backsim b & & b\backsim c & & c\backsim d & \\
1\ a & & b & & c & & d
\end{array}$$

Aut in Numeris; ſi Series ſit Numerorum cubicorum deinceps ab unitate creſcentium, differentiæ generatrices erunt numeri 0, 1, 6, 6. Voco autem *generatrices*, quia ex iis certo modo multiplicatis producuntur termini Seriei; cujus uſus tum maximè apparet, cum differentiæ generatrices ſunt finitæ, termini autem Seriei infiniti; ut in propoſito exemplo Numerorum Cubicorum.

$$\begin{array}{ccccccccccccc}
 & & & 0 & & 0 & & 0 & & & & & \\
 & & 6 & & 6 & & 6 & & 6 & & & & \\
 & 6 & & 12 & & 18 & & 24 & & 30 & & & \\
1 & & 7 & & 19 & & 37 & & 61 & & 91 & & \\
0 & 1 & & 8 & & 27 & & 64 & & 125 & & 216 &
\end{array}$$

Hoc cum audiſſet clariſſimus *Pellius*, reſpondit, id jam fuiſſe in literas relatum à D. *Mouton*, Canonico *Lugdunenſi*, ex obſervatione nobiliſſimi viri *Franciſci Regnaldi Lugdunenſis*, dudùm in literario Orbe celebris, in libro laudati D. *Mouton* de diametris apparentibus Solis & Lunæ. Ego qui ex Epiſtolâ quâdam à *Regnaldo* ad *Monconiſium* ſcriptâ, & Diario itinerum *Monconiſiano* inſertâ, nomen D. *Moutoni* & deſignata ejus duo didiceram ; Diametros Luminarium apparentes, & conſilium de menſuris rerum ad poſteros tranſmittendis ; ignorabam tamen librum ipſum prodiiſſe : quarè apud D. *Oldenburgium Societatis Regalis* Secretarium, ſumtum mutuò tumultuariè percurri, & inveni veriſſimè dixiſſe *Pellium*. Sed & mihi tamen dandam operam credidi, ne qua in animis relinqueretur ſuſpicio, quaſi, tacito * inventoris nomine, alienis meditationibus honorem mihi quærere voluiſſem ; & ſpero appariturum eſſe, non adeò egenum me meditationum propriarum ut cogar alienas emendicare. Duobus autem argumentis ingenuitatem meam vindicabo. Primò, ſi ipſas Schedas meas confuſas, in quibus non tantùm inventio mea ſed & inveniendi modus occaſióque apparet, monſtrem : deinde, ſi quædam momenti maximi *Regnaldo Moutonóque* indicta addam, quæ ab heſterno veſpere confinxiſſe me non ſit veriſimile, quaéque non poſſunt facilè expectari à Tranſcriptore.

Ex Schedis meis occaſio inventi hæc apparet : quærebam modum inveniendi differentias omnis generis poteſtatum, quemadmodum conſtat differentias Quadratorum eſſe numeros impares ; inveneràmque regulam generalem ejuſmodi.

Datâ potentiâ gradûs dati præcedente, invenire ſequentem (vel contrà) diſtantiæ datæ vel radicum datarum ; ſeu invenire potentiarum gradûs dati utcunque diſtantium differentias. Multiplicetur potentia gradûs, proximè præcedentis radicis majoris per differentiam radicum ; & differentia potentiarum gradûs proximè præcedentis multiplicetur per radicem minorem : productorum ſumma erit quæſita differentia potentiarum, quarum radices ſunt datæ. Eandem regulam ita inflexeram, ut ſufficeret, præter radices, cujuſlibet gradûs, etiamſi non proximè præcedentis, potentias datarum radicum dari, ad differentias potentiarum alterius cujuſcunque, licet altioris, gradûs inveniendas. Et oſtendi quod in Quadratis obſervatur, numeros impares eſſe eorum differentias, id non niſi regulæ propoſitæ ſubſumptionem eſſe.

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* Id eſt, celato.

His meditationibus defixus, quemadmodum in Quadratis differentiæ ſunt numeri impares, ita quoque quæſivi quales eſſent differentiæ Cuborum, quæ cum irregulares viderentur, quæſivi differentias differentiarum, donec inveni differentias tertias eſſe numeros ſenarios. Hæc obſervatio mihi aliam peperit: videbam enim ex differentiis præcedentibus generari terminos differentiáſque ſequentes, ac proinde, ex primis, quas ideò voco *generatrices*, (ut hoc loco 0 . 1 . 6 . 6,) ſequentes omnes. Hoc concluſo, reſtabat invenire, quo additionis, multiplicationíſve, aut horum complicationis, genere, termini ſequentes ex differentiis generatricibus producerentur. Atque ita reſolvendo experiundóque deprehendi primum Terminum 0 componi ex primâ differentiâ generatrice 0 ſumtâ ſemèl, ſeu vice unâ: Secundum 1 ex primâ 0 ſemèl & ſecundâ 1 ſemèl: Tertium 8 ex primâ 0 ſemèl, ſecundâ 1 bis & tertiâ 6 ſemèl: nam $0 \times 1 + 1 \times 2 + 6 \times 1 = 8$. Quartum 27, ex primâ 0 ſemèl, ſecundâ 1 tèr, tertiâ 6 tèr, quartâ 6 ſemèl: nam $0 \times 1 + 1 \times 3 + 6 \times 3 + 6 \times 1 = 27$, *&c.* ídque Analyſis mihi univerſale eſſe comprobavit. Hæc fuit occaſio obſervationis meæ, longè alia à *Moutonianâ*, qui cum in Tabulis condendis laboraret, in hoc calculandi compendium cum *Regnaldo* incidit: nec vel illi vel *Regnaldo* adimenda laus; quod & *Briggius* in Logarithmicis ſuis jam olim talia quædam, obſervante *Pellio*, ex parte advertit. Mihi hoc ſupereſt ut addam nonnulla illis indicta, ad amoliendum Tranſcriptoris nomen; neque enim intereſt Reipublicæ quis obſervaverit, intereſt quid obſervetur. Primum, ergò illud adjicio, quod apud *Moutonium* non extat, & caput tamen rei eſt: quinam ſint illi numeri, quorum Tabulam ille exhibet in infinitum continuandam, quorum ductu in differentias generatrices, productis inter ſe junctis, termini Serierum generentur. Vides enim ex ipſo modo quo tabula ab eo *pag.* 385, exhibetur, non fuiſſe id ei ſatìs exploratum; alioqui enim veriſimile eſt ita Tabulam fuiſſe diſpoſiturum, ut ea numerorum connexio atque harmonia appareret; niſi quis de induſtriâ texiſſe dicat: ita enim ſe habet pars Tabulæ.

1	1					
2	1	1				
3	1	2	1			
(4)	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

Apparet ex hujus Tabulæ conſtructione ſolam haberi rationem correſponſûs numerorum generantium cum numero Termini generati; ut cum terminus eſt quartus (4) producitur ex primâ differentiâ ſemèl, ſecundâ ter 3, tertiâ ter 3, quartâ ſemèl 1; ideò in eâdem (4) Lineâ tranſversâ locantur 1. 3. 3. 1. Sed vel non obſervavit vel diſſimulavit autor correſponſum numerorum, ſi à ſummo deorſùm eundo per columnas diſponantur hoc modo,

1	1					
2	1 ↘	1				
3	1	2 ↘	1			
4	1	3	3 ↘	1		
5	1	4	6	4 ↘	1	
6	1	5	10	10	5 ↘	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

Ita enim ſtatìm vera genuináque eorum natura ac generatio apparet; eſſe, ſcilicet, eos numeros quos *Combinatorios* appellare ſoleo, de quibus multa dixi in diſſertatiunculâ de Arte Combinatoriâ; quóſque alii appellant Ordines numericos; alii ſpecie primam columnam Unitatum; ſecundam Numerorum naturalium, tertiam Triangularium, quartam Pyramidalium, quintam Triangulo-Triangularium, *&c.* de quibus integer extat Tractatus *Paſchalii* ſub titulo Trianguli Arithmetici; in quo tamen proprietatem numerorum ejuſmodi tàm illuſtrem támque naturalem * non obſervatam ſum miratus. Sed eſt profectò caſus quidam in inveniendo, qui non ſemper maximis ingeniis maxima, ſed ſæpe etiam mediocribus nonnulla offert.

Hinc jam vera numerorum iſtorum natura, & Tabulæ conſtructio, ſive à *Regnaldo* ſive à *Moutonio* diſſimulata, intelligitur: ſemper enim terminus datus columnæ datæ componitur ex termino præcedente columnæ tàm præcedentis quàm datæ: Atque illud quoque apparet, non opus eſſe moleſto calculo ad Tabulam à *Moutonio* propoſitam continuandam, ut ipſe poſtulat; cum hæ numerorum Series paſsìm jam tradantur calculentúrque.

Cæterùm *Moutonius* obſervatione iſtâ ad interponendas medias proportionales inter duos extremos numeros datos; ego ad inveniendos ipſos numeros extremos in infinitum cum eorum differentiis, utendum cenſebam. Hinc ille, non niſi cum differentiæ ultimæ evaneſcunt (aut penè evaneſcunt) uſum regulæ invenit; ego detexi innumerabiles caſus, regulâ quâdam inobſervatâ comprehendendos; ubi poſſum ex datis numeris finitis certo modo multiplicatis producere numeros plurimarum Serierum in infinitum euntium, etſi differentiæ earum non evaneſcant.

Ex iiſdem fundamentis poſſum efficere in progreſſionibus problemata plurima; aut in Numeris ſingularibus, aut in Rationibus vel Fractionibus: poſſum enim progreſſiones ad-

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* *Imò obſervata fuit. Vide* Paſchalii *Triangulum Arithmeticum*, Pariſiis *Anno* 1665 *editum, pag.* 2. *ubi definitionum antepenultima hæc eſt.*

Le nombre de chaque cellule eſt egal à celuy de la cellule qui la précéde dans ſon rang perpendiculaire, plus à celuy de la cellule qui la précéde dans ſon rang parallèle. Ainſi la cellule F, *c'eſt à dire le nombre de la cellule* F, *égale la cellule* C *plus la cellule* E; *& ainſi des autres.*

dere ſubtraheréque, imò multiplicare quoque & dividere, idque compendiosè.

$$\begin{array}{ccccccc}
\frac{1}{3} & \cdot & \frac{1}{4} & \cdot & \frac{1}{5} & \cdot & \frac{1}{6} \\
\frac{1}{6} & \cdot & \frac{1}{10} & \cdot & \frac{1}{15} & \cdot & \frac{1}{21} \\
\frac{1}{10} & \cdot & \frac{1}{20} & \cdot & \frac{1}{35} & \cdot & \frac{1}{56} \\
\frac{1}{15} & \cdot & \frac{1}{35} & \cdot & \frac{1}{70} & \cdot & \frac{1}{126} \\
\&c. & & \&c. & & \&c. & & \&c.
\end{array}$$

Multa alia circà hos numeros obſervata ſunt à me, ex quibus illud eminet, quod modum habeo ſummam inveniendi Seriei Fractionum in infinitum decreſcentium; quarum numerator Unitas, nominatores vero numeri iſti Triangulares aut Pyramidales, aut Triangulo-Triangulares, *&c.*

End of the Extract from Mr. Leibnitz's *Letter.*

Art. 4. By the help of the foregoing table of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, as far as 100, we may find the cube-root of any number exact to two places of figures, without the trouble of any calculation whatſoever, or by the mere inſpection of the table. Thus, for example, if I wanted to find the cube-root of 2000, I need only look along the ſecond column of the table, (which contains the cubes of the numbers ſet down in the firſt column,) till I found the two cube numbers which are neareſt to the propoſed number 2000, the one above it and the other below it. Theſe numbers I ſhould find to be 2197 and 1728; of which the former is the cube of 13, and the latter is the cube of 12. And hence I might conclude with certainty that, ſince the propoſed number 2000 is greater than 1728, or the cube of 12, but leſs than 2197, or the cube of 13, its cube-root muſt be greater than 12, but leſs than 13, and conſequently that the two firſt figures of it muſt be 12. And from the two firſt figures of the cube-root of any propoſed number, we may derive the following figures of it to five, or ſix, or any greater number of figures that we may deſire, by the method of approximation invented for this purpoſe by Monſieur *De Lagney*, which has ſince been approved and adopted by Dr. *Halley* and other Mathematicians, as the moſt convenient that can be taken. This method is as follows.

Monſieur De Lagney's *Method of approximating to the Value of the Cube-root of any propoſed Number, when the Two, or Three, firſt Figures of the ſaid Cube-root are known.*

Art. 5. If the number of which the cube-root is to be extracted be called c, and a number, conſiſting of two, or more figures, that is ſomewhat leſs than the true value of the ſaid cube-root be called a, the remaining part of the ſaid cube-root will be very nearly equal to the quantity $\frac{\overline{c-a^3}\times a}{c+2a^3}$, and conſequently the whole of the ſaid cube-root will be very nearly equal to $a+\frac{\overline{c-a^3}\times a}{c+2a^3}$; but it will always be a little greater than the ſaid quantity. Alſo the ſaid remaining part of the cube-root of c, which is to be added to its firſt value a, will be very nearly equal to the quantity $\sqrt{\frac{4c-a^3}{12a}}-\frac{a}{2}$, and conſequently the whole of the ſaid cube-root will be very nearly equal to $a+\sqrt{\frac{4c-a^3}{12a}}-\frac{a}{2}$, or to $\frac{a}{2}+\sqrt{\frac{4c-a^3}{12a}}$; but it will always be a little leſs than that quantity. And this latter expreſſion will be a little nearer to the true value of the cube-root of c than the former expreſſion $a+\frac{\overline{c-a^3}\times a}{c+2a^3}$; but the difference will be ſo ſmall as to be hardly worth conſidering.

Art. 6. And, if a, or the firſt near value of the cube-root of the propoſed number c, be a little greater than its true value, the quantity to be ſubtracted from a, in order to make it equal to the ſaid true value, will be very nearly equal to the quantity $\frac{\overline{a^3-c} \times a}{c+2a^3}$, and conſequently the ſaid cube-root will be very nearly equal to $a - \frac{\overline{a^3-c} \times a}{c+2a^3}$; but it will be always a little greater than the ſaid quantity.

Alſo the ſaid difference between a and the true value of the cube-root of c, or quantity which is to be ſubtracted from a, in order to make it equal to the ſaid cube-root, will be very nearly equal to the quantity $\frac{a}{2} - \sqrt{\frac{4c-a^3}{12a}}$, and conſequently the ſaid cube-root will be very nearly equal to $a - \overline{\frac{a}{2} - \sqrt{\frac{4c-a^3}{12a}}}$, or to $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$; but it will always be a little leſs than the ſaid quantity. And this latter expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ will be a little nearer to the true value of the cube-root of c than the former expreſſion $a - \frac{\overline{a^3-c} \times a}{c+2a^3}$; but the difference will be ſo ſmall as to be hardly worth conſidering.

Art. 7. The number of figures that will be exact in the ſecond near value of the cube-root of the propoſed number c that will be obtained by either of theſe four expreſſions $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$, $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, $a - \frac{\overline{a^3-c} \times a}{c+2a^3}$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, (which laſt expreſſion, we may obſerve, is the very ſame with the ſecond expreſſion,) is uſually triple, or triple wanting one figure, and in the worſt caſes is triple wanting two figures, of the number of figures that are exact in the firſt near value, a,

An Example of the Extraction of the Cube-root of a Number, by means of the foregoing Method of Approximation.

Art. 8. Let it be required to extract the cube-root of the number 2; which anſwers to the ſolution of the Problem, ſo much celebrated amongſt the Antients, of doubling the cube, or finding the length of the ſide of a cube that ſhall be double of a given cube.

Here I obſerve, in the firſt place, that, ſince the number 2000 is equal to 1000 × 2, or to 10 × 10 × 10 × 2, the cube-root of 2000 muſt be equal to 10 times the cube-root of 2. But it appears from the foregoing table of cube numbers, that the cube-root of 2000 muſt be greater than 12, but leſs than 13, and conſequently that the two firſt figures of it muſt be 12. Therefore the cube-root of 2 muſt be $(= \frac{12}{10} = \frac{10}{10} + \frac{2}{10}) = 1.2$, or the two firſt figures of the ſaid cube-root muſt be 1.2. Here then we have $c = 2$, or 2.000,000, and $a = 1.2$, and conſequently $a^3 = 1.728$, and $2a^3 = 3.456$, and $c + 2a^3$ $(= 2 + 3.456) = 5.456$, and $c - a^3$ $(= 2.000 - 1.728) = 0.272$, and $\overline{c - a^3} \times a$ $(= 0.272 \times 1.2)$ 3.264, and $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ $(= \frac{3.264}{5.456}) = 0.059,82$. Therefore $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $= 1.2 + 0.059,82$, or 1.259,82; or the ſecond near value of the cube-root of 2, which is obtained by means of the firſt expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ given in art. 5, is 1.259,82.

The number of figures that are exact in this ſecond near value, 1.259,82, of the cube-root of 2, is four, to wit, the figures 1.259, that is, three times as many figures, wanting two, as are contained in 1.2, or a, the firſt near value of the ſaid cube-root, the more accurate value of which is 1.259,921,049, &c, which is greater than 1.259,82 by 0.000,1, &c.

The other expreſſion given in art. 5, to wit, $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, may be computed as follows.

Since a is $= 1.2$, and c is $= 2$, we ſhall have $12a$ ($= 12 \times 1.2$) $= 14.4$, and $\frac{a}{2} = 0.6$, and $4c = 8.000$, and $4c - a^3$ ($= 8.000 - 1.728$) $= 6.272$, and $\frac{4c-a^3}{12a}$ ($= \frac{6.272}{14.4}$) $= 0.435,555,555$, &c, and $\sqrt{\frac{4c-a^3}{12a}}$ ($= \sqrt{0.435,555,555}$, &c,) $= 0.659,96$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ ($= 0.6000,00 + 0.659,96$) $= 1.259,96$. Therefore $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, or the ſecond near value of the cube-root of 2, which is obtained by means of the ſecond expreſſion given in art. 5, is 1.259,96; which is exact in the firſt five figures 1.2599, and is greater than the true value of the ſaid cube-root, to wit, 1.259,92, &c, by only 0.000,04, &c, which is leſs than the difference 0.000,1, by which 1.259,82, or the ſecond value of the cube-root of 2, obtained by means of the former expreſſion $a + \frac{\overline{c-a^3} \times a}{c + 2a^3}$, fell ſhort of the true value of the ſaid cube-root. But either of theſe ſecond near values of the ſaid cube-root, 1.259,82 and 1.259,96, is a great improvement upon its firſt near value, 1.2, though leſs than it would have been if the two firſt figures of the cube-root of the propoſed number had been higher figures than 1 and 2.

Art. 9. And, if we repeat this proceſs of approximation with either of the two expreſſions $a + \frac{\overline{c-a^3} \times a}{c + 2a^3}$, and $\frac{a}{2}$ +

$\sqrt{\frac{4c-a^3}{12a}}$, taking the firſt four figures of the ſecond values of the cube-root of 2, which have been already found, (and which are the ſame in both the foregoing calculations,) to wit, the four figures 1.259, for the baſis of the next operation, we ſhall obtain the value of the ſaid cube-root to a very great degree of exactneſs. This may be done in the manner following.

Let a be ſuppoſed $= 1.259$.

Then we ſhall have a^3 $(= \overline{1.259}|^3) = 1.995,616,979$, and conſequently $2a^3$ $(= 2 \times 1.995,616,979) = 3.991,233,958$, and $c + 2a^3$ $(= 2 + 3.991,233,958) = 5.991,233,958$, and $c - a^3$ $(= 2.000,000,000 - 1.995,616,979) = 0.004,383,021$, and $\overline{c - a^3}| \times a$ $(= 0.004,383,021 \times 1.259) = 0.005,518,223,439$, and $\frac{\overline{c-a^3}| \times a}{c + 2a^3}$ $(= \frac{0.005,518,223,439}{5.991,233,958}) = 0.000,921,c49,55$, &c. Therefore $a + \frac{\overline{c-a^3}| \times a}{c + 2a^3}$ will be $(= 1.259 + 0.000,921,049,55$, &c,) $= 1.259,921,049,55$, &c; which is exact in the firſt ten figures 1.259,921,049, the more accurate value of the cube-root of 2 being 1.259,921,049,89, which is greater than 1.259,921,049,55 by only the very ſmall quantity 0.000,000,000,34.

Alſo we ſhall have $4c$ $(= 4 \times 2) = 8.000,000,000$, and $4c - a^3$ $(= 8.000,000,000 - 1.995,616,979) = 6.004,383,021$, and $12a$ $(= 12 \times 1.259) = 15.108$, and $\frac{4c - a^3}{12a}$ $(= \frac{6.004,383,021}{15.108}) = 0.397,430,900,3$, and $\sqrt{\frac{4c-a^3}{12a}}$ $(= \sqrt{0.397,430,700,3}) = 0.630,421,050,01$. Therefore $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ will be $(= \frac{1.259}{2} + 0.630,421,050,01 = 0.6295 + 0.630,421,050,01) = 1.259,921,050,01$; which exceeds the more accurate value of the cube-root of 2, to wit, 1.259,921,049,89, by only the very ſmall quantity 0.000,000,000,12, which is ſtill leſs than the ſmall quantity 0.000,000,000,34.

Another Example of the Extraction of the Cube-root of a Number, by the ſame Method of Approximation.

Art. 10. Let it be propoſed to find, in inches and decimal parts of an inch, the ſide of a cube that is equal to the Engliſh meaſure called a gallon, which contains 231 cubick inches; or, in other words, let it be required to find the cube-root of the number 231.

Now, if we look along the column of cube numbers in the foregoing table, we ſhall find that 216 is the cube of 6, and that 343 is the cube of 7. Therefore we may conclude that the cube-root of the propoſed number 231 muſt be greater than 6, but leſs than 7. We will therefore take 6 for the value of a, or for the firſt near value of $\sqrt[3]{231}$, with which we are to begin our approximation.

Now, ſince c is, in this example, equal to 231, and a is $= 6$, we ſhall have a^3 $(= \overline{6}|^3) = 216$, and $2a^3 = 432$, and $c + 2a^3$ $(= 231 + 432) = 663$, and $c - a^3$ $(= 231 - 216) = 15$, and $\overline{c - a^3}| \times a$ $(= 15 \times 6) = 90$; and $\frac{\overline{c - a^3}| \times a}{c + 2a^3}$ $(= \frac{90}{663} = 0.13$. Therefore $a + \frac{\overline{c - a^3}| \times a}{c + 2a^3}$ will be $(= 6 + 0.13) = 6.13$; which is therefore the ſecond near value of the cube-root of 231.

Now let a be taken $= 6.13$, in order to obtain a third near value of the cube-root of 231.

Then we ſhall have a^3 $(= \overline{6.13}|^3) = 230.346,397$, and $2a^3$ $(= 2 \times 230.346,397) = 460.692,794$, and $c + 2a^3$ (=

(= 231 + 460.692,794) = 691.692,794, and $c - a^3$ (= 231.000,000 − 230.346,397) = 0.653,603, and $\overline{c-a^3} \times a$ (= 0.653,603 × 6.13) = 4.006,586,39, and $\frac{\overline{c-a^3} \times a}{c+2a^3}$ (= $\frac{4.006,586,39}{691.692,794}$) = 0.005,792,436. Therefore $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$ will be (= 6.13 + 0.005,792,436) = 6.135,792,436; which is therefore the third near value of the cube-root of 231, or of the length, in inches and decimal parts of an inch, of the ſide of a cube that contains an Engliſh gallon. Q. E. I.

This number 6.135,792,436, is exact in the firſt nine figures, 6.135,792,43, the more accurate value of the cube-root of 231 being, according to Dr. Halley, (from whoſe tract upon this ſubject in the Philoſophical Tranſactions this example is taken,) 6.135,792,439,661,958, &c. Therefore the number of figures obtained exactly in this inſtance by means of the expreſſion $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$ is juſt triple of the number of figures that are contained in a, or 6.13, agreeably to what is obſerved above in art. 7.

Art. 11. The other expreſſion of the third near value of the cube-root of 231, to wit, the expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, may be computed as follows.

Since c is = 231, and a is = 6.13, we ſhall have $4c$ (= 4 × 231) = 924, and $\frac{a}{2}$ (= $\frac{6.13}{2}$) = 3.065, and $12a$ (= 12 × 6.13) = 73.56, and a^3 (= $\overline{6.13}|^3$) = 230.346,397, and $4c - a^3$ (= 924.000,000 − 230.346,397) = 693.653,603, and $\frac{4c-a^3}{12a}$ (= $\frac{693.653,603}{73.56}$) = 9.429,766,218,053,289,8, and $\sqrt{\frac{4c-a^3}{12a}}$ (= $\sqrt{9.429,766,218,053,289,8}$) = 3.070,702,441. Therefore $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ will be (= 3.065 + [illegible] =)

$+ 3.070,792,441) = 6.135,792,441$; or the third near value of the cube-root of 231, obtained by means of the expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be 6.135,792,441.

Q. E. I.

Art. 12. This number 6.135,792,441, obtained by means of the expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, exceeds the more accurate value of the cube-root of 231, to wit, 6.135,792,439, &c, by only the ſmall fraction 0.000,000,002; which is ſomewhat leſs than the ſmall fraction 0.000,000,003, by which the former number 6.135,792,436, that was obtained by means of the expreſſion $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$, falls ſhort of the ſaid more accurate value. But both theſe differences enter in the ſame place of decimal fractions, to wit, the ninth place, and therefore the ſmall difference of exactneſs in theſe two expreſſions $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ is not worth attending to. But Dr. Halley gives the preference to the latter expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ on another account, to wit, becauſe he thinks the extraction of the ſquare-root of the fraction $\frac{4c-a^3}{12a}$ a leſs laborious operation than the diviſion of $\overline{c-a^3} \times a$ by the great diviſor $c+2a^3$. His words are as follows. "And this Formula [the irrational formula $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, though he uſes a ſomewhat different notation,] is deſervedly preferable to the rational [or $a + \frac{\overline{c-a^3} \times a}{c+2a^3}$,] upon the account of the great diviſor, which is not to be managed without a great deal of labour; whereas the extraction of the ſquare-root proceeds much more eaſily, as manifold experience has taught me."

Theſe

Thoſe two examples are taken from a very ingenious and uſeful tract on this ſubject, intitled, *A new, exact, and eaſy Method of finding the Roots of Equations Generally, and that without any previous Reduction*; written by the celebrated Dr. Edmund Halley, and publiſhed firſt in the Philoſophical Tranſactions for the month of May 1694, Number 210, and afterwards in the year 1708, in the ſecond volume of the Collection of Mathematical and Philoſophical Tracts, intitled *Miſcellanea Curioſa*, in three volumes octavo. See the 2d volume of the ſaid *Miſcellanea Curioſa*, pages 70, 71, 72, 73, 74, and 75.

A Third Example of the Extraction of the Cube-root of a Number, by the ſame Method of Approximation.

Art. 13. This example ſhall be that which is given by Mr. Raphſon in his *Analyſis Æquationum Univerſalis*, Problem 2d. It is to find the cube-root of the number 37,945.

Now, if we look along the column of cube-numbers in the foregoing table, we ſhall find that 35,937 is the cube of 33, and that 39,304 is the cube of 34. Therefore, ſince the propoſed number 37,945 is greater than 35,937, but leſs than 39,304, it follows that the cube-root of 37,945 will be greater than 33, but leſs than 34, and conſequently that the two firſt figures of it will be 33.

Here then we have $c = 37,945$, and $a = 33$. Therefore a^3 will be $= 35,937$, and $2a^3$ will be $(= 2 \times 35,937) = 71,874$, and $c + 2a^3$ will be $(= 37,945 + 71,874) = 109,819$, and $c - a^3$ will be $(= 37,945 - 35,937) = 2008$, and $\overline{c - a^3} \times a$ will be $(= 2008 \times 33) = 66,264$, and conſequently $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $(= \frac{66,264}{109,819}) = 0.6033$.

Therefore $a + \frac{\overline{c-a^3}| \times a}{c+2a^3}$, or the ſecond near value of the cube-root of the propoſed number 37,945, will be (= 33 + 0.6033, or) 33.6033; of which number the five firſt figures 33.603 are exact, the more accurate value of the ſaid cube-root being 33.603,526,179,43, &c.

Now let us ſuppoſe a to be = 33.6033, or the ſecond near value of the cube-root of 37,945 that has been already found; and let us, in order to obtain a third near value of it, repeat the foregoing proceſs.

Then we ſhall have a^3 = 37,944.233,801,747,937, and $2a^3$ (= 2 × 37,944.233,801,747,937,) = 75,888.467,603,495,874, and $c + 2a^3$ (= 37,945.000,000,000,000 + 75,888.467,603,495,874) = 113,833.467,603,495,874, and $c - a^3$ (= 37,945.000,000,000,000 − 37,944.233,801,747,937) = 0.766,198,252,063, and $\overline{c-a^3}| \times a$ (= 0.766,198,252,063 × 33.6033) = 25.746,789,723,548,607,9, and conſequently $\frac{\overline{c-a^3}| \times a}{c+2a^3}$ ($= \frac{25.746,789,723,548,607,9}{113,833.467,603,495,874}$) = 0.000,226,179,437,95. Therefore $a + \frac{\overline{c-a^3}| \times a}{c+2a^3}$ will be (= 33.6033 + 0.000,226,179,437,95,) = 33.603,526,179,437,95; that is, the third near value of the cube-root of c, or 37,945, that is obtained by means of the rational expreſſion $a + \frac{\overline{c-a^3}| \times a}{c+2a^3}$, will be 33.603,526,179,437,95; which I believe to be exact in the firſt fifteen figures 33.603,526, 179,437,9, if not in the laſt, or ſixteenth, figure 5 likewiſe.

Art. 14. The other expreſſion of the third near value of the cube-root of 37,945, to wit, the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, may be computed as follows.

Since c is = 37,945, and a is = 33.6033, we ſhall have $\frac{a}{2}$ ($= \frac{33.6033}{2}$) = 16.801,65, and $12a$ (= 12 × 33.6033) =

$= 403.2396$, and $4c$ $(= 4 \times 37,945) = 151,780$, and a^3 $(= \overline{33.6033}|^3) = 37,944.233,801,747,937$, and $4c - a^3$ $(= 151,780.000,000,000,000 - 37,944.233,801,747,937)$ $= 113,835.766,198,252,063$, and $\frac{4c - a^3}{12a}$ $(= \frac{113,835.766,198,252,063}{403.2396}) = 282.303,043,149,165,069$, and $\sqrt{\frac{4c - a^3}{12a}}$ $(= \sqrt{282.303,043,149,165,069}) = 16.801,876,179,437,96$. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be $(= 16.801,65 + 16.801,876,179,437,96) = 33.603,526,179,437,96$; that is, the third near value of the cube-root of c, or 37,945, that is obtained by means of the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be 33.603,526,179,437,96.

Art. 15. This number 33.603,526,179,437,96 muſt (if there has been no miſtake made in the calculation,) be ſomewhat greater than the true value of the cube-root of c, or 37,945; and the former number 33.603,526,179,437,95, obtained by means of the rational expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, muſt be ſomething leſs than the ſaid true value. Theſe two numbers differ only by an unit in the laſt, or ſixteenth, figure. And hence it follows that the firſt ſixteen figures of the ſaid true value muſt be the ſame with thoſe of the leſſer of thoſe two numbers, or 33.603,526,179,437,95. Mr. Raphſon, however, computes it to be 33.603,526,179, 438,08. But I ſuſpect that his three laſt figures are not exact. But, whether they are exact or not, we may, at leaſt, conclude that, ſo far as theſe different calculations agree with each other, they muſt be exact, and conſequently that the firſt thirteen figures of the cube-root of the number 37,945 are 33.603,526,179,43.

A fourth

A Fourth Example of the Extraction of the Cube-root of a Number, by means of the foregoing Method of Approximation.

Art. 16. This example ſhall be one that is given by Monſieur de Lagney himſelf, in his book intitled, *Nouveaux Elements d'Arithmêtique et d'Algêbre,* which was publiſhed at Paris in Duodecimo, in the year 1697. It is to find the cube-root of the number 696,536,483,318,640,035,073, 641,037, which conſiſts of twenty-ſeven figures, and may be expreſſed in the words following, to wit, 696 quadrillions, or fourth powers of a million, 536,483 trillions, or third powers of a million, 318,640 billions, or ſecond powers of a million, 035,073 millions, and 641,037 units.

This number is ſo great that it will be convenient to divide it into theſe two parts, 696,536,483,000,000,000,000,000,000 and 318,640,035,073,641,037, and to begin by ſeeking the cube-root of the firſt part, 696,536,483,000,000,000,000, 000,000.

Now this number 696,536,483,000,000,000,000,000,000 is = 696,536,483 × 1,000,000,000,000,000,000, or 696,536,483 × the cube of 1,000,000. Therefore its cube-root will be equal to 1,000,000 × the cube-root of 696,536,483. Therefore, if we can find the cube-root of 696,536,483, we need only multiply it by 1,000,000, in order to obtain the cube root of 696,536,483,000,000,000, 000,000,000. We will therefore endeavour to find the cube-root of 696,536,483.

Now this number 696,536,483 is greater than 696,536,000, or than 696,536 × 1000, or than 696,536 × the cube of 10. Therefore the cube-root of 696,536,483 will be greater than the product of the multiplication of the cube-root of 696,536 into 10. Therefore, if we can find the cube-root of 696,536, we need only multiply it by 10, in order to obtain the cube-root of 696,536,000, which will be ſomething leſs than the cube-root of 696,536,483, and may ſerve as a baſis from which to begin our approximation to the true value of the ſaid cube-root. We will therefore now endeavour to find, to a ſmall degree of exactneſs, the cube-root of 696,536.

Art. 17. Now, if we look into the foregoing table of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, as far as 100, we ſhall find that 681,472 is the cube of 88, and that 704,969 is the cube of 89. It follows therefore that the cube-root of 696,536, (which is greater than 681,472, but leſs than 704,969,) muſt be greater than 88, but leſs than 89, and conſequently that the two firſt figures of it muſt be 88. Therefore the cube-root of 696,536,000 muſt be greater than 88 × 10, or 880, but leſs than 89 × 10, or 890, and conſequently the two firſt figures of it will be 88. Therefore 880, being leſs than the cube-root of the number 696,536,000, will be leſs alſo than the cube-root of the number 696,536,483, which is greater than 696,536,000; but it will approach ſufficiently near to it to enable us to begin a further approximation to it by means of the foregoing expreſſions of Monſieur de Lagney.

Art. 18. Let us therefore ſuppoſe a to be $= 880$; and, in order to find a ſecond near value of the cube-root of c, or 696,536,483, let us compute the expreſſion $a + \frac{\overline{c-a^{3}}\times a}{c+2a^{3}}$.

Here then we ſhall have a^{3} $(= \overline{880}|^{3}) = 681,472,000$, and $2a^{3} = 1,362,944,000$, and $c + 2a^{3}$ $(= 696,536,483 + 1,362,944,000) = 2059,480,483$, and $c - a^{3}$ $(= 696,536,483 - 681,472,000) = 15,064,483$, and $\overline{c-a^{3}} \times a$ $(= 15,064,483 \times 880) = 13,256,745,040$, and conſequently $\frac{\overline{c-a^{3}}\times a}{c+2a^{3}}$ $(= \frac{13,256,745,040}{2059,480,483}) = 6.436,9$. Therefore $a + \frac{\overline{c-a^{3}}\times a}{c+2a^{3}}$ will be $(= 880 + 6.436,9) = 886.4369$; and conſequently 886.4369 will be a ſecond near value of the cube-root of the number c, or 696,536,483.

Art. 19. Therefore (by what is ſhewn in art. 16,) 886.4369 × 1000,000, or 886,436,900 will be nearly equal to, but ſomewhat leſs than, the cube-root of the number 696,536,483,000,000,000,000,000,000, and, *à fortiori*, will be leſs than the cube-root of the propoſed number 696,536,483,318,640,035,073,641,037. And, as 886,436,900 is not much leſs than 886,437,000, it ſeems probable that 886,437,000 will likewiſe be ſomewhat leſs than the cube-root ot the ſaid number 696,536,483,318,640,035,073,641,037. And ſo upon trial we ſhall find it to be. For the cube of 886,437,000 is 696,535,206,998,055,453,000,000,000, which is leſs than the ſaid propoſed number.

Art. 20. Now let a be ſuppoſed to be $=$ 886,437,000, and let us endeavour to find a nearer value of the cube-root of the propoſed number 696,536,483,318,640,035,073,641,037, by computing the expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$.

Then, ſince c is $=$ 696,536,483,318,640,035,073,641,037, and a is $=$ 886,437,000, and conſequently a^3 is $=$ 696,535,206,998,055,453,000,000,000, we ſhall have $\frac{a}{2}$ $\left(= \frac{886,437,000}{2}\right) =$ 443,218,500, and

$$4c \; (= 4 \times 696,536,483,318,640,035,073,641,037)$$
$$= 2,786,145,933,274,560,140,294,564,148$$

and $4c - a^3$ ($=$ 2,786,145,933,274,560,140,294,564,148, $-$ 696,535,206,998,055,453,000,000,000) $=$ 2,089,610,726,276,504,687,294,564,148, and $12a$ ($=$ 12 × 886,437,000) $=$ 10,637,244,000, and $\frac{4c - a^3}{12a}$ $\left(= \frac{2,089,610,726,276,504,687,294,564,148}{10,637,244,000}\right) =$ 196,442,868,686,332,983, and $\sqrt{\frac{4c - a^3}{12a}}$ ($= \sqrt{}$ 196,442,868,686,332,983) $=$ 443,218,759. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be ($=$

443,218,500 + 443,218,759) = 886,437,259; and conſequently the cube-root of the propoſed long number 696,536,483,318,640,035,073,641,037, will be very nearly equal to 886,437,259. Q. E. I.

Monſieur de Lagney determines this cube-root to be only 886,437,166. But this is owing, as I apprehend, to his intirely neglecting the laſt eighteen figures, 318,640, 035,073,641,037, of the propoſed number, and conſequently giving us only the cube-root of the number 696,536,483,000,000,000,000,000,000, which is leſs than the propoſed number: whereas in the laſt operations of the foregoing proceſs we took notice of all the figures of the propoſed number, when we found the value of 4*c*, and extracted the ſquare-root of the fraction $\frac{4c-a^3}{12a}$.

Mr. de Lagney adds, as a proof of the great uſefulneſs of this method of extracting cube-roots, that the moſt ſkilful Arithmetician would not be able to find the cube-root of this long number, 696,536483,318640,035073,641037, to the ſame degree of exactneſs, or to nine places of figures, by the common method of extracting the cube-root, in the ſpace of a whole month. See Monſieur de Lagney's *Nouveaux Eléments d'Arithmétique et d'Algébre*, page 307.

A SCHOLIUM.

Art. 21. This very uſeful method of approximating to the cube-roots, and other roots, of numbers was firſt publiſhed by Mr. de Lagney, at Paris, in the *Journal des Sçavants* for the 14th of May 1691, and afterwards was publiſhed again at greater length, and with a demonſtration, in a ſeparate tract in quarto, in the month of May of the following year 1692. But Mr. Joſeph Raphſon had publiſhed his *Analyſis Æquationum Univerſalis*, (which contains a general method of finding the roots of all ſorts of equations by approximation,) in the year 1690: and his method of approximation is not very different from this of Mr. de Lagney; and the ground, or principle, of it is exactly the ſame. So that, if Mr. de Lagney had ſeen Mr. Raphſon's *Analyſis Æquationum* before he had diſcovered his own method of approximation, it would have been eaſy for him to have deduced his own method from Mr. Raphſon's; and in that caſe it would have been candid in him to acknowledge that he had ſeen Mr. Raphſon's book, and had been led by it to the diſcovery of his own method. This, however, he has not done; at leaſt, not in his *Nouveaux Eléments d'Arithmétique et d'Algébre*, which is the only book of his that I have ſeen. And therefore I ſuppoſe he invented his method of approximation by his own efforts, and without having met with Mr. Raphſon's book.

Art. 22. It may further be obſerved, that Mr. Raphſon himſelf was not the firſt inventor of the method of reſolving equations by approximations of the kind he has made uſe of, that is, by approximations performed by transforming the original equation into another equation that involves in it the powers of the unknown difference between the firſt near value (already obtained, by conjecture or otherwiſe,) of the root of the firſt equation and its true value, and by reſolving the ſaid transformed equation in the manner of a ſimple equation, or by dropping all the terms that involve in them any higher powers of the ſaid unknown difference, or root of the ſecond equation, than its ſimple power. But this excellent method of diſcovering the roots of high equations had been found out by the great Sir Iſaac Newton more than twenty-four years before the publication of Mr. Raphſon's *Analyſis Æquationum Univerſalis*, to wit, in the year 1666, when he wrote his learned little tract, intitled *De Analyſi per æquationes numero terminorum infinitas*, which is printed in the *Commercium Epiſtolicum* of Mr. John Collins and other Mathematicians, pages 67, 68, 69, &c. - - 93, of the 2d edition. This tract was firſt printed in the year 1712, in the firſt edition of the ſaid *Commercium Epiſtolicum*,

licum. But it had been ſhewn in manuſcript to Dr. Iſaa Barrow, and by him ſent to the ſaid Mr. John Collins, wit Mr. Newton's leave, in the month of July 1669, and ſhew to the Lord Viſcount Brouncker, (a great Mathematicia of that time,) and, probably, to many other learned Mathematicians that were Fellows of the Royal Society, to which Mr. Collins was then a Secretary. And afterwards a part o it, containing a ſhort ſpecimen of Mr. Newton's method of reſolving equations by approximation, was publiſhed by Dr. Wallis in the 94th chapter of his Algebra in the year 1685, which was five years before the publication of Mr. Raphſon's *Analyſis Æquationum Univerſalis.* Whether this ſpecimen ſuggeſted to Mr. Raphſon the diſcovery of his method of approximating to the roots of equations, (which differs but little from that of Sir Iſaac Newton,) it is difficult to determine. He has not mentioned Newton's method in his treatiſe on this ſubject, though he was a great admirer of his genius, and ever ready to commend him: and therefore I am inclined to think that the above-mentioned ſpecimen of Newton's method of approximation was not the circumſtance that led him to the diſcovery of his own. *But,* whether it was or not, it is certain that the honour of priority with reſpect to this very uſeful invention is due to Sir Iſaac Newton.

Of the Ground, or Principle, of the Inveſtigation of the foregoing Expreſſions, invented by Monſieur de Lagney, *for Approximating to the Value of the Cube-root of a given Number.*

Art. 23. The inveſtigation of all the foregoing expreſſions invented by Monſieur de Lagney for approximating to the cube-root of a given number, when a firſt near value of the ſaid cube-root that is exact to one, or two, or more, places of decimal figures, is already known, is not difficult. It reſults from the contemplation of the compound quantities that are equal to the cubes of a binomial quantity, (ſuch as $a + b$,) and a reſidual quantity, (ſuch as $a - b$,) according as a, or the firſt near value of $\sqrt[3]{c}$ which is already known, is leſs, or greater, than $\sqrt[3]{c}$; and therefore it ought properly to be divided into two parts, the one relating to the caſe in which a, or the firſt near value of the cube-root of the given number c that is already known, is leſs than the cube-root of c, and the other relating to the caſe in which a, or the ſaid firſt near value of $\sqrt[3]{c}$, is greater than the ſaid cube-root. The firſt of theſe inveſtigations, (by which we ſhall alſo obtain Mr. Raphſon's approximation to the value of the ſaid cube-root, in the ſame caſe, or when a is leſs than $\sqrt[3]{c}$,) is as follows.

An Inveſtigation of the two Expreſſions, $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, *and* $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, *given by Monſieur de* Lagney, *for a Second near Value of the Cube-root of a given Number* c, *when* a, *or a Firſt near Value of it that is already known, is leſs than its true Value.*

Art. 24. Let z be put for the unknown difference by which a, or the firſt near value of the cube-root of the given number c, falls ſhort of its true value; ſo that $a + z$ ſhall be $= \sqrt[3]{c}$.

Then will $\overline{a + z}|^3$ be $= c$. But $\overline{a + z}|^3$ is $= a^3 + 3aaz + 3azz + z^3$. Therefore $a^3 + 3aaz + 3azz + z^3$ will alſo be $= c$. This cubick equation is the foundation both of Mr. Raphſon's and of Monſieur de Lagney's methods of approximating further to the true value of $\sqrt[3]{c}$.

Art. 25. Mr. Raphſon's approximation is obtained as follows. Since z is leſs, and uſually much leſs, than a, to wit, about a 10th or a 100th part of it, or, perhaps, ſtill leſs,) it follows that both $3azz$ and z^3 will be leſs, and uſually much leſs, than $3aaz$, and conſequently that $a^3 + 3aaz$ will be nearly equal to $a^3 + 3aaz + 3azz + z^3$, and therefore to c. Let them therefore be ſuppoſed to be equal to c. Then, ſince $a^3 + 3aaz$ are $= c$, we ſhall have $3aaz = c - a^3$, and conſequently $z = \frac{c - a^3}{3aa}$, which fraction conſiſts intirely of known quantities. Therefore $a + z$ will be $= a + \frac{c - a^3}{3aa}$, and $a + \frac{c - a^3}{3aa}$ will be a ſecond near value of $\sqrt[3]{c}$, or the cube-root of the given number c. Q. E. I.

This is Mr. Raphſon's approximation to the cube-root of c, when a is leſs than the ſaid cube-root; and it is the ſimpleſt and eaſieſt approximation that can well be imagined, and approaches very conſiderably beyond a to the true value of $\sqrt[3]{c}$. For it uſually gives us twice as many figures exact as we had before in a, or the firſt near value of $\sqrt[3]{c}$. And it is leſs operoſe, or difficult to compute, than Mr. de Lagney's firſt, or rational, expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$; becauſe it is eaſier to divide $c - a^3$ by $3aa$, or three times the ſquare of a, than, firſt, to multiply $c - a^3$ by a, and then to divide the product by $c + 2a^3$, which is a longer number than $3aa$. And for theſe reaſons Mr. Raphſon, in the Appendix to the ſecond edition of his *Analyſis Æquationum Univerſalis*, (which was publiſhed in the year 1697, ſeveral years after the publication of Mr. de Lagney's method of approximation,) declares that he continued to prefer his own approximation to thoſe of Mr. de Lagney, notwithſtanding their greater exactneſs.

We will now proceed to inveſtigate Mr. de Lagney's firſt, or rational, expreſſion above-mentioned, in obtaining which Mr. Raphſon's approximation is made uſe of as a neceſſary ſtep.

Art. 26. Mr. de Lagney's firſt, or rational, expreſſion, $a + \frac{\overline{c-a^3} \times a}{c + 2a^3}$, is obtained by preſerving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 + 3aaz + 3azz + z^3 = c$, or by ſuppoſing $a^3 + 3aaz + 3azz$ to be equal to c, and reſolving the quadratick equation $a^3 + 3aaz + 3azz = c$, reſulting from that ſuppoſition, in an imperfect, or inaccurate, manner, by proceeding as follows.

Since $a^3 + 3aaz + 3azz$ is $= c$, we ſhall have $3aaz + 3azz = c - a^3$, and (dividing both ſides by $3aa + 3az$) $z = \frac{c - a^3}{3aa + 3az}$. Let us now ſubſtitute, inſtead of z, in the denominator of this fraction $\frac{c - a^3}{3aa + 3az}$, the near value of z already obtained by the reſolution of the ſimple equation $a^3 + 3aaz = c$, to wit, the fraction $\frac{c - a^3}{3aa}$; and we ſhall thereby obtain the equation $z = \frac{c - a^3}{3aa + 3a \times \frac{c - a^3}{3aa}}$; which laſt quantity is equal to $\frac{c - a^3}{3aa + \frac{c - a^3}{a}}$ $\left(= \frac{c - a^3}{\frac{3a^3 + c - a^3}{a}} = \frac{c - a^3}{\frac{c + 2a^3}{a}} = \overline{c - a^3} \times \frac{a}{c + 2a^3}\right) = \frac{\overline{c-a^3} \times a}{c + 2a^3}$. Therefore z will be $= \frac{\overline{c-a^3} \times a}{c + 2a^3}$, and conſequently $a + z$ will be $= a + \frac{c - a^3 \times a}{c + 2a^3}$. Therefore the true value of $a + z$, or the cube-root of the given number c, will be nearly $= a + \frac{\overline{c-a^3} \times a}{c + 2a^3}$. Q. E. I.

Art. 27. This expreſſion $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$ of the ſecond near value of $\sqrt[3]{c}$, will always be leſs than its true value; as may be demonſtrated in the manner following.

Since $c - a^3$ is $= 3aaz + 3azz + z^3$, and $3aaz + 3azz + z^3$ is $= 3aaz + azz + 2azz + z^3$, it follows that $c - a^3$ will be $= 3aaz + azz + 2azz + z^3$, and conſequently will be greater than $3aaz + azz$. Therefore $\frac{c-a^3}{a}$ will be greater than $\frac{3aaz + azz}{a}$, or than $3az + zz$. But $\frac{c-a^3}{a}$ is $= 3a \times \frac{c-a^3}{3aa}$. Therefore $3a \times \frac{c-a^3}{3aa}$ will be greater than $3az + zz$; and conſequently (adding $3aa$ to both ſides,) $3aa + 3a \times \frac{c-a^3}{3aa}$ will be greater than $3aa + 3az + zz$. Therefore $\frac{c-a^3}{3aa + 3a \times \frac{c-a^3}{3aa}}$ will be leſs than $\frac{c-a^3}{3aa + 3az + zz}$. But $\frac{c-a^3}{3aa + 3a \times \frac{c-a^3}{3aa}}$ is $= \frac{\overline{c-a^3}\times a}{c+2a^3}$, and $\frac{c-a^3}{3aa + 3az + zz}$ is $=$ the true value of z. Therefore $\frac{\overline{c-a^3}\times a}{c+2a^3}$ will be leſs than the true value of z. Therefore $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$ will be leſs than the true value of $a + z$, or than $\sqrt[3]{c}$.

Q. E. D.

This expreſſion, $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$, gives uſually three times as many figures of the value of $\sqrt[3]{c}$ exact as were given exactly by a, or the firſt near value of the ſaid cube-root. But in ſome caſes the figures which it gives exactly are only three times as many wanting one, and in ſome unfavourable caſes only three times as many wanting two, as were exact in a; as we have ſeen in ſome of the foregoing examples.

Art. 28. Mr. de Lagney's ſecond, or irrational, expreſſion, $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, for the ſecond near value of the cube-root of c, when a is leſs than the ſaid cube-root, is obtained by preſerving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 + 3aaz + 3azz + z^3 = c$, and reſolving the equation thence reſulting, to wit, the quadratick equation $a^3 + 3aaz + 3azz = c$, in an accurate manner. This may be done as follows.

If we ſuppoſe $a^3 + 3aaz + 3azz$ to be equal to c, we ſhall have $3aaz + 3azz = c - a^3$, and (dividing all the terms by $3a$) $az + zz = \frac{c - a^3}{3a}$. Therefore (adding $\frac{aa}{4}$ to both ſides,) we ſhall have $\frac{aa}{4} + az + zz = \frac{c - a^3}{3a} + \frac{aa}{4}$ $\left(= \frac{4c - 4a^3}{12a} + \frac{3a^3}{12a} = \frac{4c - 4a^3 + 3a^3}{12a}\right) = \frac{4c - a^3}{12a}$, and (extracting the ſquare-roots of both ſides,) $\frac{a}{2} + z = \sqrt{\frac{4c - a^3}{12a}}$. Therefore z will be $= \sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}$, and $a + z$ will be $\left(= a + \sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}\right) = \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Therefore $\sqrt[3]{c}$ will be nearly equal to the ſame quantity $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Q. E. I.

Art. 29. This expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ muſt always be greater than the true value of $\sqrt[3]{c}$, or than the true value of $a + z$ in the original cubick equation $a^3 + 3aaz + 3azz + z^3 = c$.

For it is derived from a ſuppoſition that $a^3 + 3aaz +$

$3azz$ is equal to c, or to $a^3 + 3aaz + 3azz + z^3$, or is greater than it really is: from which it will neceſſarily follow that the value of z deduced from that ſuppoſition muſt be greater than its true value, and conſequently that the value of $a + z$ deduced from that ſuppoſition, that is, the expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be greater than the true value of $a + z$, or than $\sqrt[3]{c}$. Q. E. D.

Art. 30. The irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will approach a little nearer than the rational expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ to the true value of $\sqrt[3]{c}$, becauſe it is obtained by reſolving the quadratick equation $a^3 + 3aaz + 3azz = c$ accurately, whereas the rational expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ is obtained by reſolving the ſame quadratick equation inaccurately, by ſubſtituting $\frac{c - a^3}{3aa}$ inſtead of z in the quantity $3az$ in the denominator of the fraction $\frac{c - a^3}{3aa + 3az}$ in art. 26. But the difference of the two expreſſions in point of exactneſs is not conſiderable; and the principal reaſon for preferring the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ to the rational expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, is, that there is much leſs labour in extracting the ſquare-root of the fraction $\frac{4c - a^3}{12a}$, than in dividing the numerator $\overline{c - a^3} \times a$ by the denominator $c + 2a^3$, when that denominator is a very long number.

An

An Inveſtigation of the Two Expreſſions, $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, *and* $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, *given by Monſieur de* Lagney *for a Second near Value of the Cube-root of a given Number* c, *when* a, *or a Firſt near Value of it that is already known, is greater than its true Value.*

Art. 31. Let z be put for the unknown difference by which a, or the firſt near value of the cube-root of the given number c, exceeds its true value; ſo that $a - z$ ſhall be $= \sqrt[3]{c}$.

Then will $\overline{a - z}|^3$ be $= c$. But $\overline{a - z}|^3$ is $= a^3 - 3aaz + 3azz - z^3$. Therefore $a^3 - 3aaz + 3azz - z^3$ will alſo be $= c$. This cubick equation is the foundation both of Mr. Raphſon's and of Monſieur de Lagney's methods of approximating further to the true value of $\sqrt[3]{c}$.

Art. 32. Mr. Raphſon's approximation is obtained as follows.

Since z is leſs, and uſually much leſs, than a, (to wit, about a 10th, or a 100th, part of it, or, perhaps, ſtill leſs,) it follows that both $3azz$ and z^3 will be leſs, and uſually much leſs, than $3aaz$, and conſequently that $a^3 - 3aaz$ will be nearly equal to $a^3 - 3aaz + 3azz - z^3$, and therefore to c. Let them therefore be ſuppoſed to be equal to c. Then, ſince $a^3 - 3aaz$ are $= c$, we ſhall have $a^3 = c + 3aaz$, and $3aaz = a^3 - c$, and conſequently $z = \frac{a^3 - c}{3aa}$. Therefore $a - z$ will be $= a - \left|\frac{a^3 - c}{3aa}\right.$, and conſequently the expreſſion $a - \left|\frac{\overline{a^3 - c}}{3aa}\right.$ will be a ſecond near value of $\sqrt[3]{c}$, or the cube-root of the given number c. Q. E. I.

This is Mr. Raphſon's approximation to the cube-root of c, when a is greater than the ſaid cube-root; and it is the ſimpleſt and eaſieſt approximation that can well be imagined, and approaches much nearer than a to the true value of the ſaid cube-root. For it uſually gives us the value of the ſaid cube-root exact to twice as many decimal places of figures as were exact in a, or the firſt near value of the ſaid cube-root. And it is leſs operoſe, or difficult to compute, than Mr. de Lagney's firſt, or rational, expreſſion, $a - \frac{\overline{a^3-c} \times a}{c + 2a^3}$; becauſe it is eaſier to divide $c - a^3$ by $3aa$, or three times the ſquare of a, than, firſt, to multiply $a^3 - c$ by a, and then to divide the product by $c + 2a^3$, which is a longer number than $3aa$. And for theſe reaſons Mr. Raphſon always preferred it to Mr. de Lagney's approximations, notwithſtanding their greater exactneſs.

We will now proceed to inveſtigate Mr. de Lagney's firſt expreſſion above-mentioned, to wit, the rational expreſſion $a - \frac{\overline{a^3-c} \times a}{c + 2a^3}$, in obtaining which Mr. Raphſon's approximation is made uſe of as a neceſſary ſtep.

Art. 33. Mr. de Lagney's firſt, or rational, expreſſion, $a - \frac{\overline{a^3-c} \times a}{c + 2a^3}$, is obtained by preſerving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, or by ſuppoſing $a^3 - 3aaz + 3azz$ to be equal to c, and reſolving the quadratick equation $a^3 - 3aaz + 3azz = c$, reſulting from that ſuppoſition, in an imperfect, or inaccurate, manner, by proceeding as follows.

Since $a^3 - 3aaz + 3azz$ is ſuppoſed to be equal to c, we ſhall have $a^3 + 3azz = c + 3aaz$, and $a^3 = c + 3aaz - 3azz$, and $a^3 - c = 3aaz - 3azz$, or $3aaz - 3azz = a^3 - c$, and (dividing both ſides of the equation by $3aa - 3az$,) $z = \frac{a^3 - c}{3aa - 3az}$. Mr. de Lagney then ſubſtitutes,

tutes, inſtead of z, in the denominator of the fraction $\frac{a^3 - c}{3aa - 3az}$, the near value of z already obtained by the reſolution of the ſimple equation $a^3 - 3aaz = c$, to wit, the fraction $\frac{a^3 - c}{3aa}$, and thereby obtains the equation $z = \frac{a^3 - c}{3aa - 3a \times \overline{\frac{a^3 - c}{3aa}}}$; which laſt quantity is equal to $\frac{a^3 - c}{3aa - \overline{\frac{a^3 - c}{a}}}$ $\left(= \frac{a^3 - c}{\frac{3a^3 - \overline{a^3 - c}}{a}} = \frac{a^3 - c}{\frac{3a^3 - a^3 + c}{a}} = \frac{a^3 - c}{\frac{2a^3 + c}{a}} = \overline{a^3 - c} \times \frac{a}{c + 2a^3}\right) = \frac{a^3 - c \times a}{c + 2a^3}$. Therefore z will be $= \frac{a^3 - c \times a}{c + 2a^3}$, and conſequently $a - z$ will be $= a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$. Therefore the true value of $a - z$, or of the cube-root of the given number c, will be nearly $= a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$.

Q. E. I.

Art. 34. This expreſſion, $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, of the ſecond near value of $\sqrt[3]{c}$, will always be greater than its true value; as may be demonſtrated in the manner following.

It has been ſhewn above in art. 31, that $a^3 - 3aaz + 3azz - z^3$ is $= c$. Therefore $a^3 + 3azz$ will be $= c + 3aaz + z^3$, and a^3 will be $= c + 3aaz - 3azz + z^3$, and $a^3 - c$ will be $= 3aaz - 3azz + z^3$. But $3aaz - 3azz + z^3$ is $= 3aaz - azz - 2azz + z^3$. Therefore $a^3 - c$ will be $= 3aaz - azz - 2azz + z^3$. But, becauſe z is leſs than a, z^3 will be leſs than azz, and, *à fortiori*, leſs than $2azz$. Therefore $3aaz - azz - 2azz + z^3$ will be leſs than $3aaz - azz$; and conſequently $a^3 - c$ (which is $= 3aaz - azz - 2azz + z^3$,) will be leſs than $3aaz - azz$. Therefore $\frac{a^3 - c}{a}$ will be leſs than $\frac{3aaz - azz}{a}$, or than $3az - zz$.

zz. But $\frac{a^3-c}{a}$ is $= 3a \times \overline{\frac{a^3-c}{3aa}}$. Therefore $3a \times \overline{\frac{a^3-c}{3aa}}$ will be leſs than $3az - zz$. Therefore, if both theſe quantities be ſubtracted from $3aa$, the remainder $3aa - 3a \times \overline{\frac{a^3-c}{3aa}}$ will be greater than the remainder $3aa - \overline{3az - zz}$, or than $3aa - 3az + zz$. Therefore $\frac{a^3-c}{3aa - 3a \times \overline{\frac{a^3-c}{3aa}}}$ will be leſs than $\frac{a^3-c}{3aa - 3az + zz}$. But $\frac{a^3-c}{3aa - 3a \times \overline{\frac{a^3-c}{3aa}}}$ is $= \frac{\overline{a^3-c} \times a}{c + 2a^3}$. Therefore $\frac{\overline{a^3-c} \times a}{c + 2a^3}$ will be leſs than $\frac{a^3-c}{3aa - 3az + zz}$. But $\frac{a^3-c}{3aa - 3az + zz}$ is equal to the value of z in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, or to its true value. Therefore $\frac{\overline{a^3-c} \times a}{c + 2a^3}$ will be leſs than the true value of z. And conſequently $a - \frac{\overline{a^3-c} \times a}{c + 2a^3}$ will be greater than the true value of $a - z$, or than the cube-root of c. Q. E. D.

Art. 35. The other, or irrational, expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, given by Mr. de Lagney for the ſecond near value of the cube-root of c, when a is greater than the ſaid cube-root, is obtained by preſerving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, obtained in art. 31, or by ſuppoſing the trinomial quantity $a^3 - 3aaz + 3azz$ to be equal to c, and reſolving the quadratick equation $a^3 - 3aaz + 3azz = c$ in an accurate manner. This may be done as follows.

If we ſuppoſe $a^3 - 3aaz + 3azz$ to be $= c$, we ſhall have $a^3 + 3azz = c + 3aaz$, and $a^3 = c + 3aaz - 3azz$, and $a^3 - c = 3aaz - 3azz$, or $3aaz - 3azz = a^3 - c$.

Therefore (dividing both ſides of the equation by $3a$,) we ſhall have $az - zz = \frac{a^3 - c}{3a}$.

The compound quantity $az - zz$, which forms the left-hand ſide of this equation, is $= \overline{a - z|} \times z$, and conſequently (by Euclid's Elements, Book 2d, Prop. 5,) muſt be leſs than the ſquare of $\frac{a}{2}$, or than $\frac{aa}{4}$. And conſequently the other ſide of the equation, or the quantity $\frac{a^3 - c}{3a}$, will alſo be leſs than $\frac{aa}{4}$. They may therefore both be ſubtracted from $\frac{aa}{4}$. Let them be ſo ſubtracted. And we ſhall then have $\frac{aa}{4} - \overline{|3z - zz} = \frac{aa}{4} - \overline{|\frac{a^3 - c}{3a}}$, or $\frac{aa}{4} - az + zz = \frac{aa}{4} - \overline{|\frac{a^3 - c}{3a}}\ (= \frac{3a^3}{12a} - \overline{|\frac{4a^3 - 4c}{12a}} = \frac{3a^3 - \overline{|4a^3 - 4c}}{12a} = \frac{3a^3 - 4a^3 + 4c}{12a}) = \frac{4c - a^3}{12a}$. Therefore the ſquare-root of the trinomial quantity $\frac{aa}{4} - az + zz$, will be equal to the ſquare-root of the fraction $\frac{4c - a^3}{12a}$. Now, if z could be of two different values, the one greater than $\frac{a}{2}$, and the other leſs than $\frac{a}{2}$, the trinomial quantity $\frac{aa}{4} - az + zz$ might have two ſquare-roots, to wit, $\frac{a}{2} - z$, and $z - \frac{a}{2}$. But, as z in the preſent problem is ſuppoſed to be much leſs than $\frac{a}{2}$, the latter of theſe ſquare-roots, to wit, $z - \frac{a}{2}$, cannot exiſt, and the other ſquare-root, $\frac{a}{2} - z$, will be the only one conſiſtent with the conditions of the Problem. Therefore we ſhall have $\frac{a}{2} - z = \sqrt{\frac{4c - a^3}{12a}}$, and conſequently (adding z to both ſides,) $\frac{a}{2} = \sqrt{\frac{4c - a^3}{12a}} + z$, and (ſubtracting $\sqrt{\frac{4c-a^3}{12a}}$ from both ſides,) $z = \frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$. Therefore $a - z$ will be $= a - \overline{\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}}$ $(= a - \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}})$ $= \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Therefore the ſecond near value of $\sqrt[3]{c}$ will be $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$. Q. E. I.

Art. 36. This expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will always be leſs than the true value of the cube-root of the given number c.

For, if we ſuppoſe z to increaſe continually from 0 till it becomes equal to $\frac{a}{2}$, the compound quantity $az - zz$, or $\overline{a - z} \times z$, will increaſe continually at the ſame time till it becomes equal to $\overline{\frac{a}{2}}\big|^{2}$, or $\frac{aa}{4}$; and conſequently the compound quantity $3a \times \overline{az - zz}$, or the compound quantity $3aaz - 3azz$, will increaſe continually till it becomes equal to $3a \times \frac{aa}{4}$, or $\frac{3a^3}{4}$. Therefore, when the compound quantity $3aaz - 3azz$ is equal to $a^3 - c$, the quantity z will be greater than when the compound quantity $3aaz - 3azz$ is equal to $a^3 - c - z^3$, which is leſs than $a^3 - c$; that is, the value of z in the quadratick equation $3aaz - 3azz = a^3 - c$ will be greater than the value of z in the cubick equation $3aaz - 3azz = a^3 - c - z^3$, or in the cubick equation $3aaz - 3azz + z^3 = a^3 - c$, or in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$. But the value of z in the quadratick equation $3aaz - 3azz =$

$a^3 - c$ is $\frac{a}{2} - \sqrt{\frac{4c-a^3}{12a}}$; and the value of z in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$ is the true value of z, or of the exceſs of a above $\sqrt[3]{c}$. Therefore the expreſſion $\frac{a}{2} - \sqrt{\frac{4c-a^3}{12a}}$ will be greater than the true value of z. Therefore $a - \overline{\frac{a}{2} - \sqrt{\frac{4c-a^3}{12a}}}$, or $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be leſs than the true value of $a - z$, or than the cube-root of the given number c. Q. E. D.

Art. 37. The irrational *Formula* $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ will approach a little nearer than the rational *Formula* $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ to the true value of the cube-root of c; becauſe the irrational *Formula* is derived from the quadratick equation $a^3 - 3aaz + 3azz = c$ by an accurate reſolution of it, and the rational *Formula* is derived from the ſame equation by an inaccurate reſolution of it. But the difference of exactneſs between theſe two expreſſions is not great, and either of them will uſually give us three times as many decimal figures of $\sqrt[3]{c}$ exact as were exact in a, or the firſt near value of it. But, when the given number c itſelf conſiſts of nine, or ten, or more figures, and alſo when a conſiſts of three, or four, figures, and conſequently a^3 conſiſts of nine, or ten, or more, figures, the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be found to be much eaſier to compute than the rational expreſſion $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, on account of the labour of dividing $\overline{a^3 - c} \times a$ by the long number $c + 2a^3$; and therefore Dr. Halley thinks it ought to be preferred to the other.

Art. 38. I have now given very full inveſtigations of the four expreſſions invented by Mr. de Lagney for the ſecond near value of the cube-root of a given number c, of which a firſt near value, denoted by the letter a, is already known, to wit, the four expreſſions $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{c+2a^3}}$, and $a - \frac{\overline{a^3-c}\times a}{c+2a^3}$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$; of which the two firſt relate to the caſe in which a, or the firſt near value of the ſaid cube-root, is leſs than its true value; and the two laſt relate to the caſe in which a, or the firſt near value of the ſaid cube-root, is greater than its true value. And I have given demonſtrations of what is aſſerted concerning theſe four expreſſions in art. 5 and 6, to wit, that the firſt of them, or $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$, is always leſs than the true value of $\sqrt[3]{c}$, and that the ſecond of them, or $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, is always greater than the ſaid true value, and that the third expreſſion $a - \frac{\overline{a^3-c}\times a}{c+2a^3}$ is always greater than the ſaid true value, and that the fourth expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ is always leſs than the ſaid true value. And the two firſt of theſe aſſertions have been confirmed upon trial in the examples given in art. 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20, of the extraction of the cube-roots of the numbers 2, 231, 37,945, and the long number 696,536,483,318,640,035,073,641,037; the ſaid cube-roots having been extracted by means of the two firſt expreſſions $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, which relate to the caſe in which a, or the firſt near value of $\sqrt[3]{c}$, is leſs than its true value. I will now, therefore give an example, or two, of the extraction of the cube-roots of numbers by means of the two latter expreſſions

$a =$

$a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, which relate to the caſe in which a, or the firſt near value of $\sqrt[3]{c}$, is greater than its true value.

An Example of the Extraction of the Cube-root of a given Number, by means of Mr. de Lagney's *Third and Fourth Expreſſions* $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ *and* $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, *in which* a, *or the Firſt near Value of* $\sqrt[3]{c}$, *is ſuppoſed to be greater than its true Value.*

Art. 39. Let it be required to find the cube-root of 2, which was extracted above in art. 8 and 9; and let a, or the firſt near value of the ſaid cube-root, be 1.26, which is ſomewhat greater than its true value, which is 1.259,921, 049, &c.

Here then we have $c = 2$, and $a = 1.26$. Therefore a^3 will be $= 2.000,376$, and $2a^3$ will be $= 4.000,752$, and $c + 2a^3$ will be $(= 2 + 4.000,752) = 6.000,752$, and $a^3 - c$ will be $(= 2.000,376 - 2) = 0.000,376$, and $\overline{a^3 - c} \times a$ will be $(= 0.000,376 \times 1.26) = 0.000,473,76$, and conſequently $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$ will be $(= \frac{0.000,473,76}{6.000,752}) = 0.000,078,950$. Therefore $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, or the ſecond near value of the cube-root of 2, will be $(= 1.260,000,000 - 0.000,078,950) = 1.259,921,050$; which is a little greater than its true value 1.259,921,049, &c, agreeably to what is aſſerted in art. 6, and demonſtrated in art. 34.

And we ſhall have $4c$ $(= 4 \times 2) = 8.000,000$, and $\frac{a}{2}$ $(= \frac{1.26}{2}) = 0.63$, and $12a$ $(= 12 \times 1.26) = 15.12$, and $4c - a^3$ (= 8.000,000 — 2.000,376) = 5.999,624, and $\frac{4c-a^3}{12a}$ $\left(= \frac{5.999,624}{15.12}\right)$ = 0.396,800,529,100,529,100,529, &c, and $\sqrt{\frac{4c-a^3}{12a}}$ (= $\sqrt{}$0.396,800,529,100,529,100,529, &c,) = 0.629,921,049,894,76, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ (= 0.630,000,000,000,00 + 0.629,921,049,894,76) = 1.259,921,049,894,76. Therefore the ſecond near value of the cube-root of 2, obtained by the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be 1.259,921,049,894,76; which is a little leſs than the true value of the ſaid cube-root, agreeably to what is aſſerted in art. 6, and demonſtrated in art. 36, the more accurate value of the ſaid cube-root being 1.259,921,049,894,873,164,76, &c.

Art. 40. The foregoing more accurate value of the cube-root of 2, to wit, 1.259,921,049,894,873,164,76, was obtained by taking 1.259,921,0 for a, or the firſt near value of the ſaid cube-root, and computing the expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$.

For, if a is = 1.259,921,0, or 1.259,921, we ſhall have $\frac{a}{2}$ = 0.629,960,5, and a^3 (= $\overline{1.259,921}|^3$) = 1.999,999,762,390,486,961, and $4c$ (= 4 × 2) = 8.000,000,000,000,000,000, and $4c - a^3$ = 8.000,000,000,000,000,000
— 1.999,999,762,390,486,961
= 6.000,000,237,609,513,039,
and $12a$ (= 12 × 1.259,921) = 15.119,052, and conſequently $\frac{4c-a^3}{12a}$ $\left(= \frac{6.000,000,237,609,513,039}{15.119,052}\right)$ = 0.396,850,294,423,850,982,125,069,746,436, and $\sqrt{\frac{4c-a^3}{12a}}$ (= $\sqrt{}$ 0.396850,294423,850982,125069,746436) = 0.629,960,549,894,873,164,76, &c. Therefore $\frac{a}{2}$ +

$\surd$

$\sqrt{\frac{4c-a^3}{12a}}$ will be ($=$ 0.629,960,5 + 0.629,960,549,894, 873,164,76, &c,) $=$ 1.259,921,049,894,873,164,76, &c, which therefore will be a very near value of the cube-root of 2.

All the twenty-one figures of this number 1.259,921, 049,894,873,164,76, may be depended upon as exact, if no miſtake has been made in computing the value of the expreſſion $\sqrt{\frac{4c-a^3}{12a}}$; becauſe a, or the firſt value of the cube-root of 2, to wit, 1.259,921,0, conſiſts of eight figures which are all exact, and the number of figures that are exact in $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ is always triple, or triple wanting one or two figures, of the number of figures that are exact in a; as was obſerved in art. 7.

Art. 41. And in like manner in the ſecond example, relating to the extraction of the cube-root of 231, if we take a equal to 6.14 (which is ſomewhat greater than the true value of the ſaid cube-root,) inſtead of taking it equal to 6.13, (which is ſomewhat leſs than the ſaid true value,) and compute the two expreſſions $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2}$ + $\sqrt{\frac{4c-a^3}{12a}}$, we ſhall find the former of theſe expreſſions to be ſomewhat greater, and the latter of them to be ſomewhat leſs, than the true value of the ſaid cube-root, agreeably to what is aſſerted in art. 6. Theſe computations may be performed as follows.

If a is ſuppoſed to be $=$ 6.14, we ſhall have $\frac{a}{2}$ ($= \frac{6.14}{2}$) $=$ 3.07, and $12a$ ($=$ 12 × 6.14) $=$ 73.68, and a^3 ($= \overline{6.14|}^3$) $=$ 231.475,544, and $a^3 - c$ ($=$ 231.475,544 $-$ 231) $=$ 0.475,544, and $\overline{a^3 - c|} \times a$ ($=$ 0.475,544 × 6.14) $=$ 2.919,840,16, and $2a^3$ ($=$ 2 × 231.475,544) $=$ 462.

462.951,088, and $c + 2a^3$ (= 231 + 462.951,088) = 693.951,088, and conſequently $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$ $\left(= \frac{2.919,840,16}{693.951,088}\right)$ = 0.004,207,559. Therefore $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ will be (= 6.140,000,000 − 0.004,207,559) = 6.135,792,441; which is greater than the true value of the cube-root of 231, agreeably to what is aſſerted in art. 6, and demonſtrated in art. 34, the ſaid true value being only 6.135,792, 439,661,958, &c. See above, art. 10, page 476.

And we ſhall have $4c$ (= 4 × 231) = 924, and $4c - a^3$ = 924.000,000 − 231.475,544) = 692.524,456, and $\frac{4c - a^3}{12a}$ $\left(= \frac{692.524,456}{73.68}\right)$ = 9.399,083,279,044,516,829, and $\sqrt{\frac{4c - a^3}{12a}}$ (= √9.399083,279044,516829) = 3.065,792, 439,004. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be (= 3.07 + 3.065,792,439,004,) = 6.135,792,439,004; which is leſs than the true value of the cube-root of 231, to wit, 6.135,792,439,661,958, &c, agreeably to what is aſſerted in art. 6, and demonſtrated in art. 36.

Art. 42. Theſe two examples are ſufficient to illuſtrate and confirm what is aſſerted in art. 6, and demonſtrated in art. 34 and 36, concerning the two expreſſions $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$; which are given by Monſieur de Lagney for a ſecond near value of the cube-root of a given number c, when a, or the firſt near value of it, is greater than its true value. And with them I ſhall conclude the preſent tract.

End of the Tract on the Cubes of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, &c, and on Mr. de Lagney's *Method of Extracting the Cube-roots of Numbers by Approximation.*

A GENE-

A

GENERAL METHOD

OF

EXTRACTING THE ROOTS OF NUMBERS

BY

APPROXIMATION;

INVENTED BY

MONSIEUR DE LAGNY,

A MEMBER OF THE ROYAL ACADEMY OF SCIENCES AT PARIS,

AND PUBLISHED IN THE YEAR 1697,

IN HIS

Nouveaux Elèments d'Arithmétique et d'Algébre.

A
GENERAL METHOD
OF
EXTRACTING THE ROOTS OF NUMBERS
BY
APPROXIMATION.

Article 1. IN the foregoing Tract I have inſerted Monſieur de Lagny's Method of Extracting the Cube-roots of Numbers by Approximation, and have both given full inveſtigations of it, and illuſtrated it by ſeveral examples that clearly prove its great utility. But Mr. de Lagny did not confine this method to the extraction of the cube-roots of numbers, but extended it to the extraction of their fifth roots, and their ſeventh roots, and all higher roots of them whatſoever. This he did by purſuing the ſame principle by which he had before been enabled to find his approximations to the cube-root of a given number, to wit, by conſidering the conſtitution of the compound quantity that is equal to any given power of a binomial quantity, (ſuch as $a + b$,) or of a reſidual quantity, (ſuch as $a - b$,) and ſubſtituting the ſum, or difference, between a, the firſt near value of the root ſought, (which is ſuppoſed to be already known,) and z, its unknown difference from the true value of the ſaid root ſought, inſtead of the ſaid true value itſelf in the original equation derived from the conditions of the Problem, and then reſolving the new equation, reſulting from ſuch ſubſtitution, as if it were a quadratick equation, or neglecting all the terms of it which involve any higher powers of its root, or the difference z, than the ſquare. This method I ſhall now endeavour to explain in the ſolutions of the two following Problems.

PROBLEM I.

Art. 2. Let N be any propoſed number whatſoever, and m any propoſed whole number whatſoever; and let a be a known number that is nearly equal to, but leſs than, the mth root of the given number N. It is required to find a ſecond near value of the ſaid mth root of the given number N, that ſhall approach much nearer to it than a, or the former near value of it that is already known.

SOLUTION.

Let z be put for the unknown difference between a, the firſt near value of the mth root of the given number N, and the true value of the ſaid number. Then, ſince a is ſuppoſed to be leſs than $\sqrt[m]{N}$, it follows that $a + z$ will be $= \sqrt[m]{N}$, and conſequently that $\overline{a+z}|^{m}$ will be $= N$.

But, by the binomial theorem in the firſt and ſimpleſt caſe of it, to wit, the caſe of integral powers, $\overline{a+z}|^{m}$ will be $=$ the ſeries $a^{m} + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^{2}$

$+$

$+ m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} z^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} z^4 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times a^{m-5} z^5 +$ &c, continued to $m + 1$ terms; or, if, for the ſake of brevity, we ſubſtitute the capital letters A, B, C, D, E, F, &c, inſtead of the ſeveral numeral co-efficients 1, m, $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, and $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, reſpectively, $\overline{a + z}|^{m}$ will be $=$ the ſeries $A a^{m} + B a^{m-1} z + C a^{m-2} z^2 + D a^{m-3} z^3 + E a^{m-4} z^4 + F a^{m-5} z^5 +$ &c, continued to $m + 1$ terms. Therefore the ſaid ſeries $A a^{m} + B a^{m-1} z + C a^{m-2} z^2 + D a^{m-3} z^3 + E a^{m-4} z^4 + F a^{m-5} z^5 +$ &c, continued to $m + 1$ terms, will be $= N$. This is the original equation, by the ſolution of which we are to find a near value of z, and conſequently a ſecond near value of $a + z$, or $\sqrt[m]{N}$.

Art. 3. Now, ſince z is leſs than a, and uſually much leſs, being about a 10th or 100th part of it, or ſome ſtill leſſer part of it, it is evident that all the terms in the aforeſaid ſeries that involve zz, and z^3, and z^4, and the following powers of z, will be leſs, and uſually much leſs, than the term $B a^{m-1} z$, which involves only the ſimple power of z. And therefore, if all the ſaid terms of the ſeries be neglected or omitted, the two firſt terms alone, to wit, $A a^{m} + B a^{m-1} z$, will be nearly equal to the given number N; and conſequently (if we ſubtract $A a^{m}$ from both ſides of the equation,) we ſhall have $B a^{m-1} z$ nearly $= N - A a^{m}$, and (dividing both ſides by $B a^{m-1}$) z nearly $=$

$= \frac{N - Aa^m}{Ba^{m-1}}$, or (becauſe A is $= 1$, and B is $= m$) z nearly $= \frac{N - a^m}{ma^{m-1}}$; which fraction may be derived from the known quantities N and a by the operations of Multiplication, Subtraction, and Diviſion. This therefore is an approximation to the true value of z, and conſequently $a + \frac{N - a^m}{ma^{m-1}}$ will be an approximation to the true value of $a + z$, or of $\sqrt[m]{N}$, or will be a ſecond near value of it that will approach nearer to it than a, or the firſt near value of it which was already known. But it will evidently be ſomewhat greater than the true value of $\sqrt[m]{N}$; becauſe it aroſe from a ſuppoſition that $Aa^m + Ba^{m-1}z$ were equal to the whole ſeries of which they are only the two firſt terms, or that they were greater than they really were.

This quantity, $a + \frac{N - a^m}{ma^{m-1}}$, is the expreſſion given by Mr. Raphſon for the ſecond near value of the mth root of the given number N. And it is a very uſeful approximation: for it uſually gives us twice as many figures of the true value of $\sqrt[m]{N}$ exact as were exact in a, or the firſt near value of the ſaid mth root. And it is evidently the moſt ſimple and eaſy approximation to the value of the ſaid mth root that can well be imagined.

Art. 4. But Mr. de Lagny, being deſirous of finding at once a ſtill nearer value of the mth root of the number N, retains the third term $Ca^{m-2}z^2$, as well as the two firſt terms Aa^m and $Ba^{m-1}z$, of the ſeries $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 + Da^{m-3}z^3 + Ea^{m-4}z^4 + Fa^{m-5}z^5 +$ &c, (which is equal to N,) and thereby converts the original equation $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 + Da^{m-3}z^3 + Ea^{m-4}z^4 + Fa^{m-5}z^5 +$ &c, $= N$ into a quadratick equation, to wit, the equation $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 = N$, inſtead of converting it (as Mr. Raphſon does,) into the ſimple equation $Aa^m + Ba^{m-1}z = N$. And this quadratick equation he reſolves firſt imperfectly, or inaccurately, by ſubſtituting in one of its terms, inſtead of z, the inaccurate value of z already found by the reſolution of the ſimple equation $Aa^m + Ba^{m-1}z = N$, to wit, the fraction $\frac{N-a^m}{ma^{m-1}}$, (by which ſubſtitution the quadratick equation is reduced to a ſimple equation,) and then reſolving the ſaid ſimple equation thereby obtained; which produces a ſecond value of z that is nearer than the former value $\frac{N-a^m}{ma^{m-1}}$ to its true value. And this gives him a rational expreſſion for the value of $a + z$, or the ſecond near value of $\sqrt[m]{N}$. And then he reſolves the ſame quadratick equation, $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2$, accurately, by the common methods of reſolving quadratick equations; which produces a ſurd, or irrational, expreſſion for the value of z, and conſequently another ſurd, or irrational, expreſſion for the value of $a + z$, or for the ſecond near value of $\sqrt[m]{N}$. Theſe reſolutions of the ſaid quadratick equation $Aa^m + Ba^{m-1}z - Ca^{m-2}z^2 = N$, may be performed in the following manner.

Art. 5. Since $Aa^m + Ba^{m-1}z + Ca^{m-2}$ is $= N$, we ſhall have $Ba^{m-1}z + Ca^{m-2}z^2 = N - Aa^m$; that is, $z \times Ba^{m-1} + z \times C \times a^{m-2}z$ will be $= N - Aa^m$, or $z \times \overline{Ba^{m-1} + C \times a^{m-2}z}$ will be $= N - Aa^m$. Therefore (dividing both ſides of the equation by the compound quantity $Ba^{m-1} + C \times a^{m-2}z$) we ſhall have $z = \frac{N - Aa^m}{Ba^{m-1} + Ca^{m-2}z}$.

Now let $\frac{N - Aa^m}{Ba^{m-1}}$, or $\frac{N - a^m}{ma^{m-1}}$, (which has already been ſhewn to be nearly equal to z,) be ſubſtituted inſtead of z in the ſecond term $Ca^{m-2}z$ of the denominator of the fraction laſt obtained, to wit, the fraction $\frac{N - Aa^m}{Ba^{m-1} + Ca^{m-2}z}$. And we ſhall have $z = \frac{N - Aa^m}{Ba^{m-1} + Ca^{m-2} \times \overline{\frac{N - a^m}{ma^{m-1}}}}$, or (becauſe A is $= 1$, and B is $= m$,)

$$z = \frac{N - a^m}{ma^{m-1} + Ca^{m-2} \times \overline{\frac{N - a^m}{ma^{m-1}}}}; \text{ which is}$$

$$= \frac{N - a^m}{\frac{m^2a^{2m-2} + Ca^{m-2} \times \overline{N - a^m}}{ma^{m-1}}}$$

$$= \frac{N - a^m}{\frac{m^2a^{2m-2} + CNa^{m-2} - Ca^{2m-2}}{ma^{m-1}}} =$$

$$\overline{N - a^m} \times \frac{ma^{m-1}}{m^2a^{2m-2} + CNa^{m-2} - Ca^{2m-2}} =$$

$$= \frac{\overline{N - a^m} \times m a^{m-1}}{m^2 a^{2m-2} + C N a^{m-2} - C a^{2m-2}}$$

$$= \frac{\overline{N - a^m} \times 2m a^{m-1}}{2m^2 a^{2m-2} + 2C N a^{m-2} - 2C a^{2m-2}}$$ = (becauſe C is $= m \times \frac{m-1}{2}$, and conſequently 2C is $= m \times \overline{m-1}$, which therefore may be ſubſtituted for it,)

$$\frac{\overline{N - a^m} \times 2m a^{m-1}}{2m^2 a^{2m-2} + m \times \overline{m-1} \times N a^{m-2} - m \times \overline{m-1} \times a^{2m-2}}$$

= (if we divide both the numerator and the denominator by *m*,) $\frac{\overline{N - a^m} \times 2a^{m-1}}{2m a^{2m-2} + \overline{m-1} \times N \times a^{m-2} - \overline{m-1} \times a^{2m-2}}$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{\overline{2m - \overline{m-1}} \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}}$$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{\overline{2m - m + 1} \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}}$$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{\overline{m+1} \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}}$$ = (if we multiply both the numerator and denominator into *a*,)

$$\frac{\overline{N - a^m} \times 2a^{m}}{\overline{m+1} \times a^{2m-1} + \overline{m-1} \times N \times a^{m-1}}$$ = (by dividing both the numerator and the denominator of this fraction by a^{m-1},) $\frac{\overline{N - a^m} \times 2a}{\overline{m+1} \times a^m + \overline{m-1} \times N}$, or $\frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} + a^m}$.

Therefore *z* will be $= \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, and conſe-

quently

quently $a + z$ will be $= a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, or $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ will be a ſecond near value of $a + z$, or of the mth root of the propoſed number N.

Q. E. I.

Art. 6. The quadratick equation mentioned above in art. 4, to wit, the equation $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 = N$, or (becauſe A is $= 1$, and B is $= m$, and C is $= m \times \frac{m-1}{2}$,) the equation $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}z^2 = N$, may be accurately reſolved in the manner following.

By doubling both ſides of this equation we ſhall have $2a^m + 2ma^{m-1}z + m \times \overline{m-1} \times a^{m-2}zz = 2N$; and, by multiplying both ſides of this equation into aa, we ſhall have $2a^{m+2} + 2ma^{m+1}z + m \times \overline{m-1} \times a^m \times zz = 2a^2N$; and, by ſubtracting $2a^{m+2}$ from both ſides, we ſhall have $2ma^{m+1}z + m \times \overline{m-1} \times a^m \times zz = 2aaN - 2a^{m+2}$; and, by dividing both ſides of this laſt equation by $m \times \overline{m-1} \times a^m$, we ſhall have $\frac{2ma^{m+1}z}{m \times \overline{m-1} \times a^m} + zz = \frac{2aaN - 2a^{m+2}}{m \times \overline{m-1} \times a^m}$, or $\frac{2a^{m+1}z}{\overline{m-1} \times a^m} + zz = \frac{2aaN - 2a^{m+2}}{m \times \overline{m-1} \times a^m}$, or (becauſe $\frac{a^{m+1}}{a^m}$ is $= a$,) $\frac{2az}{m-1} +$

$zz = \frac{2aa\text{N} - 2a^{m+2}}{m \times \overline{m-1} \times a^{m}}$, or $\frac{2a}{m-1} \times z + zz = \frac{2\text{N} - 2a^{m}}{m \times \overline{m-1} \times a^{m-2}}$,

or $\frac{2a}{m-1} \times z + zz = \frac{2 \times \overline{\text{N} - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}$.

Now let the ſquare of $\frac{a}{m-1}$ (which is half the co-efficient of z in the term $\frac{2az}{m-1}$) be added to both ſides of this equation. And we ſhall have $\overline{\frac{a}{m-1}}|^{2} + \frac{2a}{m-1} \times z + zz = \overline{\frac{a}{m-1}}|^{2} + \frac{2 \times \overline{\text{N} - a^{m}}}{m \times \overline{m-1} \times a^{m-2}} = \frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{\text{N} - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}$; or, if, for the ſake of brevity, we put P for the quantity $\frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{\text{N} - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}$, we ſhall have $\overline{\frac{a}{m-1}}|^{2} + \frac{2a}{m-1} \times z + zz = \text{P}$. Therefore, (extracting the ſquare-roots of both ſides,) we ſhall have $\frac{a}{m-1} + z = \surd\text{P}$, and $z = \surd\text{P} - \frac{a}{m-1}$. Therefore $a + z$, or $\surd^{m}\overline{|\text{N}}$, will be $= a + \surd\text{P} - \frac{a}{m-1}$, or $a + \surd\overline{\left|\frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{\text{N} - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}\right.} - \frac{a}{m-1}$. Q. E. I.

Art. 7. This irrational expreſſion

$$a + \surd\overline{\left|\frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{\text{N} - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}\right.} - \frac{a}{m-1}\text{, or } a +$$

$\surd\text{P} - \frac{a}{m-1}$, will approach ſomewhat nearer to the true value of $\surd^{m}\overline{|\text{N}}$ than the former, or rational, expreſſion,

$a +$

$a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$; becauſe it proceeds from the accurate reſolution of the quadratick equation $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}zz = N$, whereas the rational expreſſion $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ was derived from an inaccurate reſolution of the ſame quadratick equation. But the difference of the two expreſſions, in point of exactneſs, is not great; and either of them will uſually give us three times as many decimal figures of the true value of $\sqrt[m]{N}$ exact as were exact in a, or the preceeding near value of it.

Examples of the Extraction of the Roots of Numbers by Means of the foregoing Expreſſions.

EXAMPLE I.

Art. 8. Let it be required to find the cube-root of the number 2, having 1.259 for a, or the firſt near value of the ſaid cube-root.

Here N is = 2, and m is = 3, and conſequently $m - 1$ is (= 3 − 1) = 2, and $m + 1$ is (= 3 + 1) = 4, and the expreſſion $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ becomes $= a + \frac{2a \times \overline{N - a^3}}{2N + 4a^3}$, or $a + \frac{a \times \overline{N - a^3}}{N + 2a^3}$, or $a + \frac{\overline{N - a^3} \times a}{N + 2a^3}$, or, (if we ſubſtitute the ſmall letter s inſtead of the capital letter N,) N,) $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$; which is the firſt, or rational, expreſſion, given in the preceeding tract for the cube-root of the number c. And as a is ſuppoſed to be $= 1.259$, this expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $= 1.259 + \frac{\overline{2 - \overline{1.259}|^3} \times 1.259}{2 + 2 \times \overline{1.259}|^3}$ $= 1.259 + \frac{\overline{2 - 1.995,616,979} \times 1.259}{2 + 2 \times 1.995,616,979} = 1.259 + \frac{0.004,383,021 \times 1.259}{2 + 3.991,233,958} = 1.259 + \frac{0.005,518,223,439}{5.991,233,958} = 1.259$ $+ 0.000,921,049,55$, &c $= 1.259,921,049,55$, &c. Therefore $1.259,921,049,55$, &c, will be a near value of the cube-root of 2. Q. E. I.

See the preceeding Tract, page 474.

Art. 9. And if we compute this cube-root by means of the irrational expreſſion $a + \sqrt{}P - \frac{a}{m-1}$, or $a - \frac{a}{m-1} + \sqrt{}P$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}}$, we ſhall find that this expreſſion will, upon making the proper ſubſtitutions, co-incide with the irrational expreſſion given for the cube-root of a given number c in the foregoing tract, to wit, the expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, and conſequently give the ſame value of the cube root of 2, as was obtained in the foregoing tract by means of that expreſſion.

For, ſince m is $= 3$, we ſhall have $m - 1 = 2$, and $\overline{m-1}|^2 = 4$, and $m \times \overline{m-1}$ $(= 3 \times 2) = 6$, and $m - 2$ $(= 3 - 2) = 1$, and $a^{m-2} = a$, and conſequently $\frac{aa}{\overline{m-1}|^2} = \frac{aa}{4}$, and $\frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}} = \frac{2 \times \overline{N - a^3}}{6 \times a} = \frac{N - a^3}{3a}$, and P, or $\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N-a^m}}{m \times \overline{m-1} \times a^{m-2}}$ $(= \frac{aa}{4} + \frac{N-a^3}{3a} = \frac{3a^3}{12a} + \frac{4N-4a^3}{12a} = \frac{3a^3+4N-4a^3}{12a}) = \frac{4N-a^3}{12a}$, or, (if we ſubſtitute c inſtead of N, in order to adopt the notation uſed in the foregoing tract,) $P = \frac{4c-a^3}{12a}$, and $a - \frac{a}{m-1} + \sqrt{P}$ $(= a - \frac{a}{2} + \sqrt{P} = \frac{a}{2} + \sqrt{P}) = \frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$; which is the irrational expreſſion given in the foregoing tract for the ſecond near value of the cube-root of the number c. So that the general, irrational, expreſſion $a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N-a^m}}{m \times \overline{m-1} \times a^{m-2}}}$, which is obtained by the general inveſtigation of the mth root of the given number N in the preſent tract, agrees perfectly with the particular, irrational, expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, which was obtained in the foregoing tract, by the particular inveſtigation of the cube-root of the given number c.

And, ſince c is $= 2$, and a is $= 1.259$, we ſhall have a^3 $(= \overline{1.259}|^3) = 1.995,616,979$, and $4c - a^3$ $(= 8.000,000,000 - 1.995,616,979) = 6.004,383,021$, and $12a$ $(= 12 \times 1.259) = 15.108$, and $\frac{4c-a^3}{12a}$ $(= \frac{6.004,383,021}{15.108}) = 0.397,430,700,3$, and $\sqrt{\frac{4c-a^3}{12a}}$ $(= \sqrt{0.397430,7003}) = 0.630,421,050,01$, and $\frac{a}{2}$ $(= \frac{1.2590}{2}) = 0.6295$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ $(= 0.6295 + 0.630,421,050,01) = 1.259,$

921,050,01. Therefore 1.259,921,050,01 will be a ſecond near value of the cube-root of 2. Q. E. I.

The firſt eight figures 1.259,921,0, of this number 1.259,921,050,01, are exact, the more accurate value of the cube-root of 2 being 1.259,921,049,894,873,164,76, &c, of which I believe all the figures to be exact. See above, pages 502 and 503.

EXAMPLE II.

Art. 10. Let it be required to find the fifth root of the number 2, which the celebrated *Vieta*, or *Monſieur Viete*, has found to be 1.148,697.

Here m is $= 5$, and conſequently $m - 1$ is $= 4$, and $m + 1$ is $= 6$, and the firſt general expreſſion $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ will be $(= a + \frac{2a \times \overline{N - a^5}}{4N + 6a^5}) = a + \frac{a \times \overline{N - a^5}}{2N + 3a^5}$, or $a + \frac{\overline{N - a^5} \times a}{2N + 3a^5}$; and the other general expreſſion $a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}}$, will be $(= a - \frac{a}{4} + \sqrt{\frac{aa}{16} + \frac{2 \times \overline{N - a^5}}{5 \times 4 \times a^3}} = \frac{3a}{4} + \sqrt{\frac{aa}{16} + \frac{N - a^5}{10a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5}{160a^3} + \frac{16N - 16a^5}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5 + 16N - 16a^5}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{16N - 6a^5}{160a^3}}) = \frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$. We muſt therefore compute one of the two expreſſions $a + \frac{a \times \overline{N - a^5}}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, in order to obtain a ſecond near value of the fifth root of N, or 2, when we ſhall have firſt found *a*, or a firſt near value of the ſaid fifth root that is leſs than its true value, to a ſmall degree of exactneſs.

Art. 11. The value of *a*, or the firſt approximation to the fifth root of 2, may be found in the following manner.

The fifth root of 2 is the ſecond of ſix quantities in continued geometrical proportion, of which 1 is the firſt, or leaſt, and 2 is the ſixth, or greateſt. Now the exceſs of the ſecond of theſe proportionals above the firſt, is neceſſarily leſs than the fifth part of the exceſs of the greateſt above the leaſt, becauſe the exceſſes increaſe in the ſame proportion as the terms themſelves. The exceſs of the greateſt of theſe ſix terms, to wit, 2, above the leaſt of them, or 1, is 1; of which the fifth part is $\frac{1}{5}$, or $\frac{2}{10}$, or 0.2. Therefore the exceſs of the ſecond term above the firſt will be leſs than $\frac{2}{10}$, or 0.2; and conſequently the ſecond term itſelf will be leſs than $1 + \frac{2}{10}$, or than 1.2. Let us therefore ſuppoſe this ſecond term, or the fifth root of 2, which we are ſeeking, to be equal to 1.1; and let us raiſe this number to the fifth power, in order to try how nearly it approaches to the truth.

Now, if we raiſe 1.1 to the fifth power, or multiply it four times into itſelf, we ſhall find that $\overline{1.1}|^5$ is $=$ 1.61051; which is conſiderably leſs than 2. Therefore 1.1 muſt be conſiderably leſs than the true value of the 5th root of 2. But we have ſeen that the ſaid true value muſt be leſs than 1.2. Let us therefore ſuppoſe it to be equally diſtant from 1.1 and 1.2, or to be $=$ 1.15, and try whether this will not be pretty near the truth.

Now $\overline{1.15}|^5$ is $=$ 2.011,357,187,5; which is a little bigger than 2. Therefore 1.15 muſt be ſomething greater than the 5th root of 2. But the difference can be but ſmall.

We will therefore, in the next place, ſuppoſe the ſaid 5th root to be = 1.14, and raiſe this number to the fifth power, in order to diſcover whether the ſaid fifth power will be greater, or leſs, than 2, and conſequently whether 1.14 will be greater, or leſs, than the fifth root of 2.

Now, if we multiply 1.14 four times ſucceſſively into itſelf, we ſhall find that $\overline{1.14}|^5$ is = 1.925,414,582,4; which is ſomewhat leſs than 2. Therefore the 5th root of 2 will be greater than 1.14, but leſs than 1.15; and therefore 1.14 will be a very convenient firſt near value of the 5th root of 2, and will be very fit to be made the baſis of a further approximation to the true value of the ſaid 5th root, by ſubſtituting it inſtead of a in either of the two expreſſions $a + \frac{\overline{N - a^5}| \times a}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, which have been derived from the foregoing Problem.

Art. 12. Now, if we ſuppoſe a to be = 1.14, we ſhall have aa = 1.2996, and a^3 = 1.481,544, and a^5 = 1.925,414,582,4, and conſequently N − a^5 (= 2.000,000,000,0 − 1.925,414,582,4) = 0.074,585,417,6, and $\overline{N - a^5}| \times a$ (= 0.074,585,417,6 × 1.14) = 0.085,027,376,064, and 2N (= 2 × 2) = 4, and $3a^5$ (= 3 × 1.925,414,582,4) = 5.776,243,747,2, and 2N + $3a^5$ (= 4 + 5.776,243,747,2) = 9.776,243,747,2, and $\frac{\overline{N - a^5}| \times a}{2N + 3a^5}$ (= $\frac{0.085{,}027{,}376{,}064}{9.776{,}243{,}747{,}2}$) = 0.008,697,34. Therefore $a + \frac{\overline{N - a^5}| \times a}{2N + 3a^5}$ will be (= 1.14 + 0.008,697,34) = 1.148,697,34; and conſequently 1.148,697,34 will be very nearly equal to the 5th root of 2. Q. E. I.

This number 1.148,697,34 agrees with that found by *Vieta*, to wit, 1.148,697, in all its ſeven figures, but is carried to two more figures.

The other expreſſion $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, may be computed as follows.

Since a is $= 1.14$, we ſhall have $3a\ (= 3 \times 1.14) = 3.42$, and $\frac{3a}{4}\ (= \frac{3.42}{4}) = 0.855$, and $8N\ (= 8 \times 2) = 16$, and $3a^5\ (= 3 \times 1.925,414,582,4) = 5.776,243,747,2$, and $8N - 3a^5\ (= 16.000,000,000,0 - 5.776,243,747,2) = 10.223,756,252,8$, and $80a^3\ (= 80 \times 1.481,544) = 118.523,520$, and $\frac{8N - 3a^5}{80a^3}\ (= \frac{10.223,756,252,8}{118.523,520}) = 0.086,259,303,240,40$, and $\sqrt{\frac{8N - 3a^5}{80a^3}}\ (= \sqrt{0.086259,303240,40}) = 0.293,699,34$, and conſequently $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}\ (= 0.855 + 0.293,699,34) = 1.148,699,34$. Therefore the fifth root of 2 will be very nearly equal to 1.148,699,34.

Q. E. I.

EXAMPLE III.

Art. 13. Let it be required to find the 5th root of the number 307,68282,11067,15625.

In order to find a firſt near value of the 5th root of this long number, we may begin by comparing it with the fifth powers of the numbers 10, 100, and 1000, and the following powers of 10. Now the fifth power of 10 is 100,000; which is very much leſs than the propoſed number: and the fifth power of 100 is 10,000,000,000; which is alſo much leſs than the propoſed number: and the fifth power of 1000 is 1000,000,000,000,000; which is alſo leſs than the propoſed number: but the fifth power of 10,000 is 100,000,000,000,000,000,000; which is greater than the propoſed number. We may therefore conclude that 1000 muſt be leſs, and that 10,000 muſt be greater than the fifth root of the propoſed number. Therefore the fifth root of the ſaid number muſt be of an intermediate magnitude between 1000 and 10,000.

Further, the propoſed number 307,68282,11067,15625 is greater than 307,00000,00000,00000, or than $307 \times \overline{1000}|^5$. Therefore the fifth root of the ſaid number will be greater than the product of the multiplication of the fifth root of 307 by 1000. We will therefore inquire what is the fifth root of 307.

Now the fifth power of 2 is 32, and the fifth power of 3 is 243, and the fifth power of 4 is 1024. Therefore the fifth power of 4 is much greater than 307, and the fifth power of 3 is a little leſs than 307. We may therefore conclude that the fifth root of 307 will be much leſs than 4, and a little greater than 3; and conſequently the fifth root of the number 307,00000,00000,00000 will be much leſs than 4 × 1000, or 4000, and a little greater than 3 × 1000, or 3000. We may therefore reaſonably conjecture that the fifth root of the propoſed number 307,68282,11067,15625, will be nearly equal to 3100. And accordingly, if we raiſe this number 3100 to the fifth power, we ſhall find the ſaid power of it to be = 286,29151,00000,00000, which is pretty nearly equal to, but ſomewhat leſs than, the propoſed number 307,68282,11067,15625. Therefore 3100 will be a proper number to make the baſis of a further approximation to the true value of the fifth root of the ſaid propoſed number, by means of either of the two expreſſions above-mentioned, to wit, $a + \frac{\overline{N - a^5}| \times a}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$.

Art. 14. Here then we have N = 307,68282,11067,15625, and a = 3100, and conſequently a^5 = 286,29151,00000,00000, and $3a^5$ (= 3 × 286,29151,00000,00000) = 858,87453,00000,00000, and 2N (= 2 × 307,68282,11067,15625) = 615,36564,22134,31250, and $2N + 3a^5$

(= 615,36564,22134,31250 + 858,87453,00000,00000) = 1474,24017,22134,31250, and $N - a^5$ (= 307,68282, 11067,15625 — 286,29151,00000,00000) = 21,39131, 11067,15625, and $\overline{N - a^5} \times a$ (= 21,39131,11067,15625 × 3100) = 66313,06443,08184,37500, and $\frac{\overline{N - a^5} \times a}{2N + 3a^5}$ ($= \frac{66313,06443,08184,37500}{1474,24017,22134,31250}$) = 44.98. Therefore $a + \frac{\overline{N - a^5} \times a}{2N + 3a^5}$ will be (= 3100 + 44.98) = 3144.98, which will therefore be nearly equal to the fifth root of the propoſed number 307,68282,11067,15625. Q. E. I.

The three firſt figures, 314, of this number 3144.98, or, rather, the five firſt figures of it, 3144.9, are exact, the error being only in the ſixth figure 8, which ought to be a 9 inſtead of an 8. For the exact root of the propoſed number 307,68282,11067,15625 is 3144.999,999, *ad infinitum*, or the whole number 3145, as will appear by raiſing the ſaid number 3145 to the fifth power.

Art. 15. The other, or irrational, expreſſion for the ſecond near value of the fifth root of this number 307,68282, 11067,15625, to wit, the expreſſion $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, may be computed as follows.

Since a is = 3100, we ſhall have $3a$ (= 3 × 3100) = 9300, and $\frac{3a}{4}$ ($= \frac{9300}{4}$) = 2325, and a^3 = 2,97910,00000, and a^5 = 286,29151,00000,00000, and $80a^3$ (= 80 × 2,97910,00000) = 238,32800,00000, and $8N$ (= 8 × 307,68282,11067,15625) = 2461,46256,88537,25000, and $3a^5$ = 858,87453,00000,00000, and $8N - 3a^5$ (= 2461, 46256,88537,25000 — 858,87453,00000,00000) = 1602, 58803,88537,25000, and $\frac{8N - 3a^5}{80a^3}$ ($= \frac{1602,58803,88537,25000}{238,32800,00000}$) = 672429.6091, and $\sqrt{\frac{8N - 3a^5}{80a^3}}$ (= $\sqrt{}$ 672429.6091)

820.01,

820.01, and conſequently $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$ ($=$ 2325 + 820.01) $=$ 3145.01. Therefore 3145.01 will be a ſecond near value of the fifth root of the propoſed number 307, 68282,11067,15625. Q. E. I.

Art. 16. If we ſhould chuſe to find the fifth root of this number 307,68282,11067,15625, by means of Mr. Raphſon's expreſſion for its value, to wit, the expreſſion $a + \frac{N - a^m}{ma^{m-1}}$, or $a + \frac{N - a^5}{5a^4}$, (which certainly has the merit of being much ſimpler, and eaſier to be remembered, than either of Mr. de Lagny's expreſſions, and likewiſe much eaſier to compute,) the computation will be as follows.

Since a is $=$ 3100, we ſhall have $a^4 =$ 9235,21000,00000, and $a^5 =$ 286,29151,00000,00000, and $5a^4 =$ 46176, 05000,00000, and $N - a^5$ ($=$ 307,68282,11067,15625 $-$ 286,29151,00000,00000) $=$ 21,39131,11067,15625, and $\frac{N - a^5}{5a^4}$ ($= \frac{21,39131,11067,15625}{46176,05000,00000}$) $=$ 46. Therefore $a + \frac{N - a^5}{5a^4}$ will be ($=$ 3100 + 46) $=$ 3146; which therefore will be the ſecond near value of the fifth root of the propoſed number 307,68282,11067,15625, obtained by Mr. Raphſon's approximation. Q. E. I.

A SCHOLIUM.

This laſt near value of the fifth root of the ſaid propoſed number, which has been obtained by Mr. Raphſon's approximation, is greater than its true value, 3145, by only an unit, or the 3145th part of the ſaid true value. So that this very ſimple method of approximating to the roots of numbers

bers may be juſtly conſidered as extremely uſeful as well as eaſy. And, if this proceſs were to be repeated, by taking 3146 for the value of *a*, and ſuppoſing $a - z$ to be equal to $\sqrt[5]{N}$, (*a*, or 3146, being ſomewhat greater than the true value of the fifth root which we are in ſearch of,) and by computing the expreſſion $a - \frac{N - a^5}{5a^4}$, that would reſult from that ſuppoſition, this ſecond proceſs would double the number of figures that are exact in *a*, or 3146, or give us about four times the number of figures that were exact in 3100, or the former value of *a*; which is more than is done by either of the two expreſſions, $a + \frac{\overline{N - a^5} \times a}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, given us by Mr. *de Lagny*: So that two ſteps of Mr. *Raphſon*'s method of approximation are more than equivalent to one ſtep of Mr. *de Lagny*'s method. It may therefore be doubted, whether Mr. *Raphſon*'s method is not, upon the whole, to be preferred to Mr. *de Lagny*'s, as Mr. *Raphſon* himſelf always thought it to be. For he tells us in the Appendix to the ſecond edition of his excellent Treatiſe, intitled, *Analyſis Æquationum Univerſalis*, (which ſecond edition was publiſhed in the year 1697, ſeven years after the firſt edition of it, and five or ſix years after the publication of *Monſieur de Lagny*'s method,) that he himſelf had had thoughts of reſolving the quadratick equation $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 = N$, or $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}z^2 = N$, in the imperfect manner adopted by Mr. *Lagny*, in order to obtain his rational value of *z*, to wit, $\frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, to wit, by ſubſtituting in the term $m \times \frac{m-1}{2} \times a^{m-2}z^2$ of the ſaid quadratick equation, inſtead of *z*, the value of *z* already obtained by the reſolution of the ſimple equation $a^m + ma^{m-1}z = N$, to wit, the

$$= \frac{2 \times \overline{N - a^m}}{\frac{2ma^{2m-2} + \overline{m-1} \times a^{m-2} \times N - ma^{2m-2} + a^{2m-2}}{a^{m-1}}}$$

$$= \frac{2 \times \overline{N - a^m}}{\frac{ma^{2m-2} + a^{2m-2} + \overline{m-1} \times a^{m-2} \times N}{a^{m-1}}}$$

$$= \frac{2a \times \overline{N - a^m}}{\frac{ma^{2m-1} + a^{2m-1} + \overline{m-1} \times a^{m-1} \times N}{a^{m-1}}}$$

$$= \frac{2a \times \overline{N - a^m}}{ma^m + a^m + \overline{m-1} \times N} = \frac{2a \times \overline{N - a^m}}{\overline{m+1} \times a^m + \overline{m-1} \times N})$$

$= \frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, and consequently $a + z = a + \frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$; which is Mr. *de Lagny*'s rational expreſſion for the ſecond near value of the *m*th root of N. Mr. *Raphſon* (I ſay,) tells us that he himſelf had had thoughts of reſolving the quadratick equation $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}z^2 = N$ in this manner, in order to obtain a value of z ſomewhat nearer to the truth than the fraction $\frac{N - a^m}{ma^{m-1}}$, which he had obtained by the reſolution of the ſimple equation $a^m + ma^{m-1}z = N$; but that he did not think proper to adopt this method, becauſe he thought his own method of approximation, (which he publiſhed in the firſt edition of his *Analyſis Æquationum Univerſalis*, in the year 1690, and which is derived from the ſimple equation $a^m + ma^{m-1}z = N$,) the eaſieſt and cleareſt, and, upon the whole, the beſt, and fitteſt for practice, that could be followed. His words, in his Appendix, relating to this ſubject, are as follows. *An Dominus* de Lagny *Librum meum unquam viderat, nec-ne, prorsùs neſcio. Quibúſce modis non ſolùm ſua methodus, ſed et etiàm aliæ quam-plurimæ, eodem prorsùs proceſſu, et perpetuà inde derivatâ graduum ſcalâ, inveniri poſſint, hujúſce Appendicis eſt oſtendere; ìdque, quàm poſſimus, breviſſimè.*

Ipſe equidem de gradatìm inferendis (quas priùs rejeceram in Theoremate Vietæo,) *poteſtatibus olìm cogitavi: Sed tamen non proſecutus fui; utpote qui methodum meam, harum omnium fundamentalem, veluti facillimam ſemper exiſtimavi. Subſequenti proceſſu earum omnium inventionem indagare cuilibet liceat.* See Mr. Raphſon's *Analyſis Æquationum Univerſalis*, Edition 2d, 1697, page 49. And again, in page 55, he concludes his Appendix with theſe words: *Innumeras etiàm alias methodos et abbreviationes* (novarum *quidem* methodorum *nomine inſigniendas,) adinvenire liceat; quæ tamen omnia fundamentali huic ſuperiorum poteſtatum imprimis rejectionis methodo, poſteàque gradatìm retinendarum, innitantur. Noſtram tamen ſimpliciſſimam fore et facillimam, cuivis pateat.*

EXAMPLE IV.

Art. 17. Let it be required to find the ſeventh root of the number 34,487,717,467,307,513,182,492,153,794,673; which, Mr. *Bonnycaſtle*, of the Royal Military Academy at Woolwich, in his *Scholar's* Guide to Arithmetick, page 189, tells us, is 32017.

This number muſt, in the firſt place, be compared with the ſeventh powers of 10, 100, 1000, &c, to which it approaches neareſt, in order to know between which two of thoſe numbers 10, 100, 1000, &c, its ſeventh root will lie. Now the ſeventh power of 10 is 10,000,000, which is very much leſs than the ſaid propoſed number; and the ſeventh power of 100 is 100,000,000,000,000, which is likewiſe much leſs than the ſaid propoſed number; and the ſeventh power of 1000 is 1000,000,000,000,000,000,000, which is likewiſe leſs than the ſaid propoſed number; and the ſeventh power of 10,000 is 10,000,000,000,000,000,000,000,000,000, or 1 with twenty-eight cyphers annexed to it; which is likewiſe leſs than the ſaid propoſed number, which conſiſts of 32 figures. But the ſeventh power of 100,000 is 100,000, 000,000,000,000,000,000,000,000,000,000, or 1 with 35 cyphers annexed to it, and is therefore greater than the ſaid propoſed number, which conſiſts of only 32 figures. Therefore 10,000 will be leſs, and 100,000 will be greater, than the ſeventh root of the ſaid propoſed number.

Further, the propoſed number 34,487,717,467,307,513, 182,492,153,794,673, is greater than the number 34480, 000,000,000,000,000,000,000,000,000, or than 3448 × 10,000,000,000,000,000,000,000,000,000, or than 3448 × the ſeventh power of 10,000: but the difference between them is not great. Therefore the ſeventh root of the ſaid propoſed number will be greater than 10,000 times the ſeventh root of 3448: but the difference between them will not be great. And conſequently, if we can find the ſeventh root of the number 3448 exact to two, or three, places of figures, we need only multiply the ſaid ſeventh root by 10,000, in order to obtain the ſeventh root of 34480,000, 000,000,000,000,000,000,000,000, exact to two or three places of figures. And, when we have obtained the ſaid near value of the ſeventh root of the number 34480,000, 000,000,000,000,000,000,000,000, the ſaid near value will likewiſe be a near value of the ſeventh root of the propoſed number 34487,717,467,307,513,182,492,153,794,673, and will be leſs than the true value of the ſeventh root of the ſaid number, and therefore will ſerve as a convenient baſis of a further approximation to the true value of the ſeventh root of the ſaid propoſed number, by means of one of Monſieur *de Lagny*'s two expreſſions found above in the ſolution of the foregoing Problem. We muſt therefore now endeavour to find a near value of the ſeventh root of the number 3448. Now this may be done in the manner following.

Art. 18. The ſeventh power of the number 2 is 128, and the ſeventh power of 3 is 2187, and the ſeventh power of 4 is 16384. Therefore, ſince 3448 is greater than 2187, or the ſeventh power of 3, but is much leſs than 16384, or the ſeventh power of 4, it follows that the ſeventh root of 3448 muſt be greater than 3, but much leſs than 4. We may therefore reaſonably conjecture, that it will be nearly equal to $3\frac{1}{5}$, or 3.2. And accordingly, upon trial, we ſhall find it to be ſo. For, if we raiſe 3.2 to its ſeventh power, we ſhall find the ſaid power to be = 3435·973,836,8; which is leſs than 3448, but very nearly equal to it. Therefore 3.2 is a very near firſt value of the ſeventh root of the number 3448; and conſequently 3.2 × 10,000, or 32000, will be a very near firſt value of the number 34480,000, 000,000,000,000,000,000,000,000, and therefore will be alſo a pretty near firſt value of the propoſed number 34487,717,467,307,513,182,492,153,794,673. We will therefore ſuppoſe a to be = 32000, and proceed, upon that ſuppoſition, to compute the two expreſſions given in the ſolution of the foregoing Problem, for a ſecond value of the ſeventh root of the ſaid propoſed number that ſhall approach nearer than a, or 32000, to its true value. Theſe computations will be as follows.

Art. 19. The firſt, or rational, expreſſion, given in the ſolution of the foregoing Problem, for the ſecond near value of the mth root of any propoſed number N, is $a + \frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$. Now, when m is = 7, this expreſſion will be $\left(= a + \frac{\overline{N - a^7} \times 2a}{\overline{7-1} \times N + \overline{7+1} \times a^7} = a + \frac{\overline{N - a^7} \times 2a}{6N + 8a^7}\right) = a + \frac{\overline{N - a^7} \times a}{3N + 4a^7}$.

Now, ſince N is, in this caſe, = 34,487,717,467,307,

513,182,492,153,794,673, and a is = 32000, we ſhall have a^7 (= $\overline{32000}|^7$) = 34,359,738,368,000,000,000,000,000, 000,000, and N — a^7
(= 34,487,717,467,307,513,182,492,153,794,673
— 34,359,738,368,000,000,000,000,000,000,000)
= 127,979,099,307,513,182,492,153,794,673, and
$\overline{N-a^7}| \times a$ (= 127,979,099,307,513,182,492,153,794,673 × 32000) = 4,095,331,177,840,421,839,748,921,429, 536,000, and 3N (= 3 × 34,487,717,467,307,513,182, 492,153,794,673) = 103,463,152,401,922,539,547,476, 461,384,019, and $4a^7$
(= 4 × 34,359,738,368,000,000,000,000,000,000,000)
= 137,438,945,472,000,000,000,000,000,000,000,
and 3N + $4a^7$
(= 103,463,152,401,922,539,547,476,461,384,019
+ 137,438,945,472,000,000,000,000,000,000,000)
= 240,902,097,873,922,539,547,476,461,384,019, and
$\frac{\overline{N-a^7}| \times a}{3N + 4a^7}$ (= $\frac{4,095,331,177,840,421,839,748,921,429,436,000}{240,902,097,873,922,539,547,476,461,384,019}$ =, nearly, $\frac{4,095,331}{240,902}$) = .16.99998. Therefore $a + \frac{\overline{N-a^7}| \times a}{3N + 4a^7}$ will be (= 32000 + 16.9998) = 32016.9998, and conſequently the ſecond near value of the ſeventh root of the propoſed number 34,487,717,467,307,513,182,492,153,794, 673 will be 32016.9998. Q. E. I.

This number is true in all the figures but the laſt, which ought to be a 9 inſtead of an 8, the true value of the ſeventh root of the ſaid propoſed number being 32016.999,999,999, 999, &c, *ad infinitum*, or the whole number 32017.

Art. 20. The ſecond, or irrational, expreſſion, given in the ſolution of the foregoing Problem, for the ſecond near value of the mth root of any propoſed number N, is $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N-a^m}}{m \times \overline{m-1} \times a^{m-2}}}$; which, when m is = 7, near value of the ſeventh root of the propoſed number 34,487,717,467,307,513,182,492,153,794,673. Q. E. I.

This number 32017.000,01 is exact in the firſt nine figures, 32017.0000, and errs only in the 10th figure, 1, which ought to be a cypher, 0, inſtead of a 1, becauſe the true value of this ſeventh root is 32017.000,000,000, &c, *ad infinitum*, or the whole number 32017.

Art. 21. If we ſeek the value of this ſeventh root by Mr. Raphſon's expreſſion $a + \frac{N - a^m}{m a^{m-1}}$, or $a + \frac{N - a^7}{7a^6}$, the computation will be as follows.

Since a is $= 32,000$, we ſhall have a^6 $(= \overline{32,000}|^6) =$ 1,073,741,824,000,000,000,000,000,000, and $7a^6$ $(= 7 \times 1,073,741,824,000,000,000,000,000,000) = 7,516,192,768,000,000,000,000,000,000$, and a^7 $(= \overline{32,000}|^7) =$ 34,359,738,368,000,000,000,000,000,000,000, and $N - a^7$

$$(= 34,487,717,467,307,513,182,492,153,794,673$$
$$- 34,359,738,368,000,000,000,000,000,000,000)$$
$$= 127,979,099,307,513,182,492,153,794,673,$$

and

$$\frac{N - a^7}{7a^6} \left(= \frac{127,979,099,307,513,182,492,153,794,673}{7,516,192,768,000,000,000,000,000,000}\right) = 17.02.$$

Therefore $a + \frac{N - a^7}{7a^6}$ is $(= 32000 + 17.02) = 32017.02$; and conſequently 32017.02 will be nearly equal to the ſeventh root of the propoſed number 34,487,717,467,307,513,182,492,153,794,673. Q. E. I.

This number 32017.02 is exact in the ſix firſt figures 32017.0, and errs only in the ſeventh figure 2, which ought to be a cypher inſtead of a 2, becauſe the true value of this ſeventh root is 32017.000,000,000, &c, *ad infinitum*, or the whole number 32017.

Art. 22. This expreſſion $a + \frac{N - a^m}{m a^{m-1}}$, or $a + \frac{N - a^7}{7a^6}$, is ſo much ſimpler than either of Mr. *de Lagny*'s expreſſions above-mentioned, and ſo much leſs difficult to be computed, that I am inclined to agree with Mr. *Raphſon* in thinking it, upon the whole, preferable to them. But, perhaps, when a, or the firſt value of the root ſought, conſiſts of only one figure, it may ſometimes be adviſeable to make uſe of one of Mr. *de Lagny*'s expreſſions, in order to obtain a ſecond near value of the root ſought, and then to make uſe of Mr. *Raphſon*'s expreſſion in order to obtain a third near value of it.

Art. 23. These four examples are, I preſume, ſufficient to illuſtrate *Monſieur de Lagny*'s method of extracting the mth root of any propoſed number denoted by the letter N, by means of either of the two expreſſions $a + \frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} + \frac{N - a^m}{m \times \frac{m-1}{2} \times a^{m-2}}}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}}$, when a, or the firſt near value of $\sqrt[m]{N}$, which is ſuppoſed to be already known, is leſs than its true value; which is the caſe ſuppoſed in the foregoing Problem. I ſhall therefore now proceed to conſider the other caſe, in which a, or the firſt near value of the mth root of the propoſed number N, is greater than its true value, and to inveſtigate ſimilar expreſſions for a ſecond near value of the ſaid root that ſhall approach nearer than a to its true value. This may be done by a ſolution of the following Problem.

PROBLEM

PROBLEM II.

Art. 24. Let N be any propoſed number whatſoever, and m any propoſed whole number whatſoever; and let a be a known number that is nearly equal to, but ſomewhat greater than, the mth root of the given number N. It is required to find a ſecond near value of the ſaid mth root of the given number N, that ſhall approach much nearer to it than a, or the former near value of it that is already known.

SOLUTION.

Let z be put for the unknown difference between a, the firſt near value of the mth root of the given number N, and the true value of the ſaid root. Then, ſince a is ſuppoſed to be greater than $\sqrt{}^{m}\,N$, and to exceed it by the difference z, it follows that $a - z$ will be $= \sqrt{}^{m}\,N$, and conſequently that $\overline{a - z}|^{m}$ will be $= N$.

But, by Sir *Iſaac Newton*'s reſidual theorem in the firſt and ſimpleſt caſe of it, to wit, the caſe of integral powers, $\overline{a - z}|^{m}$ will be $=$ the ſeries $a^{m} - m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^{2} - m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} z^{3} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} z^{4} - m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times a^{m-5} z^{5} +$ &c, continued to $m + 1$ terms;

or, if, for the ſake of brevity, we ſubſtitute the capital letters A, B, C, D, E, F, &c, inſtead of the ſeveral numeral coefficients 1, m, $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, reſpectively, $\overline{a-z}|^m$ will be = the ſeries $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 - Da^{m-3}z^3 + Ea^{m-4}z^4 - Fa^{m-5}z^5$ + &c, continued to $m+1$ terms. Therefore the ſaid ſeries $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 - Da^{m-3}z^3 + Ea^{m-4}z^4 - Fa^{m-5}z^5$ + &c, continued to $m+1$ terms, will be = N. This is the original equation, by the reſolution of which we are to find a near value of z, and conſequently of $a-z$, or a ſecond near value of $\sqrt[3]{N}$, which will approach nearer to it than a, or its former near value.

Art. 25. Now, ſince z is leſs, and uſually much leſs, than a, being about a 10th, or a 100th, part of it, or ſome ſtill leſſer part of it, it is evident that all the terms in the aforeſaid ſeries that involve zz, and z^3, and z^4, and the following powers of z, will be leſs, and uſually much leſs, than the term $Ba^{m-1}z$, which involves only the ſimple power of z. And therefore, if all the ſaid terms of the ſeries be neglected or omitted, and the two firſt terms alone, to wit, the terms $Aa^m - Ba^{m-1}z$, be retained, the ſaid two terms alone will be nearly equal to the whole ſeries, and conſequently to the given number N; and therefore, if we add $Ba^{m-1}z$ to both ſides, we ſhall have Aa^m, nearly, = $N + Ba^{m-1}z$, and (ſubtracting N from both ſides,) $Aa^m - N$, nearly, $= Ba^{m-1}z$, or $Ba^{m-1}z$, nearly, $= Aa^m - N$, and (dividing both ſides by Ba^{m-1},) we ſhall have

z, nearly, $= \frac{Aa^m - N}{Ba^{m-1}}$, or (becauſe A is $= 1$, and B is $= m$,) we ſhall have z, nearly, $= \frac{a^m - N}{ma^{m-1}}$; which fraction may be derived from the known quantities N and a, by the common arithmetical operations of Multiplication, Subtraction, and Diviſion. This therefore is an approximation to the true value of z, and conſequently $a - \overline{\frac{a^m - N}{ma^{m-1}}}$ will be an approximation to the true value of $a - z$, or of $\sqrt{}^{m}N$, or will be a ſecond near value of it that will approach nearer to it than a, or the firſt near value of it which was already known. And it will be ſtill ſomewhat greater (as the former value a was,) than the true value of $\sqrt{}^{m}\ N$; as may be demonſtrated in the manner following.

The whole ſeries $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 - Da^{m-3}z^3 + Ea^{m-4}z^4 - Fa^{m-5}z^5 + \&c$, is $= N$. Therefore (adding $Ba^{m-1}z$ to both ſides,) we ſhall have the ſeries $Aa^m + Ca^{m-2}z^2 - Da^{m-3}z^3 + Ea^{m-4}z^4 - Fa^{m-5}z^5 + \&c, = N + Ba^{m-1}z$, and (ſubtracting N from both ſides,) we ſhall have the ſeries $Aa^m - N + Ca^{m-2}z^2 - Da^{m-3}z^3 + Ea^{m-4}z^4 - Fa^{m-5}z^5 + \&c, = Ba^{m-1}z$. But, becauſe $Ca^{m-2}z^2$ is greater than $Da^{m-3}z^3$, and $Ea^{m-4}z^4$ is greater than $Fa^{m-5}z^5$, and, in like manner, every following term in the ſaid ſeries that is marked with the ſign $+$, is greater than the term immediately following it, which is marked with the ſign $-$, it follows that the ſeries $Aa^m - N + Ca^{m-2}z^2 - Da^{m-3}z^3$

+

$Ea^{m-4}z^4 - Fa^{m-5}z^5 +$ &c, $= N$ into a quadratick equation, to wit, the equation $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 = N$, inſtead of converting it (as Mr. Raphſon does,) into the ſimple equation $Aa^m - Ba^{m-1}z = N$. And this quadratick equation he reſolves in two different ways, to wit, firſt, imperfectly, or inaccurately, and then accurately. By the former, or inaccurate, reſolution of it, he obtains a rational expreſſion for the value of z, and conſequently another rational expreſſion for the value of $a - z$, or for a ſecond near value of $\sqrt[3]{N}$, which is nearer to its true value than a, or the former near value of it, was; and by the accurate reſolution of the ſame quadratick equation he obtains a ſurd, or irrational, expreſſion for the value of z, and conſequently another ſurd, or irrational, expreſſion for the value of $a - z$, or for the ſecond near value of $\sqrt[3]{N}$, that approaches much nearer to its true value than its former near value, a, did. Theſe reſolutions of the ſaid quadratick equation $Aa^m - Ba^{m-1}z + Ca^{m-2}zz = N$, may be performed in the following manner.

Art. 27. Since $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2$ is $= N$, we ſhall have $Aa^m + Ca^{m-2}z^2 = N + Ba^{m-1}z$, and $Aa^m = N + Ba^{m-1}z - Ca^{m-2}z^2$, and $Aa^m - N = Ba^{m-1}z - Ca^{m-2}z^2$, or $Ba^{m-1}z - Ca^{m-2}z^2 = Aa^m - N$; that is, $z \times Ba^{m-1} - z \times Ca^{m-2}z$ will be $= Aa^m - N$, or $z \times \overline{Ba^{m-1} - Ca^{m-2}z}$ will be $= Aa^m - N$. Therefore (dividing both ſides of the equation by the compound quantity $Ba^{m-1} - Ca^{m-2}z$,) we ſhall have $z = \frac{Aa^m - N}{Ba^{m-1} - Ca^{m-2}z}$.

Now let $\frac{\mathrm{A}a^m - \mathrm{N}}{\mathrm{B}a^{m-1}}$, or $\frac{a^m - \mathrm{N}}{ma^{m-1}}$, (which has already been ſhewn to be nearly equal to z,) be ſubſtituted inſtead of z in the ſecond term, $\mathrm{C}a^{m-2}z$, of the denominator of the fraction laſt obtained, to wit, the fraction $\frac{\mathrm{A}a^m - \mathrm{N}}{\mathrm{B}a^{m-1} - \mathrm{C}a^{m-2}z}$.

And we ſhall have $z = \frac{\mathrm{A}a^m - \mathrm{N}}{\mathrm{B}a^{m-1} - \mathrm{C}a^{m-2} \times \frac{a^m - \mathrm{N}}{ma^{m-1}}}$, or (becauſe A is $= 1$, and B is $= m$,)

$$z = \frac{a^m - \mathrm{N}}{ma^{m-1} - \mathrm{C}a^{m-2} \times \frac{a^m - \mathrm{N}}{ma^{m-1}}}; \text{ which is}$$

$$(= \frac{a^m - \mathrm{N}}{\frac{m^2a^{2m-2} - \mathrm{C}a^{m-2} \times \overline{a^m - \mathrm{N}}}{ma^{m-1}}}$$

$$= \frac{a^m - \mathrm{N}}{\frac{m^2a^{2m-2} - \mathrm{C}a^{2m-2} + \mathrm{CN}a^{m-2}}{ma^{m-1}}}$$

$$= \overline{a^m - \mathrm{N}} \times \frac{ma^{m-1}}{m^2a^{2m-2} - \mathrm{C}a^{2m-2} + \mathrm{CN}a^{m-2}}$$

$$= \overline{a^m - \mathrm{N}} \times \frac{2aa \times ma^{m-1}}{2m^2a^{2m} - 2\mathrm{C}a^{2m} + 2\mathrm{CN}a^m}$$

$$= \overline{a^m - \mathrm{N}} \times \frac{2a \times ma^m}{2m^2a^{2m} - 2\mathrm{C}a^{2m} + 2\mathrm{CN}a^m}$$

$$= \overline{a^m - \mathrm{N}} \times \frac{2ma}{2m^2a^m - 2\mathrm{C}a^m + 2\mathrm{CN}}$$

=

$$=\overline{a^m - N}\times\frac{2ma}{2m^2a^m - 2\times m\times\frac{m-1}{2}\times a^m + 2\times m\times\frac{m-1}{2}\times N}$$

$$=\overline{a^m - N}\times\frac{2ma}{2m^2a^m - m\times\overline{m-1}\times a^m + m\times\overline{m-1}\times N}$$

$$=\overline{a^m - N}\times\frac{2a}{2ma^m - \overline{m-1}\times a^m + \overline{m-1}\times N}$$

$$=\overline{a^m - N}\times\frac{2a}{2ma^m - ma^m + a^m + \overline{m-1}\times N}$$

$$=\overline{a^m - N}\times\frac{2a}{ma^m + a^m + \overline{m-1}\times N}$$

$$=\overline{a^m - N}\times\frac{2a}{\overline{m+1}\times a^m + \overline{m-1}\times N})$$

$$=\frac{\overline{a^m - N}\times 2a}{\overline{m-1}\times N + \overline{m+1}\times a^m}.$$ Therefore z will be $=$ $\frac{\overline{a^m - N}\times 2a}{\overline{m-1}\times N + \overline{m+1}\times a^m}$, or $\frac{2a\times\overline{a^m - N}}{\overline{m-1}\times N + \overline{m+1}\times a^m}$; and confequently $a - z$ will be $=$

$$a - \frac{2a\times\overline{a^m - N}}{\overline{m-1}\times N + \overline{m+1}\times a^m},\ \text{or}\ a - \frac{2a\times\overline{a^m - N}}{\overline{m-1}\times N + \overline{m+1}\times a^m}$$

will be a fecond near value of $\sqrt[m]{N}$, or of the *m*th root of the propofed number N. Q. E. I.

Art. 28. The accurate refolution of the quadratick equation $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 = N$, may be performed as follows.

Since $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2$ is $= N$, we fhall have $Aa^m + Ca^{m-2}z^2 = N + Ba^{m-1}z$, and $Aa^m = N + Ba^{m-1}z - C\,{}^{m-2}z^2$, and $Aa^m - N$ $=$

be ſubtracted from $\frac{aa}{\overline{m-1}|^{2}}$. Let them be ſo ſubtracted. And then we ſhall have $\frac{aa}{\overline{m-1}|^{2}} - \frac{2az}{m-1} + zz = \frac{aa}{\overline{m-1}|^{2}} - \frac{2 \times \overline{a^{m} - N}}{m \times \overline{m-1} \times a^{m-2}}$. Therefore the ſquare-root of the trinomial quantity $\frac{aa}{\overline{m-1}|^{2}} - \frac{2az}{m-1} + zz$ will be equal to the ſquare root of the compound quantity $\frac{aa}{\overline{m-1}|^{2}} - \frac{2 \times \overline{a^{m} - N}}{m \times \overline{m-1}| \times a^{m-2}}$; or, if, for the ſake of brevity, we denote the ſaid compound quantity by the capital letter P, the ſquare-root of the trinomial quantity $\frac{aa}{\overline{m-1}|^{2}} - \frac{2az}{m-1} + zz$ will be $= \sqrt{P}$. But, whenever z is leſs than $\frac{a}{m-1}$, (as is commonly the caſe in theſe extractions of the roots of numbers,) $\frac{a}{m-1} - z$ will be the ſquare-root of the trinomial quantity $\frac{aa}{\overline{m-1}|^{2}} - \frac{2az}{m-1} + zz$. Therefore $\frac{a}{m-1} - z$ will $= \sqrt{P}$, and conſequently $\frac{a}{m-1}$ will be $= \sqrt{P} + z$, and z will be $= \frac{a}{m-1} - \sqrt{P}$. Therefore $a - z$ will be $(= a - \overline{\frac{a}{m-1} - \sqrt{P}}) = a - \frac{a}{m-1} + \sqrt{P}$; and conſequently $a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^{2}} - \frac{2 \times \overline{a^{m} - N}}{m \times \overline{m-1} \times a^{m-2}}}$ will be a ſecond near value of $\sqrt[m]{N}$, or of the mth root of the given number N. Q. E. I.

Art. 29. When m is $= 3$, the former of theſe two expreſſions,

preſſions, to wit, the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, will be $(= a - \frac{2a \times \overline{a^3 - N}}{2N + 4a^3}) = a - \frac{a \times \overline{a^3 - N}}{N + 2a^3}$; and the latter, or irrational, expreſſion, $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$ will be $(= a - \frac{a}{2} + \sqrt{\frac{aa}{4} - \frac{2 \times \overline{a^3 - N}}{3 \times 2 \times a}} = \frac{a}{2} + \sqrt{\frac{aa}{4} - \frac{a^3 - N}{3a}} = \frac{a}{2} + \sqrt{\frac{3a^3}{12a} - \frac{4a^3 - 4N}{12a}} = \frac{a}{2} + \sqrt{\frac{3a^3 - 4a^3 + 4N}{12a}}) = \frac{a}{2} + \sqrt{\frac{4N - a^3}{12a}}$. Therefore the two expreſſions for the ſecond near value of the cube-root of a given number N, when a, or the former near value of it, is greater than its true value, are $a - \frac{a \times \overline{a^3 - N}}{N + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4N - a^3}{12a}}$.

Art. 30. And, when m is $= 5$, the former of the two foregoing general expreſſions, to wit, the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, will be $(= a - \frac{2a \times \overline{a^5 - N}}{4N + 6a^5}) = a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$; and the latter, or irrational, expreſſion, $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$ will be $(= a - \frac{a}{4} + \sqrt{\frac{aa}{16} - \frac{2 \times \overline{a^5 - N}}{5 \times 4 \times a^3}} = \frac{3a}{4} + \sqrt{\frac{aa}{16} - \frac{a^5 - N}{10a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5}{160a^3} - \frac{16a^5 - 16N}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5 - (16a^5 - 16N}{160a^3}}$

=

$= \frac{3a}{4} + \sqrt{\frac{10a^5 - 16a^5 + 16N}{160a^3}} = \frac{3a}{4} + \sqrt{(\frac{16N - 6a^5}{160a^5})} = \frac{3a}{4}$ $+ \sqrt{\frac{8N - 3a^5}{80a^3}}$. Therefore the two expreſſions for the ſecond near value of the fifth root of a given number N, when a, or the former near value of the ſaid root, is greater than its true value, are $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$.

Art. 31. And, when m is $= 7$, the former of the two foregoing general expreſſions, to wit, the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, will be $(= a - \frac{2a \times \overline{a^7 - N}}{6N + 8a^7})$ $= a - \frac{a \times \overline{a^7 - N}}{3N + 4a^7}$; and the latter, or irrational, expreſſion, $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$ will be $(= a - \frac{a}{6} + \sqrt{\frac{aa}{36} - \frac{2 \times \overline{a^7 - N}}{7 \times 6 \times a^5}} = \frac{5a}{6} + \sqrt{\frac{aa}{36} - (\frac{a^7 - N}{21a^5}}$ $= \frac{5a}{6} + \sqrt{\frac{21a^7}{36 \times 21a^5} - (\frac{36a^7 - 36N}{36 \times 21a^5}} = \frac{5a}{6} + \sqrt{\frac{21a^7 - (36a^7 - 36N}{36 \times 21a^5}}$ $= \frac{5a}{6} + \sqrt{\frac{21a^7 - 36a^7 + 36N}{36 \times 21a^5}} = \frac{5a}{6} + \sqrt{\frac{36N - 15a^7}{36 \times 21a^5}} = \frac{5a}{6} +$ $\sqrt{(\frac{12N - 5a^7}{36 \times 7a^5})} = \frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}}$. Therefore the two expreſſions for the ſecond near value of the 7th root of a given number N, when a, or the former near value of the ſaid root, is greater than its true value, are $a - \frac{a \times \overline{a^7 - N}}{3N + 4a^7}$ and $\frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}}$.

conſequently $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ ($=$ 1.150,000,000 $-$ 0.001, 301,641) $=$ 1.148,698,359. Therefore 1.148,698,359 will be a ſecond near value of the fifth root of the given number 2. Q. E. I.

And, ſince a is $=$ 1.15, and a^3 is $=$ 1.520,875, and a^5 is $=$ 2.011,357,187,5, and $3a^5$ is $=$ 6.034,071,562,5, we ſhall have $80a^3$ ($=$ 80 $\times$ 1.520,875) $=$ 121.670,000, and $8N - 3a^5$ ($=$ 8 $\times$ 2 $-$ 6.034,071,562,5 $=$ 16.000,000, 000,0 $-$ 6.034,071,562,5,) $=$ 9.965,928,437,5, and $\frac{8N - 3a^5}{80a^3}$ $\left(= \frac{9.965,928,437,5}{121.670,000}\right)$ $=$ 0.081909,496486, and $\sqrt{\frac{8N - 3a^5}{80a^3}}$ ($= \sqrt{}$ 0.081909,496486,) $=$ 0.286,198,351, and $3a$ ($=$ 3 $\times$ 1.15) $=$ 3.45, and $\frac{3a}{4}$ $\left(= \frac{3.45}{4}\right) =$ 0.8625, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$ ($=$ 0.8625 $+$ 0.286,198,351) $=$ 1.148,698,351. Therefore 1.148,698,351 will be a ſecond near value of the fifth root of the given number 2.

Q. E. I.

As theſe two approximations to the fifth root of 2, to wit, 1.148,698,359 and 1.148,698,351, agree with each other in the firſt nine figures 1.148,698,35, we may reaſonably conclude that thoſe nine figures are exact, or are the firſt nine figures of a more accurate value of the ſaid fifth root.

Art. 33. If we make uſe of Mr. Raphſon's expreſſion, to wit, $a - \left[\frac{a^m - N}{ma^{m-1}}\right.$, or $a - \left[\frac{a^5 - N}{5a^4}\right.$, for the purpoſe of obtaining a ſecond near value of the fifth root of 2, after 1.15 has been taken for a, or, its firſt near value, the computation of it will be as follows.

Since a is $=$ 1.15, we ſhall have $a^4 =$ 1.749,006,25, and $a^5 =$ 2.011,357,187,5, and $5a^4$ ($=$ 5 $\times$ 1.749,006,25) $=$

= 8.745,031,25, and $a^5 - N$ (= 2.011,357,187,5 — 2) = 0.011,357,187,5, and $\frac{a^5 - N}{5a^4}$ (= $\frac{0.011,357,187,5}{8.745,031,25}$) = 0.001,298,7, and $a - \frac{a^5 - N}{5a^4}$ (= 1.150,000,0 — 0.001,298,7) = 1.148,701,3. Therefore 1.148,701,3 will be the ſecond near value of the fifth root of 2, reſulting from Mr. Raphſon's expreſſion $a - \frac{a^5 - N}{5a^4}$. Q. E. I.

This number 1.148,701,3, is greater than the true value of the 5th root of 2, to wit, the number 1.148,698,3, but exceeds it by only the very ſmall quantity 0.000,003,3.

EXAMPLE II.

Art. 34. Let it be required to find the 5th root of the number 2,327,834,559,873.

Now this number, which conſiſts of thirteen figures, is greater than 100,000, or the fifth power of 10; and it is likewiſe greater than 10,000,000,000, or the fifth power of 100: but it is leſs than 1,000,000,000,000,000, or the fifth power of 1000. Therefore its fifth root muſt be greater than 100, but leſs than 1000.

Further, this number, 2,327,834,559,873, is greater than 2,320,000,000,000, or than 232 × 10,000,000,000, or than 232 × the fifth power of 100. Therefore the fifth root of 2,327,834,559,873 will be greater than 100 × the fifth root of 232. But the difference will not be great; and conſequently, if we can find a number that ſhall be nearly equal to the fifth root of 232, we need only multiply the ſaid number into 100, and the product will be nearly equal to the fifth root of 2,320,000,000,000, and therefore will likewiſe be pretty nearly equal to the fifth root of the propoſed number 2,327,834,559,873, ſo as to be a convenient firſt near value of the ſaid fifth root, and a proper baſis to found a further approximation upon to a ſecond near value of the ſaid fifth root, by either of the two foregoing expreſſions of Mr. de Lagny, which have been inveſtigated above in the Solution of Problem II, or by Mr. Raphſon's expreſſion. We will therefore endeavour to find the fifth root of the number 232.

Art. 35. Now the fifth power of the number 2 is 32, which is much leſs than 232; and the fifth power of 3 is 243, which is a little greater than 232. Therefore the fifth root of 232 muſt be much greater than 2, and a little leſs than 3. We may therefore reaſonably conjecture that it will be nearly equal to 2 + $\frac{9}{10}$, or 2.9. We will therefore ſuppoſe it to be = 2.9, and try what the reſult of that ſuppoſition will be.

Now the fifth power of 2.9 is 205.11149, which is leſs than 232. Therefore the fifth root of 232 will be greater than 2.9. And, as 232 differs much leſs from 243, or the the fifth power of 3, than from 205.11149, or the fifth power of 2.9, we may reaſonably ſuppoſe that the fifth root of 232 will differ much leſs from 3 than from 2.9; and therefore we will ſuppoſe that it is nearly = 2.98, and will raiſe the ſaid number 2.98 to its fifth power, in order to examine the truth of the ſaid ſuppoſition.

Now $\overline{2.98}|^5$ is = 235.007,282,396,8; which is nearly equal to, but a little greater than, 232. Therefore 2.98 muſt be nearly equal to, but a little greater than, the fifth root of 232; and conſequently 100 × 2.98 muſt be nearly equal to, but a little greater than, the fifth root of 232 × 10,000,000,000; or 298 muſt be nearly equal to, but a little greater than, the fifth root of 2,320,000,000,000. And, further, ſince 2.98 is the fifth root of the number 235.007,282,396,8, the number 100 × 2.98, will be the fifth root of the number 235.007,282,396,8 × 10,000,000,000;

000; that is, the number 298 will be the fifth root of the number 2,350,072,823,968, which is greater than the propoſed number 2,327,834,559,873. Therefore 298 will be greater than the fifth root of the ſaid number 2,327,834,559,873. But it will be near enough to it to make it a very convenient baſis of a further approximation to the true value of the fifth root of the ſaid number, 2,327,834,559,873, by means of the two expreſſions $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$.

Art. 36. Here then we ſhall have N = 2,327,834,559,873, and $a = 298$. Therefore a^3 will be = 26,463,592, and a^4 will be = 7,886,150,416, and a^5 will be = 2,350,072,823,968; and conſequently $a^5 - N$ will be (= 2,350,072,823,968 — 2,327,834,559,873) = 22,238,264,095, and $\overline{a^5 - N} \times a$ will be (= 22,238,264,095 × 298) = 6,627,002,700,310, and 2N will be (= 2 × 2,327,834,559,873) = 4,655,669,119,746, and $3a^5$ will be (= 3 × 2,350,072,823,968) = 7,050,218,471,904, and $2N + 3a^5$ will be (= 4,655,669,119,746 + 7,050,218,471,904) = 11,705,887,591,650, and $\frac{\overline{a^5 - N} \times a}{2N + 3a^5}$ will be (= $\frac{6,627,002,700,310}{11,705,887,591,650}$) = 0.566,125,605, and conſequently $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ will be (= 298.000,900,000 — 0.566,125,605) = 297.433,874,395. Therefore 297.433,874,395 will be a ſecond near value of the fifth root of the propoſed number 2,327,834,559,873. Q. E. I.

And $3a$ will be (= 3 × 298) = 894, and $\frac{3a}{4}$ will be (= $\frac{894}{4}$) = 223.5, and a^3 will be (= $\overline{298}|^3$) = 26,463,592, and 8N will be (= 8 × 2,327,834,559,873) = 18,622,676,478,984, and a^5 will be (= $\overline{298}|^5$) = 2,350,072,823,968, and $3a^5$ will be (= 3 × 2,350,072,823,968) = 7,050,218,471,904, and $8N - 3a^5$ will be (= 18,622, 676,478,984 — 7,050,218,471,904) = 11,572,458,007,080, and $80a^3$ will be (= 80 × 26,463,592) = 2,117,087,360, and $\frac{8N-3a^5}{80a^3}$ will be (= $\frac{11{,}572{,}458{,}007{,}080}{2{,}117{,}087{,}360}$) = 5466.216, 569,863,2, and $\sqrt{\frac{8N-3a^5}{80a^3}}$ will be (= $\sqrt{5466.216569{,}8632}$) = 72.933,866,190, and conſequently $\frac{3a}{4}$ + $\sqrt{\frac{8N-3a^5}{80a^3}}$ will be (= 223.5 + 72.933,866,190) = 297.433,866,190. Therefore 297.433,866,190 will be a ſecond near value of the fifth root of the propoſed number 2,327,834,559,873. Q. E. I.

Theſe two numbers 297.433,874,395, and 297.433,866, 190, agree with each other in the firſt ſeven figures 297.433,8. Therefore we may conclude that theſe ſeven figures are exact, or are the firſt ſeven figures of a more accurate value of the fifth root of the propoſed number 2,327,834,559,873.

Art. 37. If we make uſe of Mr. Raphſon's expreſſion, to wit, $a - \frac{a^m - N}{ma^{m-1}}$, or $a - \frac{a^5 - N}{5a^4}$, for the purpoſe of obtaining a ſecond near value of the fifth root of the propoſed number 2,327,834,559,873, after 298 has been taken for a, or its firſt near value, which is ſomewhat greater than the truth, the computation of it will be as follows.

Since a is = 298, we ſhall have a^4 = 7,886,150,416, and a^5 = 2,350,072,823,968, and $a^5 - N$ (= 2,350,072,823,968 — 2,327,834,559,873) = 22,238,264,095, and $5a^4$ (= 5 × 7,886,150,416) = 39,430,752,080, and $\frac{a^5 - N}{5a^4}$ (= $\frac{22{,}238{,}264{,}095}{39{,}430{,}752{,}080}$) = 0.563,982, and $a - \frac{a^5 - N}{5a^4}$ (= 298.000,000 — 0.563,982) = 297.436,018. Therefore 297.436,018 will be a ſecond near value of the fifth root of the propoſed number 2,327,834,559,873. Q. E. I.

The firſt five figures, 297.43, of this number, 297.436,018, obtained by Mr. Raphſon's expreſſion, are exact.

And, if we make $a = 297.436$, and repeat the application of Mr. Raphſon's expreſſion, we ſhall obtain the value of the fifth root of the ſaid propoſed number 2,327,834, 559,873, to a much greater degree of exactneſs. This may be done in the manner following.

If a is taken $=$ 297.436, we ſhall have

$a^4 = 7{,}826{,}617{,}827.880{,}165{,}417{,}216$, and

$a^5 = 2{,}327{,}917{,}900{,}253.364{,}881{,}035{,}058{,}176$, and

$a^5 - \text{N}$ ($= 2{,}327{,}917{,}900{,}253.364{,}881{,}035{,}058{,}176$

$- 2{,}327{,}834{,}559{,}873.000{,}000{,}000{,}000{,}000$)

$= 83{,}340{,}380.364{,}881{,}035{,}058{,}176$,

and $5a^4$ ($= 5 \times 7{,}826{,}617{,}827.880{,}165{,}417{,}216$) $=$ 39,133,089,139.400,827,086,080, and conſequently $\frac{a^3 - \text{N}}{5a^4}$ $\left(= \frac{83{,}340{,}380.364{,}881{,}035{,}058{,}176}{39{,}133{,}089{,}139.400{,}827{,}086{,}080}\right) = 0.002{,}129{,}662$, and $a - \frac{a^3 - \text{N}}{5a^4}$ ($= 297.436{,}000{,}000 - 0.002{,}129{,}662$) $=$ 297.433,870,338. Therefore 297.433,870,338 will be the more accurate value of the fifth root of the propoſed number 2,327,834,559,873. Q. E. I.

Of this number, 297.433,870,338, which we have now found for the laſt near value of the fifth root of 2,327,834, 559,873, I believe the firſt ten figures 297.453,870,3 to be exact, if no miſtakes have been made in the calculation. Yet Mr *Raphſon* (from whoſe *Analyſis Æquationum Univerſalis*, Problem IV, page 12, this example is taken,) makes this fifth root equal to 297.433,874,895. But I believe the four laſt figures, 4895, of this number to be erroneous; becauſe in Mr. *Raphſon*'s laſt proceſs, (of which this number, 297.433,874,895, is the reſult,) the value of g (which anſwers to a in our notation,) was taken equal only to 297.46, which is exact only in the firſt four figures 297.4, whereas in the laſt proceſs of the foregoing computation we took a equal to 297.436, which is exact in the firſt five figures, 297.43; and conſequently the number reſulting from this ſuppoſition ought to be more exact than that which reſults from the other leſs accurate ſuppoſition made by Mr. *Raphſon*. But we may, at leaſt, conclude that the firſt eight figures, 297.433,87, of thoſe two numbers, which are the ſame in both, are exact, or are the firſt eight figures of a number approaching more nearly than either of them to the true value of the fifth root of the propoſed number 2,327,834,559,873.

Art. 38. Theſe two examples will, I preſume, be ſufficient to illuſtrate *Monſieur de Lagny*'s method of extracting the mth root of any propoſed number N, by means of either of the two expreſſions, $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$, when a, or the firſt near value of $\sqrt[m]{N}$, which is ſuppoſed to be already known, is greater than its true value; which is the caſe ſuppoſed above in Problem II, by the ſolution of which thoſe expreſſions were obtained. I have therefore nothing more to add concerning the explanation of *Monſieur de Lagny*'s method aforeſaid. But I will juſt make another obſervation, or two, concerning the ſaid method of extracting the roots of numbers, as compared with other methods of performing the ſame thing.

Obſervations on the ſeveral different Methods that may be taken for Extracting the Roots of Numbers.

Art. 39. In the 1ſt place, then, it is manifeſt that either of Mr. *de Lagny*'s two expreſſions, the rational one and the irrational one, for obtaining a ſecond near value of the root of a given number, when a former near value of it is already known, is greatly to be preferred to *the common*, or, rather, *the old*, method of extracting ſuch root, by which, with a great deal of trouble, we obtain only one new figure of the root ſought by every new proceſs; except, perhaps, in extracting the ſquare-root of a number, which is eaſy enough in the common way, (at leaſt for the firſt three or four figures of the root ſought,) to make it unneceſſary to have recourſe to other methods. But in extracting the cube-root, or the fifth root, or the ſeventh root, or any higher root, of a propoſed number, the caſe is very different, and it will be found highly expedient to have recourſe either to Mr. *de Lagny*'s method of extracting them, or to Mr. *Raphſon*'s, or to ſome other method of performing the ſaid extraction.

Secondly, if the *m*th root of any number is to be found only to four, or five, figures, it will be moſt adviſeable to have recourſe to a Table of Logarithms for this purpoſe. For, by the uſe of ſuch a Table, we may always obtain any propoſed root of a given number exact to four, or five, places of figures, with very great eaſe, and without making uſe of the proportional parts ſet down in thoſe tables, and which are neceſſary to the obtaining the ſaid roots exactly to ſix, or ſeven, or more, places of figures. Whenever therefore we want to find the *m*th root of a propoſed number only to four, or five, places of figures, it ſeems beſt to have recourſe at once to a Table of Logarithms for that purpoſe.

But, 3dly, if we wiſh to obtain the *m*th root of any number, exactly to nine, or ten, or more, places of figures, it will be convenient to have recourſe to either Mr. *de Lagny*'s or Mr. *Raphſon*'s methods of approximation for that purpoſe. And, if we wiſh to obtain the ſaid root exact only to nine places of figures, I ſhould think it would be expedient to make uſe of Mr. *Raphſon*'s expreſſion for that purpoſe, in preference to either of Mr. *de Lagny*'s expreſſions, as being ſimpler and eaſier to compute than the latter; but, if we wiſh to obtain the ſaid root exact to fourteen, or fifteen, places of figures, I ſhould think it would be moſt adviſeable to have recourſe to one of Mr. *de Lagny*'s expreſſions for that purpoſe, rather than to make uſe of Mr. *Raphſon*'s expreſſion, and repeat the proceſs a ſecond time, as was done above in art. 37. And of Mr. *de Lagny*'s two expreſſions, the latter, or irrational, expreſſion will be found leſs troubleſome to compute, and, uſually, in a ſmall degree more exact, than the rational expreſſion. But it may often be prudent to compute them both, to be checks upon each other; and the number of figures in which the reſults of both expreſſions are found to agree, may be juſtly concluded to be exact.

And, 4thly, when we make uſe of either Mr. *Raphſon*'s or Mr. *de Lagny*'s methods of extracting the *m*th root of a given number, I conceive it will be always adviſeable to make uſe of a Table of Logarithms firſt, in order to obtain the firſt near value of the root ſought, from which we are afterwards to derive a ſecond and more accurate near value of it, by means of the expreſſions invented by thoſe ingenious Gentlemen. This, indeed, is not abſolutely neceſſary, as it is always eaſy to find the propoſed root exact to one, or two, figures, by ſome very ſimple reaſonings and trials, as is ſhewn above in all the foregoing examples. But it will always be ſtill eaſier to find theſe firſt figures by the help of a Table of Logarithms, and we may find them by that means not only to two places of figures, but to five.

End of the Tract, intitled, Mr. de Lagny'*s General Method of Extracting the Roots of Numbers by Approximation.*

OBSERVATIONS

ON

MR. RAPHSON'S METHOD

OF

RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

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ON
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Article 1. IN the foregoing Tract I have given a pretty full explanation of *Monſieur de Lagny*'s Method of Extracting the Roots of Numbers by Approximation, and I have likewiſe mentioned Mr. *Raphſon*'s more ſimple and eaſy, though leſs exact, method of performing the ſame thing. But both theſe methods may be applied to the reſolution of all ſorts of equations, thoſe which are called *affected* equations*, or in which the unknown quantity occurs in more than one term, as well as thoſe which are called *pure* equations, or in which the unknown quantity occurs in only one term, and which are reſolved by the mere extraction of the roots of given numbers. And in all affected equations beyond biquadraticks, or thoſe of the fourth power, theſe methods of approximation are the only methods that can be taken for diſcovering their roots, or the values of the unknown quantities contained in them. And even in cubick and biquadratick equations, though particular methods have been invented by Mathematicians, for the accurate reſolution of moſt of the caſes of theſe equations, (to wit, the rules called *Cardan*'s *rules* for the reſolution of moſt caſes of cubick equations, and the rules invented by *Lewis Ferrari* of *Bologna* in Italy, about the year 1545, and explained at large in *Bombelli*'s Algebra, in the year 1579, and thoſe afterwards invented by *Monſieur Des Cartes*, and publiſhed in his Geometry in the year 1637,

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* This expreſſion of *affected equations* ſeems to require ſome further explanation. It was introduced by the celebrated Vieta, the great father and reſtorer of Algebra. He has many expreſſions peculiar to himſelf, and which have not been adopted by ſubſequent Algebräiſts. Amongſt theſe are the following ones. He calls a ſet of quantities in continual geometrical proportion, (ſuch as the quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c,) a ſet of *ſcalar* quantities, or *magnitudines ſcalares*; and, when there are ſeveral of theſe *ſcalar* quantities connected with each other by the ſigns + and −, or by Addition and Subtraction, (as in the compound quantity $x^5 + ax^4 - b^2x^3$,) he calls the higheſt quantity, or that which is fartheſt in the ſcale of quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, (to wit, the quantity x^5 in the ſaid compound quantity $x^5 + ax^4 - b^2x^3$,) *the power* of the fundamental quantity x, or of the ſecond term in the ſaid ſcale; and he calls the lower ſcalar quantities, which are involved in the ſecond and third terms of the ſaid compound quantity $x^5 + ax^4 - b^2x^3$, to wit, the quantities

for

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ties x^4 and x^3, (or, in our preſent language, the inferiour powers of x,) ſcalar quantities of a *parodic* degree to x^5, or *the power* of the fundamental quantity x. This word *parodic* I take to be derived (though Vieta does not tell us ſo,) from the Greek words παρὰ and ὁδὸς, which ſignify *near* and *a way*, or *road*, becauſe theſe inferiour ſcalar quantities, x^3 and x^4, lie *in the way* as you paſs along in the ſcale of the aforeſaid quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, from 1 to x^5, which he calls *the power* of x in the ſaid compound quantity $x^5 + ax^4 - b^2x^3$. Theſe inferiour ſcalar quantities x^3 and x^4 are therefore *parodic*, or *ſituated in the way to*, or are *leading to*, the ſaid *power*, or higher ſcalar quantity, x^5. He then proceeds to define *a pure power* and *an affected power*, and tells us, that *a pure power* is a ſcalar quantity that is not affected with, or mixed with, any *parodic*, or *inferiour* ſcalar quantity, and that *an affected power* is a ſcalar quantity that is mixed, or connected by Addition, or Subtraction, with one, or more, *inferiour*, or *parodic*, ſcalar quantities, combined with co-efficients that raiſe them to the ſame dimenſion as the *power* itſelf, or make them *homogeneous* to it, and conſequently capable of being added to it, or ſubtracted from it. Thus x^5 alone is *a pure power* of x, namely, its fifth power; and $x^5 + ax^4 - b^2x^3$ is *an affected power* of x, namely, its fifth power *affected by*, or *connected with*, the two *parodic*, or *inferiour* ſcalar quantities, x^3 and x^4, which are multiplied into bb and a, in order to make them *homogeneous to*, or *of the ſame dimenſion with*, x^5 itſelf, and conſequently capable of being added to it, or ſubtracted from it. See Schooten's edition of Vieta's Works, publiſhed at Leyden in Holland, in the year 1646, pages 3 and 4.

This, then, being the meaning of the expreſſions *a pure power* and *an affected power*, the meaning of the correſponding expreſſions of *a pure equation*

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for the reſolution of biquadratick equations, by the mediation of cubick equations,) it will be found that theſe methods of approximation will, for the moſt part, enable us to find the values of their roots to any propoſed degree of exactneſs, with leſs trouble than the particular and accurate methods above-mentioned, which have been invented for that purpoſe. So that theſe methods of reſolving equations by approximation ought to be conſidered as of the higheſt utility, and as being abſolutely neceſſary to the completion of the Doctrine of the Reſolution of Algebräick Equations, which is the moſt important branch of the Science of Algebra.

Art. 2. But it is not ſo eaſy to determine, which of theſe two methods of approximation, Mr. *Raphſon*'s, or Mr. *de Lagny*'s, deſerves to be preferred to the other on theſe occaſions. Mr. *Raphſon*'s is certainly much ſimpler than the other, becauſe it proceeds by conſidering the new, or tranſformed, equation, (reſulting from the ſubſtitution of $a + z$,

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tion and *an affected equation* follows from it of courſe: *a pure equation* ſignifying an equation in which a pure power of an unknown quantity is declared to be equal to ſome known quantity; ſuch as the equation $x^5 = 79$; and *an affected equation* ſignifying an equation in which a power of an unknown quantity affected by, or connected, either by Addition or Subtraction, with, ſome inferiour powers of the ſame unknown quantity, (multiplied into proper co-efficients in order to make them *homogeneous* to the ſaid higheſt power of the ſaid unknown quantity,) is declared to be equal to ſome known quantity; ſuch as the equation $x^5 + ax^4 - b^2x^3 = 79$. This I take to be the original meaning of the expreſſion *an affected equation*. But, as the language of *Vieta* has not been adopted by ſubſequent writers of Algebra, I ſhould think it would be more convenient to call them by ſome other name. And, perhaps, thoſe of *binomial*, *trinomial*, *quadrinomial*, *quinquinomial*, and, in general, that of *multinomial* equations, would be as convenient as any. Thus, $xx + ax = rr$, and $x^3 + ax^2 = r^3$, and $x^3 + a^2x = r^3$, and $x^4 + a^3x = r^4$, and $x^4 + ax^3 = r^4$, might all be called *binomial* equations, becauſe they would be equations in which a *binomial* quantity, or quantity conſiſting of two terms that involved the unknown quantity x, is declared to be equal to a known quantity; and, for a like reaſon, the equations $x^3 + ax^2 + b^2x = r^3$, and $x^4 - ax^3 + b^2x^2 = r^4$, and $x^4 - ax^3 + b^3x = r^4$, and $x^5 + ax^4 + b^2x^3 = r^5$, and $x^5 + ax^4 - b^2x^3 = r^5$, and $x^5 + b^2x^3 + c^4x = r^5$, might be called *trinomial* equations. And the like names might be given to equations of a greater number of terms. Dr. Hutton, I obſerve, in his excellent new Mathematical and Philoſophical Dictionary, juſt now publiſhed, (Feb. 2, 1795,) calls them *compound* equations; which is likewiſe a very proper name for them, and leſs obſcure than that of *affected* equations.

or $a - z$, inſtead of x, in the original equation,) as being only a ſimple equation, and reſolving it accordingly, or by the mere operation of Diviſion; whereas, in Mr. *de Lagny's* method, the ſaid new, or transformed, equation is conſidered as a quadratick equation, and reſolved accordingly; which, when *a* (or the firſt near value of the root, that is ſuppoſed to be already known,) is a number conſiſting of five, or ſix, figures, produces a great deal of labour, and often a great deal of perplexity. I am therefore inclined to give the preference to Mr. *Raphſon's* method in reſolving all affected equations, more eſpecially when the number *a* conſiſts of more than two figures: but it muſt be confeſſed that the celebrated Dr. *Halley* (who had much experience, and was an excellent judge of theſe matters,) was of a different opinion, and gave the preference to Mr. *de Lagny's* method, which he has therefore taken the pains to explain in a better manner than had been done by *Monſieur de Lagny* himſelf, and likewiſe to illuſtrate by examples, in his Tract in the Philoſophical Tranſactions, Number 210, intitled, "*A New, Exact, and Eaſy Method, of finding the* "*Roots of any Equations Generally, and that without any pre*-"*vious Reduction*," which was publiſhed in the year 1694. On the other hand we may obſerve, that Mr. *Raphſon* always continued to give his own method the preference, after the publication of the tracts of *Monſieur de Lagny* and Dr. *Halley* upon the ſubject, as well as before their publication, when he tells us he had himſelf had the thought of adopting the principle which was afterwards followed by Mr. *de Lagny* and Dr. *Halley*, of treating the transformed equation as a quadratick equation, but had deliberately rejected it on account of the greater eaſe and ſimplicity of the other method, in which the ſaid transformed equation is conſidered and treated as a ſimple equation. And Sir *Iſaac Newton* in his method of reſolving equations by approximation (which differs very little from Mr. *Raphſon's*,) ſeems alſo to prefer Mr. *Raphſon's* practice, of treating the transformed equation as a mere ſimple equation, to that of Mr. *de Lagny* and Dr. *Halley*, of treating the ſaid equation as a quadratick equation. I therefore cannot but recommend it to all young Algebräiſts to ſtudy Mr. *Raphſon's* excellent Treatiſe on this ſubject, intitled, *Analyſis Æquationum Univerſalis*, with great attention, and to endeavour to make themſelves maſters of it, by going carefully through all the examples given in it, and performing all the arithmetical operations contained in them. And I will venture to ſay that they will thereby acquire more uſeful knowledge in Algebra, towards the buſineſs of reſolving affected, or compound, or multinomial, equations, than by reading all that has been written by *Harriot* and *Des Cartes*, and his learned Commentator *Van Schooten*, and all his other Commentators, and their numerous followers, on the boaſted doctrine of the Generation of Equations one from another, by ſuppoſing $x - a$ to be $= 0$, and $x - b$ to be $= 0$, and $x - c$ to be $= 0$, and $x + d$ to be $= 0$, and $x + e$ to be $= 0$, and ſo on; and then multiplying the binomial quantities $x - a$, $x - b$, $x - c$, $x + d$, $x + e$, &c, into each other, and likewiſe all the abſtruſe and intricate matter that has been delivered by Sir *Iſaac Newton*, and Mr. *Graveſende* and Mr. *Mac Laurin*, and other learned Algebräiſts of modern times, on the invention of Diviſors, which is grounded on that doctrine of the Generation of Equations from each other.

Art. 3. Yet in reading this excellent Treatiſe of Mr. *Raphſon*, which I ſo much recommend, there will now and then occur ſome difficulties which are not inherent in the ſubject itſelf, but which might have been avoided, if Mr. *Raphſon* had not unfortunately adopted the perplexing doctrines of modern writers of Algebra, about negative quantities and negative roots of Equations. The quantities called *negative* are ſuch as it is impoſſible to form any clear idea of, being defined, by Sir *Iſaac Newton* and other Algebräiſts *, to be ſuch quantities as are *leſs than nothing*, or as *ariſe from the ſubtraction of a greater quantity from a leſſer*, which is an operation evidently impoſſible to be performed: and, as to the *negative roots* of an equation, they are in truth the real and poſitive roots of another equation conſiſting of the ſame terms as the firſt equation, but with different ſigns + and — prefixed to ſome of them; ſo that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and ſuppoſe the propoſed, or firſt, equation to have not only its own proper roots (which they call its *affirmative*, or *poſitive*, roots,) but to have likewiſe the roots of a different equation, which they call its *negative* roots. Thus, for example, they would ſay, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the poſitive, or affirmative, root, + 16, and the negative root, — 20. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$. So that this kind of abſurd and fantaſtick language only tends to the confounding together the two different equations $xx + 4x = 320$, and $xx - 4x = 320$, and conſidering them as if they were one and the ſame equation. Now this perplexing language is unfortunately uſed by Mr. *Raphſon* in this valuable Treatiſe, and tends to throw an air of myſtery and obſcurity upon ſome of the Problems ſolved in it, from which they would otherwiſe have been intirely free. As a proof of the truth of this obſervation, I ſhall here inſert one of the ſaid Problems, the ſolution of which is by this means rendered ſo obſcure, that I had a good deal of trouble to find out the meaning of it; though, if this language had been avoided, and the proper and natural language, belonging to the conditions of the Problem, had been uſed in its ſtead, there could not have been the leaſt difficulty in underſtanding it. This Problem is the 24th, in page 32 of the 2d edition of the book, and is, *verbatim et litteratim*, as follows.

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* Quantitates vel *Affirmativæ* ſunt, ſeu majores Nihilo, vel *Negativæ*, ſeu Nihilo minores.——Newton's *Arithmetica Univerſalis*, page 3.

When a greater quantity is taken from a leſſer of the ſame kind, the remainder becomes of the oppoſite kind.——*Mac Laurin's Algebra*, page 5.

An affirmative quantity is a quantity greater than nothing, and is known by this ſign, +; a negative quantity is a quantity leſs than nothing, and is known by this ſign, −.——*Saunderſon's Algebra*, Vol. I. page 50, article 2.

PROBLEMA XXIV.

Æquationum Quintæ Poteſtatis Adfectarum Solutio.

Proponatur $-aaaaa + 7aaaa - 20aaa + 155aa = 10{,}000.$

Hoc eſt, $-aaaaa + baaaa - caaa + daa = f.$

$$\text{Theor. } e = \frac{f + ggggg + cggg - bgggg - dgg}{4bggg + 2dg - 5gggg - 3cgg}$$

Sit $g = -5$

$f + ggggg + cggg - bgggg - dgg = -3875$

$4bggg + 2dg - 5gggg - 3cgg = -9675)\ (-3875{,}0\ (+{,}4 = e$

$$\begin{array}{r} -5, \\ +{,}4 \\ \hline \end{array}$$

$g = -4{,}6$

$f + ggggg + cggg - bgggg - dgg = -420{,}36896$

$4bggg + 2dg - 5gggg - 3cgg = -7659{,}736)\ -420{,}36896\ (+{,}055 = e$

$$\begin{array}{r} -4{,}6 \\ +\ {,}055 \\ \hline \end{array}$$

$g = -4{,}545$

$f + ggggg + cggg - bgggg - dgg = -5{,}960359465465625$

$4bggg + 2dg - 5gggg - 3cgg = -7410{,}748)\ -5{,}9603594\ (+{,}00080428 = e$

$$\begin{array}{r} -4{,}545 \\ +{,}000{,}804{,}28 \\ \hline \end{array}$$

$a = -4{,}544{,}195{,}72$

To this ſolution I have, in my copy of Mr. Raphſon's Tract, ſubjoined the following Note.

Numerus 4.544,195,72 eſt radix æquationis $a^5 + 7a^4 + 20a^3 + 155a^2 = 10{,}000$; quod hic obſcurè innuitur ſub ſpecie radicis negativæ æquationis $-a^5 + 7a^4 - 20a^3 + 155a^2 = 10{,}000$. Omnes ferè difficultates quibus permulti cultioris ingenii viri ab Algebrâ diſcendâ et excolendâ deterrentur, ex hiſce radicibus negativis et aliis quantitatibus negativis,

gativis, ſeu (ut hodierni Algebræ ſcriptores abſurdè loquuntur,) nihilo minoribus, ortum habent.

In this Problem the letter *a* is uſed for the unknown quantity, or root of the equation, which is uſually denoted by the letter *x*; and the letter *g* is uſed for the firſt near value of the root of the equation, which in the two foregoing Tracts has been denoted by the letter *a*; and the letter *x* is uſed for the difference between *g*, the firſt near value of the root of the equation, and *a*, its true value, which difference has been denoted in the two foregoing Tracts by the letter *z*. So that, if we expreſs the enunciation of the foregoing Problem in the notation that has been uſed in the two foregoing Tracts, it will be as follows.

$$\text{Proponatur } -xxxxx + 7xxxx - 20xxx + 155xx = 10{,}000,$$
$$\text{Sive } -x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000,$$

$$\text{Hoc eſt, } -xxxxx + bxxxx - cxxx + dxx = f,$$
$$\text{Sive } -x^5 + bx^4 - cx^3 + dx^2 = f.$$

$$\text{Theor. } z = \frac{f + aaaaa + caaa - baaaa - daa}{4baaa + 2da - 5aaaa - 3caa},$$
$$\text{or } z = \frac{f + a^5 + ca^3 - ba^4 - da^2}{4ba^3 + 2da - 5a^4 - 3ca^2}.$$

Art. 4. Here, then, the equation propoſed by Mr. *Raphſon* to be reſolved, is ſaid to be $-x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000$, or $155xx - 20x^3 + 7x^4 - x^5 = 10{,}000$. But this is not the equation he reſolves; and, indeed, it is not a poſſible equation, becauſe the greateſt poſſible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ is that which it has when the infinitely ſmall increment of the binomial quantity $20x^3 + x^5$ becomes equal to the contemporary increment of the binomial quantity $155xx + 7x^4$, that is, (if we put $\dot{x}$, or *x* with a point placed over it, for the infinitely ſmall increment of *x*,) when $20 \times 3x^2\dot{x} + 5x^4\dot{x}$ becomes equal to $155 \times 2x\dot{x} + 7 \times 4x^3\dot{x}$, or when $60x^2 + 5x^4$ is $= 310x + 28x^3$, or when $60x + 5x^3$ is $= 310 + 28xx$, or when $5x^3 - 28xx + 60x$ is $= 310$, or when $x^3 - \frac{28xx}{5} + 12x$ is $= 62$, or when $x^3 - 5.6xx + 12x$ is $= 62$; and that is when x is nearly $= 5.5$; at which time the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ will be nearly equal to 2733, as will appear by ſubſtituting 5.5 inſtead of x in the terms of the ſaid quantity $155xx - 20x^3 + 7x^4 - x^5$: and this quantity 2733 (which is the greateſt poſſible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$,) is very much leſs than 10,000, or the abſolute term of the equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, and conſequently the ſaid equation is impoſſible. But Mr. *Raphſon*, though he ſets down this equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, as the equation that is to be reſolved, yet really means to reſolve a quite different equation, to wit, the equation that reſults from ſuppoſing x to be a negative quantity, or from ſubſtituting the powers of $-x$, to wit, $+xx$, $-x^3$, $+x^4$, and $-x^5$, in the terms of the ſaid equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, inſtead of the like powers of $+x$, to wit, $+xx$, $+x^3$, $+x^4$, and $+x^5$; by which ſubſtitution the ſaid equation will be converted into the equation $155 \times +xx - 20 \times -x^3 + 7 \times +x^4 - 1 \times -x^5 = 10,000$, or $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, which is evidently a poſſible equation, and of which the root is 4.544, 195,72, or the ſame number which he obtains by his ſolution of the Problem, and which, with the ſign $-$ prefixed to it, he calls the negative root of the propoſed equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$. Now all this perplexity would have been avoided, if Mr. *Raphſon* had propoſed at firſt to find the root, or, in the language of modern writers of Algebra, the *affirmative*, or *poſitive*, root, of the equation $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, or $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, which equation is evidently poſſible, and can have only one root. And then all the ſteps of his ſolution would have been clear and eaſy, as will appear by reſolving this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$ according to the principles of his method; which may be done in the manner following.

The Reſolution of the Affected Equation $x^5 + 7x^4 + 20x^3 + 155x = 10,000$, *by Mr.* Raphſon's *Method of Approximation.*

Art. 5. In conſidering this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, it is, in the 1ſt place, eaſy to ſee that x muſt be greater than 1. For, if we ſuppoſe x to be $= 1$, we ſhall have $xx = 1$, and $x^3 = 1$, and $x^4 = 1$, and $x^5 = 1$; and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ will be $= 1 + 7 + 20 + 155 = 183$; which is very much leſs than the abſolute term 10,000. Therefore 1 muſt be much leſs than x.

In the ſecond place, if we ſuppoſe x to be $= 10$, we ſhall have $xx = 100$, and $x^3 = 1000$, and $x^4 = 10,000$, and $x^5 = 100,000$; ſo that x^4 alone will be equal to the abſolute term 10,000, and conſequently $x^5 + 7x^4 + 20x^3 + 155x$ muſt be very much greater than the ſaid abſolute term; and conſequently 10 muſt be much greater than x.

Thirdly, ſince x is leſs than 10 and greater than 1, let us ſuppoſe it to be equal to 5. Then we ſhall have $xx = 25$, and $x^3 = 125$, and $x^4 = 625$, and $x^5 = 3125$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 3125 + 7 \times 625 + 20 \times 125 + 155 \times 25 = 3125 + 4375 + 2500 + 3875$) $= 13,875$; which is greater than the abſolute term 10,000. Therefore 5 is greater than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

We will therefore, in the 4th place, ſuppoſe x to be $= 4$. And then we ſhall have $xx = 16$, and $x^3 = 64$, and $x^4 = 256$, and $x^5 = 1024$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 1024 + 7 \times 256 + 20 \times 64 + 155 \times 16 = 1024 + 1792 + 1280 + 2480$) $= 6576$; which is leſs than the abſolute term 10,000. Therefore 4 is leſs than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

It appears therefore that the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, is greater than 4, but leſs than 5. And either of theſe values might very well ſerve for a firſt near value of the ſaid root, or for the baſis of a further approximation to it. Mr. Raphſon makes choice of 5, which is greater than the truth.

Art. 6. Let us then ſuppoſe a, or the firſt near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, to be $= 5$; and let z be the difference by which it exceeds the true value of x. Then will x be $= a - z$, and conſequently xx will be $(= \overline{a-z}|^2) = aa - 2az + \&c$, and x^3 will be $(= \overline{a-z}|^3) = a^3 - 3a^2z + \&c$, and x^4 will be $(= \overline{a-z}|^4) = a^4 - 4a^3z + \&c$, and x^5 will be $(= \overline{a-z}|^5) = a^5 - 5a^4z + \&c$. Therefore $x^5 + 7x^4 + 20x^3$

$$+ 155xx \text{ will be } = \left\{ \begin{array}{rrl} & & \overline{a^5 - 5a^4z + \&c,} \\ + & 7 \times & \overline{a^4 - 4a^3z + \&c,} \\ + & 20 \times & \overline{a^3 - 3a^2z + \&c,} \\ + & 155 \times & \overline{aa - 2az \;\; + \&c,} \end{array} \right\}$$

$$= \left\{ \begin{array}{rrl} & a^5 - & 5a^4z + \&c, \\ + & 7a^4 - & 28a^3z + \&c, \\ + & 20a^3 - & 60a^2z + \&c, \\ + & 155aa - & 310az \;\; + \&c. \end{array} \right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z - 28a^3z - 60a^2z - 310az + \&c$, will alſo be $= 10,000$, and conſequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both ſides,) we ſhall have $a^5 + 7a^4 + 20a^3 + 155aa = 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$, or (becauſe a is $= 5$, and conſequently $a^5 + 7a^4 + 20a^3 + 155aa$ is $= 13,875$, as has been ſhewn in art. 5,) we ſhall have $13,875 = 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$, and conſequently (ſubtracting 10,000 from both ſides,) $3875 = 5a^4z + 28a^3z + 60a^2z + 310az = z \times \overline{5a^4 + 28a^3 + 60a^2 + 310a}$.

Therefore z will be $= \frac{3875}{5a^4 + 28a^3 + 60a^2 + 310a}$ $(= \frac{3875}{5 \times \overline{5}|^4 + 28 \times \overline{5}|^3 + 60 \times \overline{5}|^2 + 310 \times 5} = \frac{3875}{5 \times 625 + 28 \times 125 + 60 \times 25 + 310 \times 5} = \frac{3875}{3125 + 3500 + 1500 + 1550} = \frac{3875}{9675}) = 0.4$. Therefore $a - z$, or x, will be $(= a - 0.4 = 5.0 - 0.4) = 4.6$; and 4.6 will be a ſecond near value of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

We muſt next try whether this ſecond near value of x is greater or leſs than its true value; and for this purpoſe we muſt ſubſtitute it, inſtead of x, in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$.

Now, if we ſuppoſe x to be $= 4.6$, we ſhall have xx $(= \overline{4.6}|^2) = 21.16$, and x^3 $(= \overline{4.6}|^3) = 97.336$, and x^4 $(= \overline{4.6}|^4) = 447.7456$, and x^5 $(= \overline{4.6}|^5) = 2059.62976$, and $155xx$ $(= 155 \times 21.16) = 3279.80$, and $20x^3$ $(= 20 \times 97.336) = 1946.720$, and $7x^4$ $(= 7 \times 447.7456) = 3134.2192$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 2059.62976 + 3134.2192 + 1946.720 + 3279.80)$ $= 10,420.36896$; which is greater than 10,000, or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.6 will be greater than the true value of x in that equation.

Art. 7. To find a third near value of the root of this equation, let a be ſuppoſed to be $= 4.6$, and z be the difference by which a, or 4.6, exceeds the true value of the ſaid root.

Then we ſhall have, as before, $x = a - z$, and conſequently xx $(= \overline{a - z}|^2) = aa - 2az +$ &c, and x^3 $(= \overline{a - z}|^3) = a^3 - 3a^2z +$ &c, and x^4 $(= \overline{a - z}|^4) = a^4 - 4a^3z +$ &c, and x^5 $(= \overline{a - z}|^5) = a^5 - 5a^4z +$ &c,

and $7x^4$ ($= 7 \times \overline{a^4 - 4a^3z + \&c,}$) $= 7a^4 - 28a^3z +$ &c, and $20x^3$ ($= 20 \times \overline{a^3 - 3a^2z + \&c,}$) $= 20a^3 - 60a^2z$ + &c, and $155xx$ ($= 155 \times \overline{aa - 2az + \&c,}$) $= 155aa - 310az$ + &c, and $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rrrl} & a^5 & - \;\; 5a^4z & + \;\&c, \\ + & 7a^4 & - \;\; 28a^3z & + \;\&c, \\ + & 20a^3 & - \;\; 60a^2z & + \;\&c, \\ + & 155aa & - \;\; 310az & + \;\&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $\left\{\begin{array}{rrrl} & a^5 & - \;\; 5a^4z & + \;\&c, \\ + & 7a^4 & - \;\; 28a^3z & + \;\&c, \\ + & 20a^3 & - \;\; 60a^2z & + \;\&c, \\ + & 155aa & - \;\; 310az & + \;\&c, \end{array}\right\}$ will likewiſe be $= 10{,}000$, and conſequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both ſides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been ſhewn in the laſt article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.6}|^5 + 7 \times \overline{4.6}|^4 + 20 \times \overline{4.6}|^3 + 155 \times \overline{4.6}|^2$, is $= 10{,}420.36896$.

Therefore 10,420.36896 will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and conſequently (ſubtracting 10,000 from both ſides of the equation,) 420.36896 will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ ($= 5 \times 4.6|^4 \times z + 28 \times \overline{4.6}|^3 \times z + 60 \times \overline{4.6}|^2 \times z + 310 \times 4.6 \times z = 5 \times 447.7456 \times z + 28 \times 97.336 \times z + 60 \times 21.16 \times z + 310 \times 4.6 \times z = 2238.7280 \times z + 2725.408 \times z + 1269.60 \times z + 1426.0 \times z$) $= 7659.7360 \times z$, and conſequently z will be ($= \frac{420.36896}{7659.7360}$) $= 0.0548$, or nearly 0.055. Therefore x, or $a - z$, or $4.6 - z$, will be nearly ($= 4.6 - 0.055$,) $= 4.545$; and conſequently this number 4.545 will be a third near value of the root of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Q. E. I.

Now let this number 4.545 be ſubſtituted inſtead of x in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$, in order to diſcover whether the reſult will be greater, or leſs, than 10,000, or the abſolute term of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Now, if x be ſuppoſed to be $= 4.545$, we ſhall have xx $(= \overline{4.545}|^2) = 20.657,025$, and x^3 $(= \overline{4.545}|^3) = 93.886,178,625$, and x^4 $(= \overline{4.545}|^4) = 426.712,681,850,625$, and x^5 $(= \overline{4.545}|^5) = 1939.409,139,011,090,625$, and conſequently $7x^4$ $(= 7 \times 426.712,681,850,625) = 2986.988,772,954,375$, and $20x^3$ $(= 20 \times 93.886,178,625) = 1877.723,572,500$, and $155xx$ $(= 155 \times 20.657,025) = 3201.838,875$, and $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 1939.409,139,011,090,625 + 2986.988,772,954,375 + 1877.723,572,500 + 3201.838,875) = 10,005.960,359,465,465,625$; which is greater than 10,000, or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.545 will be greater than the true value of x in that equation.

Art. 8. To find a fourth near value of the root of this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, let a be ſuppoſed to be $= 4.545$, and z be ſuppoſed to be the difference by which a, or 4.545, exceeds the true value of the ſaid root.

Then we ſhall, as before, have $x = a - z$, and conſequently xx $(= \overline{a - z}|^2) = aa - 2az + \&c$, and x^3 $(= \overline{a - z}|^3) = a^3 - 3a^2z + \&c$, and x^4 $(= \overline{a - z}|^4) = a^4 - 4a^3z + \&c$, and x^5 $(= \overline{a - z}|^5) = a^5 - 5a^4z + \&c$, and $7x^4$ $(= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z + \&c$, and $20x^3$ $(= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z + \&c$, and $155xx$ $(= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az + \&c$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx =$

a^5

$$\left\{\begin{array}{rrrl} & a^5 - & 5a^4z & + \&c, \\ + & 7a^4 - & 28a^3z & + \&c, \\ + & 20a^3 - & 60a^2z & + \&c, \\ + & 155a^2 - & 310az & + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z + \&c, - 28a^3z + \&c, - 60a^2z + \&c, - 310az + \&c$, will likewiſe be $= 10,000$, and conſequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both ſides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been ſhewn in the laſt article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.545}|^5 + 7 \times \overline{4.545}|^4 + 20 \times \overline{4.545}|^3 + 155 \times \overline{4.545}|^2$, is $= 10,005.960,359,465,465,625$.

Therefore $10,005.960,359,465,465,625$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and conſequently (ſubtracting 10,000 from both ſides,) $5.960,359,465,465,625$ will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ $(= 5 \times \overline{4.545}|^4 \times z + 28 \times \overline{4.545}|^3 \times z + 60 \times \overline{4.545}|^2 \times z + 310 \times 4.545 \times z = 5 \times 426.712,681,850,625 \times z + 28 \times 93.886,178,625 \times z + 60 \times 20.657,025 \times z + 310 \times 4.545 \times z = 2133.563,409,253,125 \times z + 2628.813,001,500 \times z + 1239.421,500 \times z + 1408.950 \times z)$ $= 7410.747,910,753,125 \times z$. Therefore z will be $(= \frac{5.960,359,465,465,625}{7410.747,910,753,125}) = 0.000,804,28$, and x, or $a - z$, or $4.545 - z$, will be $(= 4.545,000,00 - 0.000,804,28) = 4.544,195,72$. Therefore $4.544,195,72$ will be a fourth near value of the root of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

This number 4.544,195,72, agrees with the number found by Mr. Raphſon, in all its figures.

Art. 9. The foregoing reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, has been performed at great length, in order to ſet forth, in as clear a manner as poſſible, the ſeveral reaſonings upon which the arithmetical operations uſed in it are grounded, as well as the ſaid operations themſelves. And by ſo doing the ſubject is rendered ſo much eaſier than in Mr. Raphſon's very conciſe and compreſſed way of treating it, (in which all the reaſonings are dropped, and only the arithmetical operations are exhibited,) that, though the above reſolution of the ſaid equation is three, or four, times as long as Mr. Raphſon's, yet I am fully perſuaded that it may be read and underſtood in a third, or fourth, part of the time that is neceſſary to a thorough comprehenſion of Mr. Raphſon's reſolution of it; even if he had not puzzled the matter by talking of the negative root of the equation $-x^5 + 7x^4 - 20x^3 + 155xx = 10,000$. But that this may appear the more clearly, I will now repeat the foregoing reſolution of this equation in the ſtyle and manner of Mr. Raphſon, by omitting the ſeveral reaſonings ſet forth in the foregoing articles, and making uſe of a Canon, or Theorem, for the purpoſe of computing the ſecond, third, and fourth values of z, in the ſame manner as Mr. Raphſon has done.

Art. 10. Since each of the three firſt ſucceſſive near values of x, or the root of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, from which the next near values of it are derived, to wit, the three numbers 5, 4.6, and 4.545, and which are ſucceſſively denoted by the letter a, is greater than the true value of x in the ſaid equation, or than the root of the ſaid equation, it follows that the ſecond, and third, and fourth near values of x will, each of them, be ſucceſſively denoted by the reſidual quantity $a - z$; and conſequently, by applying the reaſonings uſed in art. 6, in order to obtain the values of z, and of $a - z$, or x, we ſhall find that z will be, ſucceſſively, nearly equal to the value of the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a}$, and, therefore, that $a - z$, or x, will be, ſucceſſively, nearly equal to the value of the quantity $a -$ the fraction

$a^5 +$

$\frac{a^5 + 7a^4 + 20a^3 + 155a^2 - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a}$. This, then, is the Theorem, or Canon, by the application of which we are to compute the ſecond, and third, and fourth, near values of $a - z$, or x, after taking 5 for the firſt near value of it, or for the firſt value of a.

Now, if a is $= 5$, we ſhall have $z =$ the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{3875}{9675} = 0.4$. Therefore $a - z$ will be $(= 5 - 0.4) = 4.6$; which will therefore be the ſecond near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Secondly, if a be $= 4.6$, we ſhall have $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{420.36896}{7659.7360} = 0.0548$, or, nearly, 0.055. Therefore $a - z$ will be $(= 4.6 - 0.055) = 4.545$; which will therefore be the third near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$.

Thirdly, if a be $= 4.545$, we ſhall have $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{5.960,359,465,465,625}{7410.747,910,753,125} = 0.000,804,28$. Therefore $a - z$ will be $(= 4.545 - 0.000,804,28) = 4.544,195,72$; which will therefore be the fourth near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

Art. 11. Mr. Raphſon's Canon, or Theorem, for the value of z, is expreſſed more conciſely than the foregoing Theorem, $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a}$. For he uſes the letters b, c, d, and f, for the co-efficients 7, 20, and 155, of the fourth, third, and ſecond, power of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, and for 10,000, the abſolute term of that equation, reſpectively; which produces the following Canon, or Theorem, for the value of z, to wit, $z = \frac{a^5 + ba^4 + ca^3 + da^2 - f}{5a^4 + 4ba^3 + 3ca^2 + 2da}$. But it appears to me that, though we may ſeem to gain ſomething in point of brevity by uſing this very general notation, we loſe as much in the article of perſpicuity, which is a matter of much greater importance. However, this latter reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, which is expreſſed in Mr. Raphſon's conciſe ſtyle and manner, and the foregoing more explicit reſolution of it in art. 5, 6, 7, and 8, (in which the reaſonings, on which the ſeveral arithmetical operations are grounded, are diſtinctly ſet forth and repeated,) are, both of them, the ſame in ſubſtance, and are, as I believe, the very beſt method that can be taken for diſcovering the root of the ſaid equation.

Art. 12. It has been obſerved above in art. 2, that Sir Iſaac Newton's method of reſolving numeral equations by approximation differed but little from Mr. Raphſon's, both methods being founded on the ſame principle of conſidering the new, or transformed, equation, (reſulting from the ſubſtitution of $a + z$, or $a - z$, inſtead of x, in the original equation,) as a mere ſimple equation, or neglecting, or omitting, all the terms of it which involved in them any higher power of z than its ſimple power; which reduces the reſolution of all equations, of whatever orders, to the reſolution of a ſimple equation, or, rather, to the reſolution of ſeveral ſucceſſive ſimple equations, by which we make continual approaches to the true value of the root of the original equation. In this grand principle Sir Iſaac Newton's method and Mr. Raphſon's method perfectly agree; and, in finding the ſecond near value of x, or in making the firſt approximation to the true value of x, after having obtained, by conjecture, or trial, or in ſome other manner, the value of what has been here called a, or a firſt near value of x, or the root ſought, there is not the ſmalleſt difference between them. But in the inveſtigation of the third, and fourth, and other following near values of x, there is a little difference in their manner of proceeding, which the reader may be glad to ſee examined. I ſhall therefore now compare the two methods together, in the caſe of a very eaſy equation, by which Sir Iſaac Newton himſelf has thought proper to illuſtrate his method.

A Compariſon between Sir Iſaac Newton's *and Mr.* Raphſon's *Methods of Reſolving Numeral Equations by Approximation.*

Art. 13. Sir Iſaac Newton's Method of Reſolving Numeral Equations by Approximation, is explained by himſelf in his curious little Tract, intitled, *Analyſis per Æquationes Numero Terminorum Infinitas*, (which was written in the year 1666, and communicated to Dr. Iſaac Barrow, and to Mr. John Collins, and to other learned men of that time, in the year 1669,) by an example; which is as follows.

Art. 14. Let it be required to reſolve the cubick equation $x^3 - 2x = 5$.

Here, in the firſt place, it is eaſy to ſee that x is ſomewhat greater than 2, but much leſs than 3. For, if x is taken equal to 2, we ſhall have $2x = 4$, and $x^3 = 8$, and conſequently $x^3 - 2x$ $(= 8 - 4) = 4$; which is leſs than 5, or the true value of $x^3 - 2x$ in the propoſed equation: and, if x is taken equal to 3, we ſhall have $2x = 6$, and $x^3 = 27$, and conſequently $x^3 - 2x$ $(= 27 - 6) = 21$; which is very much greater than 5, or the true value of $x^3 - 2x$ in the propoſed equation. Therefore the true value of x in that equation muſt be much leſs than 3, and a little greater than 2. Let it therefore be ſuppoſed to be equal to the quantity $2 + z$, in which z denotes the unknown quantity by which the true value of x exceeds 2. And let $2 + z$ be ſubſtituted, inſtead of x, in the propoſed equation $x^3 - 2x = 5$. This may be done as follows.

Since x is $= 2 + z$, we ſhall have x^3 $(= \overline{2}|^3 + 3 \times \overline{2}|^2$
$\times z$

$\times z + 3 \times 2 \times zz + z^3 = 8 + 3 \times 4 \times z + 3 \times \overline{2zz + z^3}) = 8 + 12z + 6zz + z^3$, and $2x\ (= 2 \times \overline{2 + z}) = 4 + 2z$, and conſequently $x^3 - 2x\ (= 8 + 12z + 6zz + z^3 - 4 - 2z) = 4 + 10z + 6zz + z^3$. But $x^3 - 2x$ is $= 5$. Therefore $4 + 10z + 6zz + z^3$ will alſo be $= 5$, and conſequently (ſubtracting 4 from both ſides,) $10z + 6zz + z^3$ will be $= 1$; and, (ſubtracting $6zz + z^3$ from both ſides,) $10z$ will be $= 1 - 6zz - z^3$. Therefore z will be $= \frac{1 - 6zz - z^3}{10} = \frac{1}{10} - \frac{6zz - z^3}{10} = 0.1 \frac{-6zz - z^3}{10}$, that is, z is leſs than $\frac{1}{10}$, or 0.1, by the quantity $\frac{6zz + z^3}{10}$. Therefore x, or $2 + z$, is leſs than $2 + \frac{1}{10}$, or $2 + 0.1$, or 2.1, by the ſaid quantity $\frac{6zz + z^3}{10}$; which, on account of the ſmallneſs of z, (which is leſs than $\frac{1}{10}$,) will be a very ſmall quantity in compariſon of z, or of $\frac{1}{10}$, and, *à fortiori*, in compariſon of 2, and conſequently may be neglected. And therefore 2.1 will be a ſecond near value of x, or the root of the propoſed equation $x^3 - 2x = 5$, that will be a little greater than its true value, but nearer to it than any other number that conſiſts of only two places of figures.

Q. E. I.

This is the firſt ſtep of Sir Iſaac Newton's approximation to the root of the equation $x^3 - 2x = 5$, after the aſſumption of the number 2, by conjecture and trial, for its firſt near value. And in this firſt ſtep of the approximation Sir Iſaac Newton's and Mr. Raphſon's methods exactly co-incide.

Art. 15. But in the next ſtep of the approximation to the value of x, in the ſaid equation $x^3 - 2x = 5$, the two methods are ſomewhat different from each other, though the number of new figures of the true value of x, that are exact in the next near values of it reſulting from both methods,

thods, is the ſame. The difference between the methods in this ſecond ſtage of the approximation is as follows.

Mr. Raphſon corrects the value of x, or the root of the original equation $x^3 - 2x = 5$, already found, to wit, 2.1, (and which is known to be ſomewhat greater than the truth,) by ſubtracting from it the unknown quantity by which it exceeds x, and which we may call v, and ſubſtituting $2.1 - v$ inſtead of x in the ſaid original equation, $x^3 - 2x = 5$, whereby it is transformed into another cubick equation, in which v will be the only unknown quantity; and then he finds a near value of v by reſolving the ſaid transformed equation as if it were only a ſimple equation, or by neglecting the terms which involve the ſquare and cube of v, on account of their ſmallneſs, juſt as we before neglected the terms $6zz$ and z^3 in the foregoing transformed equation $10z + 6zz + z^3 = 1$ for the ſame reaſon. But Sir Iſaac Newton takes no further notice of the original equation $x^3 - 2x = 5$, till he has compleated the whole proceſs of his approximation; but, inſtead of the ſaid original equation, he conſiders the former transformed equation, $10z + 6zz + z^3 = 1$, which was derived from it, and inveſtigates the value of its root, z, to a greater degree of exactneſs than that to which it was before obtained. And this he does in the manner following.

Since it has been ſeen that z is leſs than 0.1, let the quantity by which 0.1 exceeds it be called v, ſo that z ſhall be $= 0.1 - v$; and let $0.1 - v$ be ſubſtituted, inſtead of z, in the transformed equation $10z + 6zz + z^3 = 1$, This may be done as follows,

Since z is $= 0.1 - v$, we ſhall have

$$zz\ (= \overline{0.1 - v}|^{2}) = 0.01 - 0.2v + vv,$$

$$\text{and } z^3\ (= \overline{0.1 - v}|^{3} = 0.001 - 3 \times 0.01 \times v + 3 \times 0.1 \times vv$$
$$- v^3) = 0.001 - 0.03v + 0.3vv - v^3,$$

$$\text{and } 10z\ (= 10 \times \overline{0.1 - v}) = 1 - 10v,$$

$$\text{and } 6zz = 0.06 - 1.2v + 6vv,$$

and conſequently

$$\left\{\begin{array}{l} 10z \\ +\ 6zz \\ +\ z^3 \end{array}\right\} = \left\{\begin{array}{llll} 1.00 & -\ 10v & & \\ +\ 0.06 & -\ 1.2v & +\ 6vv & \\ +\ 0.001 & -\ 0.03v & +\ 0.3vv & -\ v^3 \end{array}\right\}$$

$$= 1.061 - 11.23v + 6.3vv - v^3.$$

But $10z + 6zz + z^3$ is $= 1$.

Therefore $1.061 - 11.23v + 6.3vv - v^3$ will likewiſe be $= 1$. And conſequently (adding $11.23v$ to both ſides,) we ſhall have $1.061 + 6.3vv - v^3 = 1 + 11.23v$; and, (ſubtracting 1 from both ſides,) we ſhall have $0.061 + 6.3vv - v^3 = 11.23v$, and (neglecting $6.3vv$ and v^3 as inconſiderable in compariſon of 0.061 and $11.23v$) we ſhall have $0.061 = 11.23v$, or $11.23v = 0.061$; and conſequently (dividing both ſides by 11.23,) we ſhall have v $(= \frac{0.061}{11.23}) = 0.0054$. Therefore z, or $0.1 - v$, will be $(= 0.1 - 0.0054) = 0.0946$, and conſequently x, or $2 + z$, will be $(= 2 + 0.0946) = 2.0946$. Q. E. I.

In this manner Sir Iſaac Newton finds the root of the propoſed equation $x^3 - 2x = 5$ to be equal to 2.0946, which is as near the truth as five figures can expreſs it.

Art. 16. He then carries the inveſtigation one ſtep further, by which he obtains the value of x exact to nine places of figures; and for this purpoſe he proceeds in the manner following.

The laſt transformed equation was $11.23v = 0.061 + 6.3vv - v^3$; from which it follows that v is accurately equal to $\frac{0.061}{11.23} + \frac{6.3vv - v^3}{11.23}$, or $0.0054 + \frac{6.3vv - v^3}{11.23}$, which is greater than 0.0054 alone, becauſe $6.3vv$ is greater than v^3. Since, therefore, v is greater than 0.0054, let us ſuppoſe it to be $= 0.0054 + w$; and let this binomial quantity be ſubſtituted, inſtead of v, in the laſt transformed equation $11.23v = 0.061 + 6.3vv - v^3$, or, rather, in the equation $11.23v - 6.3vv + v^3 = 0{,}061$, conſiſting of the ſame terms as the former, but in which the terms involving the unknown quantity v are all brought to the ſame ſide of the equation, and ranged according to the powers of v, beginning from its loweſt power, or the ſimple power of v. This may be done in the manner following.

Since v is $= 0.0054 + w$, we ſhall have

$$vv\ (= \overline{0.0054 + w}|^2 = \overline{0.0054}|^2 + 2 \times 0.0054 \times w + w^2)$$
$$= 0.000{,}029{,}16 + 0.0108 \times w + w^2,$$

and

$$v^3\ (= \overline{0.0054 + w}|^3 = \overline{0.0054}|^3 + 3 \times \overline{0.0054}|^2 \times w + 3 \times 0.0054 \times w^2 + w^3$$
$$= 0.000{,}000{,}157{,}464 + 3 \times 0.000{,}029{,}16 \times w + 0.0162 \times w^2 + w^3)$$
$$= 0.000{,}000{,}157{,}464 + 0.000{,}087{,}48 \times w + 0.0162 \times w^2 + w^3,$$

and

$$11.23v\ (= 11.23 \times \overline{0.0054 + w}) = 0.060{,}642 + 11.23 \times w,$$

and

$$6.3vv\ (= 6.3 \times \overline{0.000{,}029{,}16 + 0.0108 \times w + w^2})$$
$$= 0.000{,}183{,}708 + 0.068{,}04 \times w + 6.3ww;$$

and conſequently $11.23v - 6.3vv + v^3$ will be $=$

$$\left\{\begin{array}{l} \ 0.060{,}642 + 11.23 \times w \\ -\ 0.000{,}183{,}708 - 0.068{,}04 \times w - 6.3\,ww \\ +\ 0.000{,}000{,}157{,}464 + 0{,}000{,}087{,}48 \times w + 0.0162w^2 + w^3) \end{array}\right\}$$

$$= \left\{\begin{array}{llll} \ 0.060{,}642{,}157{,}464 & + 11.230{,}087{,}48 \times w & + 0.0162w^2 & + w^3 \\ -\ 0.000{,}183{,}708 & - 0.068{,}04 \times w & - 6.3 \times w^2 & \end{array}\right\}$$

$$= 0.060{,}458{,}449{,}464 + 11.162{,}047{,}48w - 6.2838w^2 + w^3.$$

But $11.23v - 6.3vv + v^3$ is $= 0.061$.

Therefore $0.060{,}458{,}449{,}464 + 11.162{,}047{,}48 \times w - 6.2838 \times ww + w^3$ will likewiſe be $= 0.061$; and conſequently (ſubtracting $0.060{,}458{,}449{,}464$ from both ſides,) $11.162{,}047{,}48 \times w - 6.2838ww + w^3$ will be $(= 0.061{,}000{,}000{,}000 - 0.060{,}458{,}449{,}464) = 0.000{,}541{,}550{,}536$; and (neglecting the terms $6.2838ww$ and w^3, as inconſiderable in compariſon of $11.162{,}047{,}48 \times w$,) we ſhall have $11.162{,}047{,}48 \times w = 0.000{,}541{,}550{,}536$, and conſequently w $\left(= \frac{0.000,541,550,536}{11.162,047,48}\right.$ $= 0.000,048,52$. Therefore v, or $0.0054 + w$, will be $(= 0.0054 + 0.000,048,52) = 0.005,448,52$, and z, or $0.1 - v$, will be $(= 0.100,000,00 - 0.005,448,58) = 0.094,551,48$, and x, or $2 + z$, will be $(= 2 + 0.094,551,48) = 2.094,551,48$; that is, the root of the propoſed equation $x^3 - 2x = 5$ will be $= 2.094,551,48$. Q. E. I.

This number 2.094,551,48 is exact in all the figures, as will be ſhewn in a ſubſequent article.

Art. 17. Having thus ſet forth Sir Iſaac Newton's method of inveſtigating the root of the propoſed equation $x^3 - 2x = 5$ to nine places of figures, we muſt now perform the ſame thing by Mr. Raphſon's method, in order to make a compariſon between the neceſſary operations of the two methods.

Now Mr. Raphſon's method of approximating further to the root of the equation $x^3 - 2x = 5$, after having found it to be equal to $2 + 0.1 - \frac{6zz - z^3}{10}$, or to be ſomewhat leſs than 2.1, is to put v for the unknown quantity by which it falls ſhort of 2.1, and then to ſubſtitute the reſidual quantity $2.1 - v$ in the terms of the original equation $x^3 - 2x = 5$, whereby the ſaid equation will be transformed into another cubick equation, in which v will be the only unknown quantity: and then he determines the value of v by reſolving the ſaid transformed equation as if it was a mere ſimple equation, or by neglecting the terms in which the ſquare or the cube of v occur. This may be done in the manner following.

Since x is $= 2.1 - v$, we ſhall have xx $(= \overline{2.1 - v}|^2 = \overline{2.1}|^2 - 2 \times 2.1 \times v + \&c) = 4.41 - 4.2v + \&c$, and x^3 $(= \overline{2.1 - v}|^3 = \overline{2.1}|^3 - 3 \times \overline{2.1}|^2 \times v + \&c = 9.261 - 3 \times 4.41 \times v + \&c) = 9.261 - 13.23 \times v + \&c$,

&c, and $2x$ ($= 2 \times \overline{2.1 - v}$) $= 4.2 - 2v$, and conſequently $x^3 - 2x$ ($= 9.261 - 13.23 \times v + \&c - 4.2 + 2v$) $= 5.061 - 11.23 \times v$ &c.

But $x^3 - 2x$ is $= 5$.

Therefore $5.061 - 11.23 \times v$ &c, will likewiſe be $= 5$, and conſequently (adding $11.23 \times v$ to both ſides,) we ſhall have $5.061 = 5 + 11.23 \times v$, and (ſubtracting 5 from both ſides,) we ſhall have $11.23 \times v = 0.061$, and conſequently v ($= \frac{0.061}{11.23}$) $= 0.0054$. Therefore x, or $2.1 - v$, will be ($= 2.1 - 0.0054$) $= 2.0946$; or 2.0946 will be a third near value of the root of the propoſed equation $x^3 - 2x = 5$. Q. E. I.

This third near value of x is the very ſame with the third near value of it obtained above, in art. 15, by Sir Iſaac Newton's method.

Art. 18. In this ſtep of the approximation, by which we obtain the number 2.0946 for the third near value of the root of the propoſed equation $x^3 - 2x = 5$, the principal difference between the two methods ſeems to conſiſt in this, to wit, that by Mr. Raphſon's method we are obliged to raiſe the two firſt terms of the powers of the compound quantity $2.1 - v$, and conſequently to raiſe the powers of the number 2.1, which conſiſts of two figures; whereas in Sir Iſaac Newton's method of proceeding, we had occaſion only to raiſe the powers of the compound quantity $0.1 - v$, and conſequently to raiſe the powers of the number 0.1, which conſiſts of only one figure; which is ſomewhat eaſier than to raiſe the powers of 2.1. But both operations are ſo eaſy, that the difference of the labour of performing them is hardly worth conſidering. And, with reſpect to the ſimplicity of conception in the two methods, Mr. Raphſon's method ſeems to be preferable to Sir Iſaac Newton's; becauſe the former always refers to the original equation $x^3 - 2x = 5$, whereas the latter method refers to the preceeding transformed equation $10z + 6zz + z^3 = 1$, which has more terms and larger co-efficients than the original equation $x^3 - 2x = 5$.

Art. 19. But in the next ſtep of the approximation by Mr. Raphſon's method, we ſhall find the labour of raiſing the powers of the value of x already found, to wit, the powers of 2.0946, to be conſiderably greater than that of raiſing the powers of the laſt preceeding ſupplement of it according to Sir Iſaac Newton's method, that ſupplement being only the decimal fraction 0.0054, in which there are only two ſignificant figures. This will appear by performing this ſtep of the approximation by Mr. Raphſon's method; which may be done as follows.

Art. 20. The laſt near value we found for x, or the root of the equation $x^3 - 2x = 5$, by Mr. Raphſon's method, was 2.0946. Now this near value of x is greater than its true value. For, if we ſuppoſe x to be $= 2.0946$, we ſhall have x^3 $(= \overline{2.0946}|^3) = 9.189,741,550,536$, and $2x$ $(= 2 \times 2.0946) = 4.1892$, and conſequently $x^3 - 2x$ $(= 9.189,741,550,536 - 4.1892) = 5.000,541,550,536$; which is greater than 5, or the abſolute term of the equation $x^3 - 2x = 5$: and conſequently 2.0946 muſt be greater than the true value of the root of the ſaid equation.

We will therefore ſuppoſe x to be $= 2.0946 - w$, and ſubſtitute this reſidual quantity inſtead of x in the terms of the equation $x^3 - 2x = 5$.

Now, ſince x is $= 2.0946 - w$, we ſhall have xx $(= \overline{2.0946 - w}|^2 = \overline{2.0946}|^2 - 2 \times 2.0946 \times w + \&c) = 4.387,349,16 - 4.1892 \times w + \&c$, and x^3 $(= \overline{2.0946 - w}|^3 = \overline{2.0946}|^3 - 3 \times \overline{2.0946}|^2 \times w + \&c = 9.189,741,550,536 - 3 \times 4.387,349,16 \times w + \&c) = 9.189,741,550,536 - 13.162,047,48 \times w + \&c$, and $2x$ $(= 2 \times \overline{2.0946 - w}) = 4.1892 - 2w$, and conſequently $x^3 - 2x =$

$$\left\{\begin{array}{lll} 9.189,741,550,536 & - & 13.162,047,48 \times w + \&c \\ -\,4.189,2 & + & 2.000,000,00 \times w \end{array}\right\}$$

$$= 5.000,541,550,536 - 11.162,047.48 \times w + \&c.$$

But $x^3 - 2x$ is $= 5$.

Therefore $5{,}000{,}541{,}550{,}536 - 11.162{,}047{,}48 \times w +$ &c, will be $= 5$; and conſequently (adding $11.162{,}047{,}48 \times w$ to both ſides,) we ſhall have $5.000{,}541{,}550{,}536 = 5 + 11.162{,}047{,}48 \times w$, and (ſubtracting 5 from both ſides,) $0.000{,}541{,}550{,}536 = 11.162{,}047{,}48 \times w$, or $11.162{,}047{,}48 \times w = 0.000{,}541{,}550{,}536$. Therefore w will be $(= \frac{0.000{,}541{,}550{,}536}{11.162{,}047{,}48}) = 0.000{,}048{,}52$; and conſequently x, or $2.0946 - w$, will be $(= 2.094{,}600{,}00 - 0.000{,}048{,}52) = 2.094{,}551{,}48$. Therefore $2.094{,}551{,}48$ will be a fourth near value of x, or the root of the propoſed equation $x^3 - 2x = 5$. Q. E. I.

This fourth near value of x is the very ſame with the fourth near value of it obtained above, in art. 16, by Sir Iſaac Newton's method.

Art. 21. In this laſt ſtage of Mr. Raphſon's approximation to the root of the propoſed equation $x^3 - 2x = 5$, we have been obliged to raiſe the powers of the number 2.0946, which conſiſts of five places of figures; whereas in Sir Iſaac Newton's way of proceeding we only raiſed the powers of the decimal fraction 0.0054, which contains only two ſignificant figures. But then in that way of proceeding we were obliged to multiply v, or $0.0054 + w$, into 11.23, and vv, or $0.000{,}029{,}16 + 0.0108 \times w + w^2$, into 6.3; whereas in Mr. Raphſon's way of proceeding we have only to multiply x, or $2.0946 - w$, into the very ſimple co-efficient 2. So that, upon the whole, the difference of the labour of computation in the two methods is not very conſiderable, though it is rather leſs in Sir Iſaac Newton's method than in Mr. Raphſon's. But in point of ſimplicity of conception Mr. Raphſon's method ſeems much ſuperiour to Sir Iſaac's, becauſe it never loſes ſight of the original equation $x^3 - 2x = 5$, which is to be reſolved.

And, further, we may obſerve, in favour of Mr. Raphſon's method, that it never requires us to raiſe any more than the two firſt terms of the binomial and reſidual quantities $2 + z$, and $2.1 - v$, and $2.0946 - w$, which are ſubſtituted inſtead of x in the original equation $x^3 - 2x = 5$; whereas in Sir Iſaac Newton's method it is neceſſary to raiſe the other terms of the binomial and reſidual quantities $2 + z$, and $0.1 - v$, and $0.0054 + w$; which increaſes the number and intricacy of the operations of the inveſtigation. And therefore, upon the whole, I conſider Mr. Raphſon's method of approximating to the values of the roots of ſuch equations as preferable to Sir Iſaac Newton's.

A Proof of the Exactneſs of the Number 2.094,551,48, *that has been found by the foregoing Methods of Approximation for the Root of the Equation* $x^3 - 2x = 5$.

Art. 22. It remains that we prove the work to have been rightly performed, or that we ſhew that the laſt number 2.094,551,48, obtained by both theſe methods, is a very near value of the root x of the propoſed equation $x^3 - 2x = 5$, and that we determine to how many figures it is exact.

Now the plaineſt and beſt method of doing this is to ſubſtitute the number 2.094,551,48, inſtead of x, in the compound quantity $x^3 - 2x$, in order to diſcover whether the quantity reſulting from this ſubſtitution will be greater, or leſs, than 5, or the abſolute term of the propoſed equation $x^3 - 2x = 5$: and, if it ſhall appear that the ſaid reſult is greater than 5, we may conclude that the ſaid number 2.094,551,48 is greater than the true value of x in the ſaid equation; and, if it ſhall appear that the ſaid reſult is leſs than 5, we may conclude that the ſaid number is leſs than the true value of x. And, when this has been thus diſcovered, we muſt, in the next place, endeavour to determine to how many figures this number 2.094,551,48 coincides with the more accurate value of x : and, for this purpoſe, we muſt, if this number be leſs than x, increaſe it by the addition of an unit in the laſt place of figures; and, if it be greater than x, we muſt diminiſh it by the ſame ſmall quantity, and then ſubſtitute the new number thereby obtained, to wit, 2.094,551,49, or 2.094,551,47, inſtead of x, in the compound quantity $x^3 - 2x$. And, if it ſhall appear that the value of that compound quantity reſulting from that ſubſtitution is greater, or leſs, than 5, we may conclude that the number 2.094,551,49, or 2.094,551,47, is accordingly greater, or leſs, than the true value of x, in the equation $x^3 - 2x = 5$, and conſequently that the ſaid true value is of an intermediate magnitude between 2.094,551,49 and 2.094,551,48, or between 2.094,551,48 and 2.094,551,47.

Now, if we take $x = 2.094{,}551{,}48$, we ſhall have

$xx = 4.387{,}145{,}902{,}370{,}190{,}4$,

and $x^3 = 9.189{,}102{,}942{,}785{,}417{,}810{,}201{,}792$,

and $2x = 4.189{,}102{,}96$,

and conſequently

$x^3 - 2x = 4.999{,}999{,}982{,}785{,}417{,}810{,}201{,}792$; which number is ſomewhat leſs than 5, or the abſolute term of the propoſed equation $x^3 - 2x = 5$. Therefore 2.094,551,48 muſt be ſomewhat leſs than the true value of x in the ſaid equation.

Secondly, ſince x is greater than 2.094,551,48, we muſt now compare it with 2.094,551,49, by ſubſtituting that number inſtead of it in the compound quantity $x^3 - 2x$.

Now, if x is taken $= 2.094{,}551{,}49$, or $2.094{,}551{,}48 + 0.000{,}000{,}01$, we ſhall have x^3 $(= \overline{2.094{,}551{,}48}|^3 + 3 \times \overline{2.094{,}551{,}48}|^2 \times 0.000{,}000{,}01 + 3 \times 2.094{,}551{,}48 \times \overline{0.000{,}000{,}01}|^2 + \overline{0.000{,}000{,}01}|^3 = 9.189{,}102{,}942$, &c. $+$

3 × 4.387,145,902, &c × 0.000,000,01 + 0.000,000,000, &c + 0.000,000,000, &c = 9.189,102,942, &c + 13.161,437,706, &c × 0.000,000,01 + 0.000.000,000, &c + 0.000,000,000, &c = 9.189,102,942, &c + 0.000, 000,131, &c + 0.000,000,000, &c + 0.000,000,000, &c) = 9.189,103,073, &c; and $2x$ (= 2 × 2.094,551,49) = 4.189,102,98; and conſequently $x^3 - 2x$ (= 9.189,103,07, &c − 4.189,102,98) = 5.000,000,09, &c; which is greater than 5. Therefore 2.094,551,49 muſt be greater than the true value of x in the equation $x^3 - 2x = 5$.

But it has been ſhewn that 2.094,551,48 is leſs than the ſaid true value.

Therefore the true value of x in the equation $x^3 - 2x = 5$, will be of an intermediate magnitude between 2.094, 551,48 and 2.094,551,49; and conſequently all the figures of the number 2.094,551,48, which we found by the foregoing proceſſes of Sir Iſaac Newton's and Mr. Raphſon's methods of approximation for a fourth near value of the root of the equation $x^3 - 2x = 5$, are exact. Q. E. D.

Of the Difficulty of finding a, *or the Firſt near Value of the Root of an Affected Equation, in certain Caſes.*

Art. 23. There is another difficulty that occurs ſometimes in reſolving high equations by approximation, whether by Sir Iſaac Newton's method or by Mr. Raphſon's; which indeed are ſubſtantially the ſame. The difficulty I mean, is that of finding the firſt near value of the root ſought (which we have called a in this diſcourſe,) to one, or two places of figures, in order to make it the baſis of a further approximation to the true value of the root by either of theſe methods of approximation. Now, when the equation is known to have but one root, that is, but one real and affirmative root, (for all other roots are not worth conſidering,) this difficulty will not be great; becauſe it will always be eaſy to find a tolerably near value of the root by conjectures and trials, and particularly by ſuppoſing *x*, or the root of the propoſed equation, firſt, to be equal to 1, and 2dly, to be = 10, and 3dly, to be equal to ſome ſhort intermediate number conſiſting of only one figure, or, if the root appears to be greater than 10, by ſuppoſing it to be equal to 100, or 1000, and afterwards ſuppoſing it to be equal to ſome ſhort intermediate number conſiſting of two figures; as was done above in art. 5, in finding the firſt near value of *x* in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. But, when the equation conſiſts of terms connected together partly by the ſign +, and partly by the ſign —, and conſequently it may, for aught we know to the contrary, have two, or three, or four, or more real and affirmative roots, which may be of very different magnitudes, the aforeſaid method of conjectures and trials (though by no means uſeleſs,) is leſs expeditious and ſatisfactory in aſſiſting us to find the firſt near value of one of the roots than [illegible] the former caſe; and we are often puzzled to know which of the roots it would be moſt expedient to begin to inveſtigate. Now, in moſt of theſe caſes, I believe, it will be adviſeable to begin by inveſtigating the leaſt root, and for that purpoſe to expunge from the equation all the terms that have the ſign — prefixed to them, and to find, to about two places, or, at moſt, to three places, of figures, the root of the remaining equation. For this root will always be leſs than the leaſt root of the original equation, if it really has (as it appears to have,) more than one real and affirmative root; or it will be leſs than the only root of the original equation, if (notwithſtanding the appearances to the contrary,) it really has but one root. When the root of this ſecond, or curtailed, equation, has been diſcovered, it may be called *a*, and made the ground-work of an approximation to the leaſt root of the original equation, and the binomial quantity $a + z$ may be ſubſtituted in the original equation inſtead of *x*, and the transformed equation thence ariſing may be reſolved as if it was a mere ſimple equation, agreeably to Mr. Raphſon's ſon's method of approximation; and the value of z thereby obtained, being added to *a*, will give us a known value of $a + z$, or a ſecond near value of the leaſt, or the only, root of the propoſed equation: after which we may proceed to find the ſaid leaſt, or only, root of the propoſed equation by a further proſecution of Mr. Raphſon's method of approximation above-deſcribed. This method of finding a firſt near value, *a*, of the leaſt root of a propoſed equation that ſeems to have more than one real and affirmative root, is explained more at length in the third volume of the Collection of Mathematical Tracts, called *Scriptores Logarithmici*, in my Diſcourſe on the Reverſion of Infinite Serieſes publiſhed in that Volume; to which I refer the reader. See the ſaid 3d Volume, pages 724, 725, 726, 727, &c, ... to page 761. And, with this improvement of it in the caſe of equations that have, or ſeem to have, more than one real and poſitive root, I believe it may ſafely be affirmed that Mr. Raphſon's Method of Reſolving Affected Equations is the beſt General Method of effecting that purpoſe in all equations above quadratícks that has hitherto been diſcovered.

End of the Obſervations on Mr. Raphſon'*s Method of Reſolving Affected Equations by Approximation.*

A TABLE

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TABLE

OF THE

SQUARE AND CUBE ROOTS OF THE NATURAL NUMBERS 1, 2, 3, 4, 5, &c, to 180;

Being Table XIX. *of Mr.* James Dodſon'*s valuable Tables of Computation, intitled* The Calculator, *that were publiſhed in the Year* 1747.

No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.
1	1.000,000	1.000,000	31	5.567,764	3.141,381	61	7.810,250	3.936,497
2	1.414,214	1.259,921	32	5.656,854	3.174,802	62	7.874,008	3.957,892
3	1.732,051	1.442,250	33	5.744,563	3.207,534	63	7.937,254	3.979,057
4	2.000,000	1.587,401	34	5.830,952	3.239,612	64	8.000,000	4.000,000
5	2.236,068	1.709,976	35	5.916,080	3.271,066	65	8.062,258	4.020,726
6	2.449,490	1.817,121	36	6.000,000	3.301,927	66	8.124,038	4.041,240
7	2.645,751	1.912,933	37	6.082,763	3.332,222	67	8.185,353	4.061,548
8	2.828,427	2.000,000	38	6.164,414	3.361,975	68	8.246,211	4.081,656
9	3.000,000	2.080,084	39	6.244,998	3.391,211	69	8.306,624	4.101,566
10	3.162,278	2.154,435	40	6.324,555	3.419,952	70	8.366,600	4.121,285
11	3.316,625	2.223,980	41	6.403,124	3.448,217	71	8.426,150	4.140,818
12	3.464,102	2.289,428	42	6.480,741	3.476,027	72	8.485,281	4.160,168
13	3.605,551	2.351,335	43	6.557,439	3.503,398	73	8.544,004	4.179,339
14	3.741,657	2.410,142	44	6.633,250	3.530,348	74	8.602,325	4.198,336
15	3.872,983	2.466,212	45	6.708,204	3.556,893	75	8.660,254	4.217,163
16	4.000,000	2.519,842	46	6.782,330	3.583,048	76	8.717,798	4.235,824
17	4.123,106	2.571,282	47	6.855,655	3.608,826	77	8.774,964	4.254,321
18	4.242,641	2.620,741	48	6.928,203	3.634,241	78	8.831,761	4.272,659
19	4.358,899	2.668,402	49	7.000,000	3.659,306	79	8.888,194	4.290,841
20	4.472,136	2.714,418	50	7.071,068	3.684,031	80	8.944,272	4.308,870
21	4.582,576	2.758,923	51	7.141,428	3.708,430	81	9.000,000	4.326,749
22	4.690,416	2.802,039	52	7.211,103	3.732,511	82	9.055,385	4.344,481
23	4.795,832	2.843,867	53	7.280,110	3.756,286	83	9.110,434	4.362,071
24	4.898,979	2.884,499	54	7.348,469	3.779,763	84	9.165,151	4.379,519
25	5.000,000	2.924,018	55	7.416,198	3.802,953	85	9.219,544	4.396,830
26	5.099,020	2.962,496	56	7.483,315	3.825,862	86	9.273,618	4.414,005
27	5.196,152	3.000,000	57	7.549,834	3.848,501	87	9.327,379	4.431,047
28	5.291,503	3.036,589	58	7.615,773	3.870,877	88	9.380,832	4.447,960
29	5.385,165	3.072,317	59	7.681,146	3.892,996	89	9.433,981	4.464,745
30	5.477,226	3.107,232	60	7.745,967	3.914,867	90	9.486,833	4.481,405
No	Sq. Root.	Cube Rt.	No.	Sq Root.	Cube Rt.	No	Sq. Root.	Cube Rt.

No	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.
91	9.539,392	4.497,942	121	11.000,00	4.946,088	151	12.288,21	5.325,074
92	9.591,663	4.514,357	122	11.045,36	4.959,675	152	12.328,83	5.336,803
93	9.643,651	4 530,655	123	11.090,54	4.973,190	153	12.369,32	5.348,481
94	9.695,360	4.546,836	124	11.135,53	4.986,631	154	12.409,67	5.360,108
95	9.746,794	4.562,903	125	11.180,34	5.000,000	155	12.449,90	5.371,685
96	9.797,959	4.578,857	126	11.224,97	5.013,298	156	12.490,00	5.383,213
97	9.848,858	4.594,701	127	11.269,43	5.026,526	157	12.529,96	5.394,690
98	9.899,495	4.610,436	128	11.313,71	5.039,684	158	12.569,81	5.406,120
99	9.949,874	4.626,065	129	11.357,82	5.052,774	159	12.609,52	5.417,501
100	10.000,00	4.641,589	130	11.401,75	5.065,797	160	12.649,11	5.428,835
101	10.049,88	4.657,010	131	11.445,52	5.078,753	161	12.688,58	5.440,122
102	10.099,50	4.672,330	132	11.489,12	5.091,643	162	12.727,92	5.451,362
103	10.148,89	4.687,548	133	11.532,56	5.104,469	163	12.767,15	5.462,555
104	10.198,04	4.702,669	134	11.575,84	5.117,230	164	12.806.25	5.473,703
105	10.246,95	4.717,694	135	11.618,95	5.129,928	165	12.845,23	5.484,806
106	10.295,63	4.732,624	136	11.661,90	5.142,563	166	12.884,10	5.495,865
107	10.344,08	4.747,459	137	11.704,70	5.155,137	167	12.922,85	5.506,879
108	10.392,30	4.762,203	138	11.747,34	5.167,649	168	12.961,48	5.517,848
109	10.440,31	4.776,856	139	11.789,83	5.180,101	169	13.000,00	5.528,775
110	10.488,09	4.791,420	140	11.832,16	5.192,494	170	13.038,40	5.539,658
111	10.535,65	4.805,896	141	11.874,34	5.204,828	171	13.076,70	5.550,499
112	10.583,01	1.820,284	142	11.916,38	5.217,103	172	13.114,88	5.561,298
113	10.630,15	4.834,588	143	11.958,26	5.229,321	173	13.152,95	5.572,054
114	10.677,08	4.848,808	144	12.000,00	5.241,482	174	13.190,91	5.582,770
115	10.723,8	4.862,944	145	12.041,59	5.253,588	175	13.228,76	5.593,445
116	10.770,33	4.876,999	146	12.083,06	5.265,637	176	13.266,50	5.604,079
117	10.816,65	4.890,973	147	12.124,36	5.277,632	177	13.304,13	5.614,673
118	10.862,78	4.904,868	148	12,165,53	5.289,572	178	13.341,66	5.625,226
119	10.908,71	4.918,685	149	12.206,56	5.301,459	179	13.379,09	5.635,741
120	10.954,45	4.932,424	150	12.247,45	5.313,293	180	13.416,41	5.646,216
No	Sq. Root	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.

A

TABLE

OF THE

SQUARE-ROOTS AND RECIPROCALS OF ALL NUMBERS,

From 1 to 1000.

Computed by Dr. CHARLES HUTTON, *Profeſſor of Mathematicks at the Royal Military Academy at Woolwich in Kent.*

4 G

No.	Reciprocal	Square Root.	No.	Reciprocal	Sqnare Root.
1	1.0	1.000,000,000,0	51	0.019,607,8	7.141,428,428,5
2	0.5	1.414,213,562,4	52	0.019,230,8	7.211,102,550,9
3	0.333,333,3	1.733,050,807,6	53	0.018,867,9	7.280,109,889.3
4	0.25	2.000,000,000,0	54	0.018,518,5	7.348,469,228,3
5	0.2	2.236,067,977,5	55	0.018,181,8	7.416,198,487,1
6	0.166,666,6	2.449,489,742,8	56	0.017,857,1	7.483,314,773,5
7	0.142,857,1	2.645,751,311,1	57	0.017,543,9	7.549,834,435,3
8	0.125	2.828,427,124,7	58	0.017,241,4	7.615,773,105,9
9	0.111,111,1	3.000,000,000,0	59	0.016,949,0	7.681,145,747,9
10	0.1	3.162,277,660,2	60	0.016,666,6	7.745,966,692,4
11	0.090,909,0	3.316,624,790,4	61	0.016,393,4	7.810,249,675,9
12	0.083,333,3	3.464,101,615,1	62	0.016,129,0	7.874,007,874,0
13	0.076,923,0	3.605,551,275,5	63	0.015,873,0	7.937,253,933,2
14	0.071,428,5	3.741,657,386,8	64	0.015,625,	8.000,000,000,0
15	0.066,666,6	3.872,983,346,2	65	0.015,384,6	8.062,257,748,3
16	0.062,5	4.000,000,000,0	66	0.015,151,5	8.124,038,404,6
17	0.058,823,5	4.123,105,625,6	67	0.014,925,4	8.185,352,771,9
18	0.055,555,5	4.242,640,687,1	68	0.014,705,9	8.246,211,251,2
19	0.052,631,6	4.358,898,943,5	69	0.014,492,8	8.306,623,862,9
20	0.05	4.472,135,955,0	70	0.014,285,7	8.366,600,265,3
21	0.047,619,0	4.582,575,695,0	71	0.014,084,5	8.426,149,773,2
22	0.045,454,5	4.690,415,759.8	72	0.013,888,8	8.485,281,374,2
23	0.043,478,3	4.795,831,523,3	73	0.013,698,6	8.544,003,745,3
24	0.041,666,6	4.898,979,485,6	74	0.013,513,5	8.602,325,267,0
25	0.04	5.000,000,000,0	75	0.013,333,3	8.660,254,037,8
26	0.038,461,5	5.099,019,513,6	76	0.013,157,9	8.717,797,887,1
27	0.037,037,0	5.196,152,422,7	77	0.012,987,0	8.774,964,387,4
28	0.035,714,3	5.291,502,622,1	78	0.012,820,5	8.831,760,866,3
29	0.034,482,8	5.385,164,807,1	79	0.012,658,2	8.888,194,417,3
30	0.033,333,3	5.477,225,575,1	80	0.012,5	8.944,271,910,0
31	0.032,258,1	5.567,764,362,8	81	0.012,345,7	9.000,000,000,0
32	0.031,25	5.656,854,249,5	82	0.012,195,0	9.055,385,138,1
33	0.030,303,0	5.744,562,646,5	83	0.012,048,2	9.110,433,579,1
34	0.029,411,8	5.830,951,894,8	84	0.011,904,8	9.165,251,389,9
35	0.028,571,4	5.916,079,783,1	85	0.011,764,7	9.219,544,457,3
36	0.027,777,7	6.000,000,000,0	86	0.011,627,9	9.273,618,495,5
37	0.027,027,0	6.082,762,530,3	87	0.011,494,3	9.327,379,053,1
38	0.026,315,8	6.164,414,003,0	88	0.011,363,6	9.380,831,519,6
39	0.025,641,0	6.244,997,998,4	89	0.012,236,0	9.433,981,132,1
40	0.025,	6.324,555,320,3	90	0.011,111,1	9.486,832,980,5
41	0.024,390,2	6.403,124,237,4	91	0.010,989,0	9.539,392,014,2
42	0.023,809,5	6.480,740,698,4	92	0.010,869,6	9.591,663,046,6
43	0.023,255,8	6.557,438,524,3	93	0.010,752,7	9.643,650,761,0
44	0.022,727,2	6.633,249,580,7	94	0.010,638,3	9.695,359,714.8
45	0.022,222,2	6.708,203,932,5	95	0.010,526,3	9.746,794,344,8
46	0.021,739,1	6.782,329,983,1	96	0.010,416,6	9.797,958,971,1
47	0.021,276,6	6.855,654,600,4	97	0.010,309,3	9.848,857,801,8
48	0.020,833,3	6.928,203,230,3	98	0.010,204,1	9.899,494,936,6
49	0.020,408,2	7.000.000,000,0	99	0.010,101,0	9.949,874,371,1
50	0.02	7.071,067,811,9	100	0.01	10.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
101	0.009,900,9	10.049,875,621,1	151	0.006,622,5	12.288,205,727,4
102	0.009,803,9	10.099,504,938,4	152	0.006,578,9	12.328,828,005,9
103	0.009,708,7	10.148,891,565,1	153	0.006,535,9	12.369,316,876,9
104	0.009,615,4	10.198,039,027,2	154	0.006,493,5	12.409,673,646,0
105	0.009,523,8	10.246,950,766,0	155	0.006,451,6	12.449,899,598,0
106	0.009,434,0	10.295,630,141,0	156	0.006,410,3	12.489,995,996,8
107	0.009,345,8	10.344,080,432,8	157	0.006,369,4	12.529,964,086,1
108	0.009,259,2	10.392,304,845,4	158	0.006,329,1	12.569,805,090,0
109	0.009,174,3	10.440,306,508,9	159	0.006,289,3	12.609,520,212,9
110	0.009,090,9	10.488,088,481,7	160	0.006,25	12.649,110,640,7
111	0.009,009,0	10.535,653,752,9	161	0.006,211,2	12.688,577,540,4
112	0.008,928,6	10.583,005,244,3	162	0.006,172,8	12.727,922,061,4
113	0.008,849,6	10.630,145,812,7	163	0.006,135,0	12.767,145,334,8
114	0.008,771,9	10.677,078,252,0	164	0.006,097,5	12.806,248,474,9
115	0.008,695,7	10.723,805,294,8	165	0.006,060,6	12.845,232,578,7
116	0.008,620,7	10.770,329,614,3	166	0.006,024,1	12.884,098,726,7
117	0.008,547,0	10.816,653,826,4	167	0.005,988,0	12.922,847,983,3
118	0.008,474,5	10.862,780,491,2	168	0.005,952,4	12.961,481,396,8
119	0.008,403,4	10.908,712,114,6	169	0.005,917,2	13.000,000,000,0
120	0.008,333,3	10.954,451,150,1	170	0.005,882,4	13.038,404,810,4
121	0.008,264,5	11.000,000,000,0	171	0.005,848,0	13.076,696,830,6
122	0.008,196,7	11.045,361,017,2	172	0.005,814,0	13.114,877,048,6
123	0.008,130,0	11.090,536,506,4	173	0.005,780,3	13.152,946,438,0
124	0.008,064,5	11.135,528,725,7	174	0.005,747,1	13.190,905,958,3
125	0.008,	11.180,339,887,5	175	0.005,714,3	13.228,756,555,3
126	0.007,936,5	11.224,972,160,3	176	0.005,681,8	13.266,499,161,4
127	0.007,874,0	11.269,427,669,6	177	0.005,649,7	13.304,134,695,7
128	0.007,812,5	11.313,708,499,0	178	0.005,618,0	13.341,664,064,1
129	0.007,751,9	11.357,816,691,6	179	0.005,586,6	13.379,088,160,3
130	0.007,692,3	11.401,754,251,0	180	0.005,555,5	13.416,407,865,0
131	0.007,633,6	11.445,523,142,3	181	0.005,524,9	13.453,624,047,1
132	0.007,575,7	11.489,125,293,1	182	0.005,494,5	13.490,737,563,2
133	0.007,518,8	11.532,562,594,7	183	0.005,464,5	13.527,749,258,5
134	0.007,462,7	11.575,836,902,8	184	0.005,434,8	13.564,659,966,3
135	0.007,407,4	11.618,950,038,6	185	0.005,405,4	13.601,470,508,7
136	0.007,352,9	11.661,903,789,7	186	0.005,376,3	13.638,181,697,0
137	0.007,299,3	11.704,699,911,1	187	0.005,347,6	13.674,794,331,2
138	0.007,246,4	11.747,344,380,8	188	0.005,319,1	13.711,309,200,8
139	0.007,194,2	11.789,826,122,6	189	0.005,291,0	13.747,727,084,9
140	0.007,142,9	11.832,159,566,2	190	0.005,263,2	13.784,048,752,1
141	0.007,092,2	11.874,342,087,0	191	0.005,235,6	13.820,274,961,1
142	0.007,042,3	11.916,375,287,8	192	0.005,208,3	13.856,406,460,6
143	0.006,993,0	11.958,260,743,1	193	0.005,181,3	13.892,443,989,4
144	0.006,944,4	12.000,000,000,0	194	0.005,154,6	13.928,388,277,2
145	0.006,896,6	12.041,594,578,8	195	0.005,128,2	13.964,240,043,8
146	0.006,849,3	12.083,045,973,6	196	0.005,102,0	14.000,000,000,0
147	0.006,802,7	12.124,355,653,0	197	0.065,076,1	14.035,668,844,1
148	0.006,756,7	12.165,525,060,6	198	0.005,050,5	14.071,247,279,5
149	0.006,711,4	12.206,555,615,3	199	0.005,025,1	14.106,735,979,7
150	0.006,666,6	12.247,448,713,9	200	0.005,	14.142,135,623,7

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
201	0.004,975,1	14.177,446,878,8	251	0.003,984,1	15.842,979,517,8
202	0.004,950,4	14.212,670,403,6	252	0.003,968,3	15.874,507,866,4
203	0.004,926,1	14.247,806,848,8	253	0.003,952,6	15.905,973,720,6
204	0.004,902,0	14.282,856,857,1	254	0.003,937,0	15.937,377,450,5
205	0.004,875,0	14.317,821,063,3	255	0.003,921,6	15.968,719,422,7
206	0.004,854,4	14.352,700,094,4	256	0.003,906,3	16.000,000,000,0
207	0.004,830,9	14.387,494,569,9	257	0.003,891,1	16.031,219,541,9
208	0.004,807,7	14.422,205,101,9	258	0.003,876,0	16.062,378,404,2
209	0.004,784,7	14.456,832,294,8	259	0.003,861,0	16.093,476,939,4
210	0.004,761,9	14.491,376,746,2	260	0.003,846,2	16.124,515,496,6
211	0.004,739,3	14.525,839,046,3	261	0.003,831,4	16.155,494,421,4
212	0.004,717,0	14.560,219,778,6	262	0.003,816,8	16.186,414,056,2
213	0.004,694,8	14.594,519,519,3	263	0.003,802,3	16.217,274,740,2
214	0.004,672,9	14.628,738,838,3	264	0.003,787,8	16.248,076,809,2
215	0.004,651,2	14.662,878,298,6	265	0.003,773,6	16.278,820,596,1
216	0.004,629,6	14.696,938,456,7	266	0.003,759,4	16.309,506,430,3
217	0.004,608,3	14.730,919,862,7	267	0.003,745,3	16.340,134,638,4
218	0.004,587,2	14.764,823,060,2	268	0.003,731,3	16.370,705,543,7
219	0.004,566,2	14.798,648,586,9	269	0.003,717,5	16.401,219,466,9
220	0.004,545,4	14.832,396,974,2	270	0.003,703,7	16.431,676,725,2
221	0.004,524,9	14.866,068,747,3	271	0.003,690,0	16.462,077,633,2
222	0.004,504,5	14.899,664,425,8	272	0.003,676,5	16.492,422,502,5
223	0.004,484,3	14.933,184,523,1	273	0.003,663,0	16.522,711,641,9
224	0.004,464,3	14.966,629,547,1	274	0.003,649,6	16.552,945,356,9
225	0.004,444,4	15.000,000,000,0	275	0.003,636,3	16.583,123,951,8
226	0.004,424,8	15.033,296,378,4	276	0.003,623,2	16.613,247,725,8
227	0.004,405,3	15.066,519,173,3	277	0.003,610,1	16.643,316,977,1
228	0.004,386,0	15.099,668,870,5	278	0.003,597,1	16.673,332,000,5
229	0.004,366,8	15.132,745,950,4	279	0.003,584,2	16.703,293,088,5
230	0.004,347,8	15.165,750,888,1	280	0.003,571,4	16.733,200,530,7
231	0.004,329,0	15.198,684,153,6	281	0.003,558,7	16.763,054,614,2
232	0.004,310,3	15.231,546,211,7	282	0.003,546,1	16.792,855,623,7
233	0.004,291,8	15.264,337,522,5	283	0.003,533,6	16.822,603,841,3
234	0.004,273,5	15.297,058,540,8	284	0.003,521,1	16.852,299,546,4
235	0.004,255,3	15.329,709,716,8	285	0.003,508,8	16.881,943,016,1
236	0.904,237,3	15.362,291,495,7	286	0.003,496,5	16.911,534,525,3
237	0.004,219,4	15.394,804,318,3	287	0.003,484,3	16.941,074,346,1
238	0.004,201,7	15.427,248,620,9	288	0.003,472,2	16.970,562,748,5
239	0.004,184,1	15.459,624,833,7	289	0.003,460,2	17.000,000,000,0
240	0.004,166,6	15.491,933,384,8	290	0.003,448,3	17.029,386,365,9
241	0.004,149,4	15.524,174,696,3	291	0.003,436,4	17.058,722,109,2
242	0.004,132,2	15.556,349,186,1	292	0.003,424,6	17.088,007,490,6
243	0.004,115,2	15.588.457,268,1	293	0.003,413,0	17.117,242,768,6
244	0.004,098,4	15.620,499,351,8	294	0.003,401,4	17.146,428,199,5
245	0.004,081,6	15.652,475,842,5	295	0.003,389,8	17.175,564,037,3
246	0.004,065	15.684,387,141,4	296	0.003,378,3	17.204.650,534,1
247	0.004,048,6	15.716,233,645,5	297	0.003,367,0	17.233,687,939,6
248	0.004,032,3	15.748,015,748,0	298	0.003,355,7	17.262,676,,01,6
249	0.004,016,1	15.779,733,838,1	299	0.003,344,5	17.291,616,465,8
250	0.004,	15.811,388,300,8	300	0.003,333,3	17.320,508,075,7

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
301	0.003,322,3	17.349,351,572,9	351	0.002,849,0	18.734,993,995,2
302	0.003,311,3	17.378,147,196,9	352	0.002,840,9	18.761,663,039,3
303	0.003,300,3	17.406,895,185,5	353	0.002,832,9	18.788,294,228,1
304	0.003,289,5	17.435,595,774,2	354	0.002,824,8	18.814,887,722,2
305	0.003,278,7	17.464,249,196,6	355	0.002,816,9	18.841,443,681,4
306	0.003,268,0	17.492,855,684,5	356	0.002,809,0	18.867,962,264,1
307	0.003,257,3	17.521,415,467,9	357	0.002,801,1	18.894,443,627,7
308	0.003,246,8	17.549,928,774,8	358	0.002,793,3	18.920,887,928,4
309	0.003,236,2	17.578,395,831,2	359	0.002,785,5	18.947,295,321,5
310	0.003,225,8	17.606,816,861,7	360	0.002,777,7	18.973,665,961,0
311	0.003,215,4	17.635,192,088,5	361	0.002,770,1	19.000,000,000,0
312	0.003,205,1	17.663,521,732,7	362	0.002,762,4	19.026,297,590,4
313	0.003,194,9	17.691,806,013,0	363	0.002,754,8	19.052,558,883,3
314	0.003,184,7	17.720,045,146,7	364	0.002,747,3	19.078,784,028,3
315	0.003,174,6	17.748,239,349,3	365	0.002,739,7	19.104,973,174,5
316	0.003,164,6	17.776,388,834,6	366	0.002,732,2	19.131,126,469,7
317	0.003,154,6	17.804,493,814,8	367	0.002,724,8	19.157,244,060,7
318	0.003,144,7	17.832,554,500,1	368	0.002,717,4	19.183,326,093,3
319	0.003,134,8	17.860,571,099,5	369	0.002,710,0	19.209,372,712,3
320	0.003,125,	17.888,543,820,0	370	0.002,702,7	19.235,384,061,7
321	0.003,115,3	17.916,472,867,2	371	0.002,695,4	19.261,360,284,3
322	0.003,105,6	17.944,358,444,9	372	0.002,688,2	19.287,301,522,0
323	0.003,096,0	17.972,200,755,6	373	0.002,681,0	19.313,207,915,8
324	0.003,086,4	18.000,000,000,0	374	0.002,673,8	19.339,053,751,4
325	0.003,076,9	18.027,756,377,3	375	0.002,666,6	19.364,916,731,0
326	0.003,067,5	18.055,470,085,3	376	0.002,659,6	19.390,719,429,7
327	0.003,058,1	18.083,141,320,0	377	0.002,652,5	19.416,487,838,9
328	0.003,048,8	18.110,770,276,3	378	0.002,645,5	19.442,222,095,2
329	0.003,039,5	18.138,357,147,2	379	0.002,638,5	19.467,922,333,9
330	0.003,030,3	18.165,902,124,6	380	0.002,631,6	19.493,588,689,6
331	0.003,021,1	18.193,405,398,7	381	0.002,624,7	19.519,221,295,9
332	0.003,012,0	18.220,867,158,3	382	0.002,617,8	19.544,820,285,7
333	0.003,003,0	18.248,287,590,9	383	0.002,611,0	19.570,385,790,8
334	0.002,994,0	18.275,666,882,5	384	0.002,604,2	19.595,917,942,3
335	0.002,985,1	18.303,005,217,7	385	0.002,597,4	19.621,416,870,3
336	0.002,976,2	18.330,302,779,8	386	0.002,590,7	19.646,882,704,4
337	0.002,967,4	18.357,559,750,7	387	0.002,584,0	19.672,315,572,9
338	0.002,958,6	18.384,776,310,9	388	0.002,577,3	19.697,715,603,6
339	0.002,949,9	18.411,952,639,5	389	0.002,570,7	19.723,082,923,1
340	0.002,941,2	18.439,088,914,6	390	0.002,564,1	19.748,417,658,1
341	0.002,932,6	18.466,185,312,6	391	0.002,557,5	19.773,719,933,3
342	0.002,924,0	18.493,242,008,9	392	0.002,551,0	19.798,989,873,2
343	0.002,915,5	18.520,259,177,5	393	0.002,544,5	19.824,227,601,6
344	0.002,907,0	18.547,236,991,0	394	0.002,538,1	19.849,433,241,3
345	0.002,898,6	18.574,175,621,0	395	0.002,531,6	19.874,606,914,4
346	0.002,890,2	18.601,075,237,7	396	0.002,525,2	19.899,748,742,1
347	0.002,881,8	18.627,936,010,2	397	0.002,518,9	19.924,858,845,2
348	0.002,873,6	18.654,758,106,2	398	0.002,512,6	19.949,937,343,3
349	0.002,865,3	18.681,541,692,3	399	0.002,506,3	19.974,984,355,4
350	0.002,857,1	18.708,286,933,9	400	0.002,5	20.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
401	0.002,493,8	20.024,984,394,5	451	0.002,217,3	21.236,760,581,6
402	0.002,487,6	20.049,937,655,8	452	0.002,212,4	21.260,291,625,5
403	0.002,481,4	20.074,859,899,9	453	0.002,207,5	21.283,796,653,8
404	0.002,475,2	20.099,751,242,2	454	0.902,202,6	21.307,275,752,7
405	0.002,469,1	20.124,611,797,5	455	0.002,197,8	21,330,729,007,7
406	0.002,463,1	20.149,441,679,6	456	0.002,193,0	21.354,156,504,1
407	0.002,457,0	20.174,241,001,8	457	0.002,188,2	21.377,558,326,4
408	0.002,451,0	20.199,009,876,7	458	0.002,183,4	21.400,934,559,0
409	0.002,445,0	20.223,748,416,2	459	0.002,178,6	21.424,285,285,6
410	0.002,439,0	20.248,456,731,3	460	0.002,173,9	21.447,610,589,5
411	0.002,433,1	20.273,134,932,7	461	0.002,169,2	21.470,910,553,6
412	0.002,427,2	20.297,783,130,2	462	0.002,164,5	21.494,185,257,9
413	0.002,421,3	20.322,401,432,9	463	0.002,159,8	21.517,434,791,4
414	0.002,415,5	20.346,989,949,4	464	0.002,155,2	21.540,659,228,5
415	0.002,409,6	20.371,548,787,5	465	0.002,150,5	21.563,858,652,8
416	0.002,403,8	20.396,078,054,4	466	0.002,145,9	21.587,033,144,9
417	0.002,398,1	20.420,577,856,7	467	0.002,141,3	21.610,182,785,0
418	0.002,392,3	20.445,048,300,3	468	0.002,136,8	21.633,307,652,8
419	0.002,386,6	20.469,489,490,5	469	0.002,132,2	21.656,407,827,7
420	0.002,381,0	20.493,901,531,9	470	0.002,127,7	21.679,483,388,7
421	0.002,375,3	20.518,284,528,7	471	0.002,123,1	21.702,534,414,2
422	0.002,369,7	20.542,638,584,2	472	0.002,118,7	21.725,560,982,4
423	0.002,364,1	20.566,963,801,2	473	0.002,114,2	21.748,563,170,9
424	0.002,358,5	20.591,260,282,0	474	0.092,109,7	21.771,541,057,1
425	0.002,352,9	20.615,528,128,1	475	0.002,105,3	21.794,494,717,7
426	0.002,347,4	20.639,767,440,6	476	0.002,100,8	21.817,424,229,3
427	0.002,341,9	20.663,978,319,8	477	0.002,096,4	21.840,329,667,8
428	0.002,336,4	20.688,160,865,6	478	0.002,092,1	21.863,21[illegible]09,1
429	0.002,331,0	20.712,315,177,2	479	0.002,087,7	21.886,068,628,2
430	0.002,325,6	20.736,441,353,3	480	0.002,083,3	21.908,902,300,2
431	0.002,320,2	20.760,539,492,0	481	0.002,079,0	21.931,712,199,5
432	0.002,314,8	20.784,609,690,8	482	0.002,074,7	21.954,498,402,4
433	0.002,309,5	20.808,652,046,7	483	0.002,070,4	21.977,260,975,8
434	0.002,304,2	20.832,666,656,0	484	0.002,066,1	22.000,000,000,0
435	0.002,298,9	20.856,653,614,6	485	0.002,061,9	22.022,715,545,5
436	0.002,293,6	20.880,613,017,8	486	0.002,057,6	22.045,407,685,0
437	0.002,288,3	20.904,544,960,4	487	0.002,053,4	22.068,076,490,7
438	0.002,283,1	20.928,449,536,5	488	0.002,049,2	22.090,722,034,4
439	0.002,277,9	20.952,326,839,8	489	0.002,045,0	22.113,344,387,5
440	0.002,272,7	20.976,176,963,4	490	0.002,040,8	22.135,943,621,2
441	0.002,267,6	21.000,000,000,0	491	0.002,036,7	22.158,519,806,2
442	0.002,262,4	21.023,796,041,6	492	0.002,032,5	22.181,073,012,8
443	0.002,257,3	21.047,565,179,8	493	0.002,028,4	22.203,603,311,2
444	0.002,252,2	21.071,307,505,7	494	0.002,024,3	22.226,110,770,9
445	0.002,247,2	21.095,023,109,7	495	0.002,020,2	22.248,595,461,3
446	0.002,242,2	21.118,712,031,9	496	0.002,016,2	22.271,057,451,3
447	0.002,237,1	21.142,374,511,9	497	0.002,012,1	22.293,496,809,6
448	0.002,232,1	21.166,010,488,5	498	0.002,008,0	22.315,913,604,4
449	0.002,227,2	21.189,620,100,4	499	0.002,004,0	22.338,307,903,9
450	0.002,222,2	21.213,203,435,6	500	0.002.	22.360,679,775,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
501	0.001,996,0	22.383,029,285,6	551	0.001,814,9	23.473,389,188,6
502	0.001,992,0	22.405,356,502,4	552	0.001,811,6	23.494,680,248,9
503	0.001,988,1	22.427,661,492,0	553	0.001,808,3	23.515,952,032,6
504	0.001,984,1	22.449,944,320,6	554	0.001,805,1	23.537,204,591,9
505	0.001,980,1	22.472,205,054,2	555	0.001,801,8	23.558,437,978,8
506	0.001,976,3	22.494,443,758,4	556	0.001,798,6	23.579,652,245,1
507	0.001,972,4	22.516,660,498,4	557	0.001,795,3	23.600,847,442,4
508	0.001,968,5	22.538,855,339,2	558	0.001,792,1	23.622,023,622,0
509	0.001,964,6	22.561,028,345,4	559	0.001,788,9	23.643,180,835,1
510	0.001,960,8	22.583,179,581,3	560	0.001,785,7	23.664,319,132,4
511	0.001,956,9	22.605,309,110,9	561	0.001,782,5	23.685,438,564,7
512	0.001,953,1	22.627,416,998,0	562	0.001,779,4	23.706,539,182,3
513	0.001,949,3	22.649,503,305,8	563	0.001,776,2	23.727,621,035,4
514	0.001,945,5	22.671,568,097,5	564	0.001,773,0	23.748,684,174,1
515	0.001,941,7	22.693,611,435,8	565	0.001,769,9	23.769,728,648,0
516	0.001,938,0	22.715,633,383,2	566	0.001,766,8	23.790,754,506,7
517	0.001,934,2	22.737,634,001,8	567	0.001,763,7	23.811,761,799,6
518	0.001,930,5	22.759,613,353,5	568	0.001,760,6	23.832,750,575,6
519	0.001,926,8	22.781,571,499,8	569	0.001,757,5	23.853,720,883,8
520	0.001,923,1	22.803,508,502,0	570	0.001,754,4	23.874,672,772,6
521	0.001,919,4	22.825,424,421,0	571	0.001,751,3	23.895,606,290,7
522	0.001,915,7	22.847,319,317,6	572	0.001,748,3	23.916,521,486,2
523	0.001,912,0	22.869,193,252,1	573	0.001,745,2	23.937,418,407,2
524	0.001,908,4	22.891,046,284,5	574	0.001,742,2	23.958,297,101,4
525	0.001,904,8	22.912,878,474,8	575	0.001,739,1	23.979,157,616,6
526	0.001,901,1	22.934,689,882,4	576	0.001,736,1	24.000,000,000,0
527	0.001,897,5	22.956,480,566,5	577	0.001,733,1	24.020,824,298,9
528	0.001,893,9	22.978,250,586,2	578	0.001,730,1	24.041,630,560,3
529	0.001,890.4	23.000,000,000,0	579	0.001,727,1	24.062,418,831,0
530	0.001,886,8	23.021,728,866,4	580	0.001,724,1	24.083,168,396,2
531	0.001,883,2	23.043,437,243,6	581	0.001,721,2	24.103,941,586,4
532	0.001,879,7	23.065,125,189,3	582	0.001,718,2	24.124,676,163,6
533	0.001,876,2	23.086,792,761,2	583	0.001,715,3	24.145,392,935,3
534	0.001,872,7	23.108,440,016,6	584	0.001,712,3	24.166,091,947,2
535	0.001,869,2	23.130,067,012,4	585	0.001,709,4	24.186,773,244,9
536	0.001,865,7	23.151,673,805,6	586	0.001,706,5	24.207,436,873,6
537	0.001,862,2	23.173,260,452,5	587	0.001,703,6	24.228,082,879,2
538	0.001,858,7	23.194,827,009,5	588	0.001,700,7	24.248,711,306,0
539	0.001,855,3	23.216,373,532,5	589	0.001,697,8	24.269,322,199,0
540	0.001,851,8	23.237,900,077,2	590	0.001,694,9	24.289,915,603.0
541	0.001,848,4	23.259,406,699,2	591	0.001,692,0	24.310,491,562,3
542	0.001,845,0	23.280,893,453,6	592	0.001,689,1	24.331,050,121,2
543	0.001,841,6	23.302,360,395,5	593	0.001,686,3	24.351,591,323,8
544	0.001,838,2	23.323,807,579,4	594	0.001,683,5	24.372,115,213,9
545	0.001,834,9	23.345,235,059,9	595	0.001,680,7	24.392,621,835,3
546	0.001,831,5	23.366,642,891,1	596	0.001,677,9	24.413,111,231,5
547	0.001,828,2	23.388,031,127,1	597	0.001,675,0	24.433,583,445,7
548	0.001,824,8	23.409,399,821,4	598	0.001,672,2	24.454,038,521,3
549	0.001,821,5	23.430,749,027,7	599	0.001,669,4	24.474,476,501,0
550	0.001,818,1	23.452,078,799,1	600	0.001,666,6	24.494,897,427,8

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
601	0.001,663,9	24.515,301,344,3	651	0.001,536,1	25.514,701,644,3
602	0.001,661,1	24.535,688,292,8	652	0.001,533,7	25.534,290,669,6
603	0.001,658,4	24.556,058,315,6	653	0.001,531,4	25.553,864,678,4
604	0.001,655,6	24.576,411,454,9	654	0.001,529,1	25.573,423,705,1
605	0.001,652,9	24.596,747,752,5	655	0.001,526,7	25.592,967,784,1
606	0.001,650,1	24.617,067,250,2	656	0.001,524,4	25.612,496,949,7
607	0.001,647,4	24.637,369,989,5	657	0.001,522,1	25.632,011,236,0
608	0.001,644,7	24.657,656,011,9	658	0.001,519,8	25.651,510,676,8
609	0.001,642,0	24.677,925,358,5	659	0.001,517,5	25.670,995,306,0
610	0.001,639,3	24.698,178,070,5	660	0.001,515,1	25.690,465,157,3
611	0.001,636,7	24.718,414,188,6	661	0.001,512,9	25.709,920,264,4
612	0.001,634,0	24.738,633,753,7	662	0.001,510,6	25.729,360,660,5
613	0.001,631,3	24.758,836,806,3	663	0.001,508,3	25.748,786,379,2
614	0.001,628,7	24.779,023,386,7	664	0.001,506,0	25.768,197,453,5
615	0.001,626,0	24.799,193,535,3	665	0.001,503,8	25.787,593,916,5
616	0.001,623,4	24.819,347,292,0	666	0.001,501,5	25.806,975,801,1
617	0.001,620,7	24.839,484,696,7	667	0.001,499,3	25.826,343,140,3
618	0.001,618,1	24.859,605,789,3	668	0.001,497,0	25.845,695,966,6
619	0.001,615,5	24.879,710,609,2	669	0.001,494,8	25.865,034,312,8
620	0.001,612,9	24.899,799,196,0	670	0.001,492,5	25.884,358,211,1
621	0.001,610,3	24.919,871,588,8	671	0.001,490,3	25.903,667,694,0
622	0.001,607,7	24.939,927,826,7	672	0.001,488,1	25.922,962,793,6
623	0.001,605,1	24.959,967,948,7	673	0.001,485,9	25.942,243,542,1
624	0.001,602,6	24.979,991,993,6	674	0.001,483,7	25.961,509,971,5
625	0.001,6	25.000,000,000,0	675	0.001,481,4	25.980,762,113,5
626	0.001,597,4	25.019,992,006,4	676	0.001,479,3	26.000,000,000,0
627	0.001,594,9	25.039,968,051,1	677	0.001,477,1	26.019,223,662,5
628	0.001,592,4	25.059,928,172,3	678	0.001,474,9	26.038,433,132,6
629	0.001,589,8	25.079,872,408,0	679	0.001,472,8	26.057,628,441,6
630	0.001,587,3	25.099,800,796,0	680	0.001,470,6	26.076,809,620,8
631	0.001,584,8	25.119,713,374,2	681	0.001,468,4	26.095,976,701,4
632	0.001,582,3	25.139,610,180,0	682	0.001,466,3	26.115,129,714,4
633	0.001,579,8	25.159,491,250,8	683	0.001,464,1	26.134,268,690,7
634	0.001,577,3	25.179,356,620,1	684	0.001,462,0	26.153,393,661,2
635	0.001,574,8	25.199,206,336,7	685	0.001,459,9	26.172,504,656,6
636	0.001,572,3	25.219,040,425,8	686	0.001,457,7	26.191,601,707,4
637	0.001,569,9	25.238,858,928,2	687	0.001,455,6	26.210,684,844,2
638	0.001,567,4	25.258,661,880,6	688	0.001,453,5	26.229,754,097,2
639	0.001,564,9	25.278,449,319,5	689	0.001,451,4	26.248,809,496,8
640	0.001,562,5	25.298,221,281,3	690	0.001,449,3	26.267,851,073,1
641	0.001,560,1	25.317,977,802,3	691	0.001,447,2	26.286,878,856,2
642	0.001,557,6	25.337,718,918,6	692	0.001,445,1	26.305,892,875,9
643	0.001,555,2	25.357,444,666,2	693	0.001,443,0	26.324,893,162,2
644	0.001,552,8	25.377,155,080,9	694	0.001,440,9	26.343,879,744,6
645	0.001,550,4	25.396,850,198,4	695	0.001,438,8	26.362,852,652,9
646	0.001,548,0	25.416,530,054,3	696	0.001,436,8	26.381,811,916,5
647	0.001,545,6	25.436,194,684,0	697	0.001,434,7	26.400,757,564,9
648	0.001,543,2	25.455,844,122,7	698	0.001,432,7	26.419,689,627,2
649	0.001,540,8	25.475,478,405,7	699	0.001,430,6	26.438,608,132,8
650	0.001,538,5	25.495,097,568,0	700	0.001,428,6	26.457,513,110,6

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
701	0.001,426,5	26.476,404,589,7	751	0.001,331,6	27.404,379,212,1
702	0.001,424,5	26.495,282,599,0	752	0.001,329,8	27.422,618,401,6
703	0.001,422,5	26.514,147,167,1	753	0.001,328,0	27.440,845,468,0
704	0.001,420,5	26.532,998,322,8	754	0.001,326,3	27.459,060,435,5
705	0.001,418,4	26.551,836,094,7	755	0.001,324,5	27.477,263,328,1
706	0.001,416,4	26.570,660,511,2	756	0.001,322,8	27.495,454,169,7
707	0.001,414,4	26.589,471,600,6	757	0.001,321,0	27.513,632,984,4
708	0.001,412,4	26.608,269,391,3	758	0.001,319,3	27.531,799,795,9
709	0.001,410,4	26.627,053,911,4	759	0.001,3 7,5	27.549,954,627,9
710	0.001,408,5	26.645,825,188,9	760	0.001,315,8	27.568,097,504,2
711	0.001,406,5	26.664,582,251,9	761	0.001,314,1	27.586,228,448,3
712	0.001,404,5	26.683,328,128,3	762	0.001,312,3	27.604,347,483,7
713	0 001,402,5	26.702,059,845,6	763	0.001,310,6	27.622,454,633,9
714	0.001,400,6	26.720,778,431,8	764	0.001,308,9	27.640,549,922,2
715	0.001,398,6	26.739,483,914,2	765	0.001,307,2	27.658,633,371,9
716	0.001,396,6	26.758,176,320,5	766	0.001,305,5	27.676,705,006,2
717	0.001,394,7	26.776,855,678,0	767	0.001,303,8	27.694,764,848,3
718	0.001,392,8	26.795,522;013,9	768	0.001,302,1	27.712,812,921,1
719	0.001,390,8	26.814,175,355,6	769	0.001,300,4	27.730,849,247,7
720	0.001,388,8	26.832,815,730,0	770	0.001,298,7	27.748,873,851,0
721	0,001,387,0	26.851,443,164,2	771	0.001,297,0	27.766,886,753,8
722	0.001,385,0	26.870,057,685,1	772	0.001,295,3	27.784,887,978,9
723	0.001,383,1	26.888,659,319,5	773	0.001,293,7	27.802,877,548,9
724	0.001,381,2	26 907,248,094,1	774	0.001,292,0	27.820,855,486,5
725	0.001,379,3	26.925,824,035,7	775	0.001,290,3	27.838,821,814,2
726	0.001,377,4	26.944,387,170,6	776	0.001,288,7	27.856,776,554,4
727	0.001,375,5	26.962,937,525,4	777	0.001,287,0	27.874,719,729,5
728	0.001,373,6	26.981,475,126,5	778	0.001,285,3	27.892,651,362,0
729	0.001,371,7	27.000,000,000,0	779	0.001,283,7	27.910,571,473,9
730	0.001,369,9	27.018,512,172,2	780	0.001,282,1	27.928,480,087,5
731	0.001,368,0	27.037,011,669,2	781	0.001,280,4	27.946,377,225,0
732	0.001,366,1	27.055,498,516,9	782	0.001,278,8	27.964,262,908,2
733	0.001,364,3	27.073,972,741,4	783	0.001,277,1	27.982,137,159,3
734	0.001,362,4	27.092,434,368,3	784	0.001,275,5	28.000,000,000,0
735	0.001,360,5	27.110,883,423,5	785	0.001,273,9	28.017,851,452,2
736	0.001,358,7	27.129,319,932,5	786	0.001,272,3	28.035,691,537,8
737	0.001,356,9	27.147,743,921,0	787	0.001,270,6	28.053,520,278,2
738	0.001,355,0	27.166,155,414,4	788	0.001,269,0	28.071,337,688,1
739	0.001,353,2	27.184,554,438,1	789	0.001,267,4	28.089,143,810,4
740	0.001,351,3	27.202,941,017,5	790	0.001,265,8	28.106,938,645,1
741	0.001,349,5	27.221,315,177,6	791	0.001,264,2	28.124,722,220,9
742	0.001,347,7	27.239,676,943,8	792	0.001,262,6	28.142,494,558,9
743	0.001,345,9	27.258,026,340,9	793	0.001,261,0	28.150,255,680,7
744	0.001,344,1	27.276,363,394,0	794	0.001,259,4	28.178,005,607,2
745	0.001,342,3	27.294,688,127,9	795	0.001,257,9	28.195,744,359,7
746	0.001,340,5	27.313,000,567,5	796	0.001,256,3	28.213,471,959,3
747	0.001,338,7	27.331,300,737,4	797	0.001,254,7	28.231,188,427,0
748	0.001,336,9	27.349,588,662,4	798	0.001,253,1	28.248,893,783,7
749	0.001,335,1	27.367,864,366,8	799	0.001,251,6	28.266,588,050,2
750	0.001,333,3	27.386,127,875,3	800	0.001,25	28.284,271,247,5

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
801	0.001,248,4	28.301,943,396,2	851	0.001,175,1	29.171,904,291,6
802	0.001,246,9	28.319,604,517,0	852	0.001,173,7	29.189,039,038,7
803	0.001,245,3	28.337,254,630,6	853	0.001,172,3	29.206,163,733,0
804	0.001,243,8	28.354,893,757,5	854	0.001,171,0	29.223,278,392,4
805	0.001,242,2	28.372,521,918,2	855	0.001,169,6	29.240,383,034,4
806	0.001,240,7	28.390,139,133,2	856	0.001,168,2	29.257,477,676,7
807	0.001,239,2	28.407,745,422,7	857	0.001,166,9	29.274,562,336,6
808	0.001,237,6	28.425,340,807,1	858	0.001,165,5	29.291,637,031,8
809	0.001,236,1	28.442,925,306,7	859	0.001,164,1	29.308,701,779,5
810	0.001,234,6	28.460,498,941,5	860	0.001,162,8	29.325,756,597,2
811	0.001,233,0	28.478,061,731,8	861	0.001,161,4	29.342,801,502,2
812	0.001,231,5	28.495,613,697,6	862	0.001,160,1	29.359,836,511,8
813	0.001,230,0	28.513,154,858,8	863	0.001,158,7	29.376,861,643,1
814	0.001,228,5	28.530,685,235,4	864	0.001,157,4	29.393,876,913,4
815	0.001,227,0	28.548,204,847,2	865	0.001,156,1	29.410,882,339,7
816	0.001,225,5	28.565,713,714,2	866	0.001,154,7	29.427,877,939,1
817	0.001,224,0	28.583,211,855,9	867	0.001,153,4	29.444,863,728,7
818	0.001,222,5	28.600,699,292,2	868	0.001,152,1	29.461,839,725,3
819	0.001,221,0	28.618,176,042,5	869	0.001,150,7	29.478,805,946,0
820	0.001,219,5	28.635,642,126,6	870	0.001,149,4	29.495,762,407,5
821	0.001,218,0	28.653,097,563,8	871	0.001,148,1	29.512,709,126,7
822	0.001,216,5	28.670,542,373,7	872	0.001,146,8	29.529,646,120,5
823	0.001,215,1	28.687,976,575,6	873	0.001,145,5	29.546,573,405,4
824	0.001,213,6	28.705,400,188,8	874	0.001,144,2	29.563,490,998,2
825	0.001,212,1	28.722,813,232,7	875	0.001,142,9	29.580,398,915,5
826	0.001,210,6	28.740,215,726,4	876	0.001,141,6	29.597,297,173,9
827	0.001,209,2	28.757,607,689,1	877	0.001,140,3	29.614,185,789,9
828	0.001,207,7	28.774,989,139,9	878	0.001,139,0	29.631,064,780,1
829	0.001,206,3	28.792,360,097,8	879	0.001,137,7	29.647,932,474,3
830	0.001,204,8	28.809,720,581,8	880	0.001,136,3	29.664,793,948,4
831	0.001,203,4	28.827,070,610,8	881	0.001,135,1	29.681,644,159,3
832	0.001,201,9	28.844,410,203,7	882	0.001,133,8	29.698,484,809,8
833	0.001,200,5	28.861,739,379,3	883	0.001,132,5	29.715,315,916,2
834	0.001,199,0	28.879,058,156,4	884	0.001,131,2	29.732,137,494,6
835	0.001,197,6	28.896,366,553,6	885	0.001,129,9	29.748,949,561,3
836	0.001,196,2	28.913,664,589,6	886	0.001,128,7	29.765,752,132,3
837	0.001,194,7	28.930,952,283,0	887	0.001,127,4	29.782,545,223,7
838	0.001,193,3	28.948,229,652,3	888	0.001,126,1	29.799,328,851,5
839	0.001,191,9	28.965,496,715,9	889	0.001,124,9	29.816,103,031,8
840	0.001,190,5	28.982,753,492,4	890	0.001,123,6	29.832,867,780,4
841	0.001,189,1	29.000,000,000,0	891	0.001,122,3	29.849,623,113,2
842	0.001,187,6	29.017,236,257,1	892	0.001,121,1	29.866,369,046,1
843	0.001,186,2	29.034,462,281,9	893	0.001,119,8	29.883,105,595,0
844	0.001,184,8	29.051,678,092,7	894	0.001,118,6	29.899,832,775,5
845	0.001,183,4	29.068,883,707,5	895	0.001,117,3	29.916,550,603,3
846	0.001,182,0	29.086,079,144,5	896	0.001,116,1	29.933,259,094,2
847	0.001,180,6	29.103,264,421,7	897	0.001,114,8	29.949,958,263,7
848	0.001,179,2	29.120,439,557,1	898	0.001,113,6	29.966,648,127,5
849	0.001,177,9	29.137,604,568,7	899	0.001,112,3	29.983,328,701,1
850	0.001,176,5	29.154.759,474,2	900	0.001,111,1	30.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
901	0.001,109,9	39.016,662,039,6	951	0.001,051,5	30.838,287,890,2
902	0.001,108,6	30.033,314,835,4	952	0.001,050,4	30.854,497,241,7
903	0.001,107,4	30.049,958,402,6	953	0.001,049,3	30.870,698,080,9
904	0.001,106,2	30.066,592,756,7	954	0.001,048,2	30.886,890,423,0
905	0.001,105,0	30.083,217,913,0	955	0.001,047,1	30.903,074,280,7
906	0.001,103,8	30.099,833,886,6	956	0.001,046,0	30.919,249,667,5
907	0.001,102,5	30.116,440,692,8	957	0.001,044,9	30.935,416,596,5
908	0.001,101,3	30.133,038,346,6	958	0.001,043,8	30.951,575,081,1
909	0.001,100,1	30.149,626,863,4	959	0.001,042,8	30.967,725,134,4
910	0.001,098,9	30.166,206,258,0	960	0.001,041,6	30.983,866,769,7
911	0.001,097,7	30.182,776,545,6	961	0.001,040,6	31.000,000,000,0
912	0.001,096,5	30.199,337,741,1	962	0.001,039,5	31.016,124,838,5
913	0.001,095,3	30.215,889,859,5	963	0.001,038,4	31.032,241,298,4
914	0.001,094,1	30.232,432,915,7	964	0.001,037,3	31.048,349,392,5
915	0.001,092,9	30.248,966,924,5	965	0.001,036,3	31.064,449,134,0
916	0.001,091,7	30.265,491,900,8	966	0.001,035,2	31.080,540,535,8
917	0.001,090,5	30.282,007,859,5	967	0.001,034,1	31.096,623,610,9
918	0.001,089,3	30.298,514,815,1	968	0.001,033,1	31.112,698,372,2
919	0.001,088,1	30.315,012,782,4	969	0.001,032,0	31.128,764,832,5
920	0.001,087,0	30.331,501,776,2	970	0.001,030,9	31.144,823,004,8
921	0.001,085,8	30.347,981,811,0	971	0.001,029,9	31.160,872,901,8
922	0.001,084,6	30.364,452,901,4	972	0.001,028,8	31.176,914,536,2
923	0.001,083,4	30.380,915,061,9	973	0.001,027,7	31.192,947,921,0
924	0.091,082,3	30.397,368,307,1	974	0.001,026,7	31.208,973,068,7
925	0.001,081,0	30.413,812,651,5	975	0.001,025,6	31.224,989,992,0
926	0.001,079,9	30.430,248,109,4	976	0.001,024,6	31.240,998,703,6
927	0.001,078,7	30.446,674,695,3	977	0.001,023,5	31.256,999,216,2
928	0.001,077,6	30.463,092,423,5	978	0.001,022,5	31.272,991,542,2
929	0.001,076,4	30.479,501,308,3	979	0.001,021,5	31.288,975,694,3
930	0.001,075,3	30.495,901,364,0	980	0.001,020,4	31.304,951,685,0
931	0.001,074,1	30.512,292,604,8	981	0.001,019,4	31.320,919,526,7
932	0.001,073,0	30.528,675,044,9	982	0.001,018,3	31.336,879,232,0
933	0.001,071,8	30.545,048,698,6	983	0.001,017,3	31.352,830,813,2
934	0.001,070,7	30.561,413,579,9	984	0.001,016,3	31.368,774,282,7
935	0.001,069,5	30.577,769,702,8	985	0.001,015,2	31.384,709,653,0
936	0.001,068,4	30.594,117,081,6	986	0.001,014,2	31.400,636,936,2
937	0.001,067,2	30.610,455,730,0	987	0.001,013,2	31.416,556,144,8
938	0.001,066,1	30.626,785,662,2	988	0.001,012,1	31.432,467,291,0
939	0.001,065,0	30.643,106,892,1	989	0.001,011,1	31.448,370,387,0
940	0.001,063,8	30.659,419,433,5	990	0.001,010,1	31.464,265,445,1
941	0.001,062,7	30.675,723,300,4	991	0.001,009,1	31.480,152,477,4
942	0.001,061,6	30.692,018,506,4	992	0.001,008,1	31.496,031,496,0
943	0.001,060,4	30.708,305,065,6	993	0.001,007,0	31.511,902,513,2
944	0.001,059,3	30.724,582,991,5	994	0.001,006,0	31.527,765,540,9
945	0.001,058,2	30.740,852,297,9	995	0.001,005,0	31.543,620,591,2
946	0.001,057,1	30.757,112,998,5	996	0.001,004,0	31.559,467,676,1
947	0.001,056,0	30.773,365,106,9	997	0.001,003,0	31.575,306,807,7
948	0.001,054,9	30.789,608,636,7	998	0.001,002,0	31.591,137,997,9
949	0.001,053,7	30.805,843,601,5	999	0.001,001,0	31.606,961,258,6
950	0.001,052,6	30.822,070,014,8	1000	0.001,	31.622,776,601,7

Dr. Hutton's *Account of the foregoing Table of the Reciprocals and the Square-Roots of the Natural Numbers* 1, 2, 3, 4, 5, 6, 7, *&c, to* 1,000, *given at the end of the Fourth Volume of his Collection of Mathematical Problems and Tracts, intitled* Miſcellanea Mathematica, *publiſhed in Four little Volumes, Duodecimo, in the Year* 1775.

OF the preceeding Table, the uſe is evidently to ſhorten arithmetical calculations, and will appear eminently great to thoſe mathematicians and others who are frequently concerned in ſuch kinds of computations. The ſtructure of the table is evident; the firſt column contains the natural ſeries of numbers from 1 to 1,000, the 2d the reciprocals, and the 3d the ſquare-roots of the ſame numbers, very accurately calculated and printed. Theſe reciprocals and roots are the reſults preſerved of many years occaſional and accidental calculations in various ſubjects: in frequently making ſuch computations, I found that I had often to make diviſions by, and to extract the roots of, the ſame numbers; and as it ſeemed probable that this might be the caſe with me for many years longer, I formed the reſolution of preſerving all ſuch roots and reciprocals as I ſhould occaſionally produce in my calculations, that I might have them always ready on any future occaſion; which I did, by entering them always in a little book, ruled for the purpoſe, till I have at laſt collected to the number of 1,000, as above; and I now publiſh them here in this cheap and eaſy manner, that they may be of like uſe to other perſons as to myſelf. In the numerical calculations of ſuch kinds of problems as have appeared in this Miſcellany and the Diary, the uſe of this table will be found to be very great, becauſe of the frequent diviſions and extractions of roots, which are to be made: and the manner and caſes of applying theſe numbers are generally evident; only it may be remarked, that the column of reciprocals (which are no other than the decimal values of the quotients reſulting from the diviſion of unity, or 1, by each of the ſeveral numbers, from 1 to 1,000), is not only uſeful in ſhewing by inſpection the quotient when the dividend is unity, but is alſo applied with much advantage in turning many diviſions into multiplications, which are much eaſier performed than the equivalent diviſions. For, if we multiply any propoſed dividend by the reciprocal of the diviſor (as found in the table,) the product will be the quotient ſought; which is the caſe mentioned in p. 54 of my Menſuration, where this table of reciprocals was promiſed to be inſerted at the end of that Work; *but* it was then ſuppreſſed, as the book had been unavoidably extended to ſo great a ſize, and becauſe it could properly enough be omitted, as being no part of the ſubject of the book. This table of reciprocals may alſo be applied to good purpoſe in ſumming the terms of many converging ſerieſes, as in the 2d ſolution of Queſt. 106 of this Miſcellany, in which a few of the firſt terms are to be found by diviſion, and then ſummed; for the quotients of ſuch diviſions are here ſhewn by inſpection.

The reciprocals are carried on to 7, and the roots to 10 places of decimals, each being put down to the neareſt figure in the laſt place, that is, when the next figure beyond the laſt put down in the table came out a 5, or more, the laſt figure was increaſed by 1, otherwiſe not; excepting in the repetends which occurred among the reciprocals, where the real laſt figure is always put down. The reciprocals which in the table conſiſt of leſs than ſeven figures, are thoſe which terminate and are complete within that number; ſuch as ·5 the reciprocal of 2, .25 the reciprocal of 4, &c.

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ANALYSIS

ÆQUATIONUM.

William
AUCTORE (*GUIL.*) *HALES*, D. D.

COLL. S. TRIN. DUBLIN. SOCIO.

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P R Æ F A T I O.

ÆQUATIONUM Theoria, utpote permagni in matheſi momenti, trecentos fermè per annos, a tempore Lucæ de Burgo *ad uſque præſens, Analyſtarum ingenia exercuit. Veruntamen fatendum eſt, quod Analyſtæ maximi, inventis ſuis algebræ fines extendere utplurimùm ſatagunt, de inventorum evidentiâ demonſtrationibus validis atque legitimis ſtabiliendâ parùm ſolliciti; "quoniam hæ ſatis faciles ſibi viſæ ſunt, & nonnunquam abſque nimiis ambagibus tradi non poſſent"—quales memorat cauſas* Newtonus, *in* Arithmetica Univerſali, *cur, æquationum proprietates docens, demonſtrationes non ſemper adjunxit. Hanc itaque curam poſteris legant. Inde vero fit, quod regulas æquationum præſtantiſſimas, vel imperfecte tradunt, vel quaſi demonſtrationibus idoneis deſtitutas omittunt omnes fere algebræ inſtitutiones. Sic, ut pauca exempla e multis proferam,* Newtoni *celeberrimum theorema binomium, necnon germanum illud de inveniendis ſummis poteſtatum radicum æquationis; vel nullis, vel minus accuratis, vel denique alieno ex fonte petitis demonſtrationibus, omnibus fere in libris elementariis extant:* Carteſii *theorema egregium de numero radicum affirmativarum & negativarum æquationis dignoſcendo, plerique abſque demonſtratione tradunt, nonnulli negligunt: Methodus vero detegendi radices æquationum impoſſibiles, ſi quæ ſint, earumque numerum, exceptâ regulâ quadam imperfectâ a*

Newtono *traditâ, ex omnibus exulat. Hunc igitur defectum, aliquatenus ſaltem, ſupplere, noſtri inſtituti eſt ratio; ne ſcientia præſtantiſſima quidem, & quæ ipſi geometriæ vix forſan evidentiâ cederet, ſi modo lucido ordine & methodo veterum Geometrarum inſiſterent Analyſtæ, falſo incertitudinis crimine premi videatur.* Newtoni Arithmeticam Univerſalem *potiſſimum, ut interpres ſecutus ſum, in conſilium ſubinde adhibitis, peritiſſimis in re mathematicâ ingeniis,* Wallis, M^{c}Laurin, Saunderſon, De Moivre, Simpſon, Clairaut, D'Alembert, Euler, La Grange, Waring, Bertrand, Landen, Hutton, *&c. &c. qui aut ſcriptis* Newtoni *illuſtrandis, aut algebræ limitibus latius proferendis feliciſſime operam dederunt. Curſim quoque adſtruitur brevis hiſtoria præcipuarum inventionum analyticarum, ad levandum diſquiſitionis abſtruſioris tædium; & notantur auctores quorum ex ſcriptis pleniorem harum rerum cognitionem haurire licet. Quid ipſe præſtiterim aliorum eſto judicium. Hoc ſaltem præmonitos velim lectores, quod opus elementarium methodo elegantiſſimâ abſolvere, analyſi breviſſimâ & ſimul perſpicuâ, ſyntheſi concinnâ & minime operosâ, longe difficillimum eſt.* Un petit génie pourra s'imaginer qu'il eſt facile de faire un livre élémentaire clair & précis—les vrais Savans en connoiſſent la difficulté.

Sauri, Inſtitutions Mathematiques.

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DE NATURA ÆQUATIONUM.

§. 1. ÆQUATIONES ſunt quantitatum aut ſibi mutuo æqualium, aut ſimul nihilo æquipollentium congeries. Priores appellantur *canonicæ*, poſteriores *originales*.

2. Duobus modis conſiderandæ veniunt æquationes, vel ut ultimæ concluſiones ad quas in Problematis ſolvendis deventum eſt, vel ut media quorum ope finales æquationes acquirendæ ſunt. Æquatio finalis ex unica tantum quantitate incognita cognitis involuta conflatur, modò Problema definitum ſit et aliquid certi quærendum innuat. Mediæ autem, quæ problematum conditiones deſignant, plures involvunt quantitates incognitas, atque ideo debent inter ſe comparari & ita connecti ut ex omnibus una tandem emergat æquatio nova cui ineſt unica quam quærimus incognita quantitas admiſta cognitis: cujus reductione quantitatis incognitæ valor comparandus eſt.

3. Propoſita igitur quæſtione aliquâ, artificis ingenium in eo præſertim requiritur ut omnes ejus conditiones totidem æquationibus deſignet. Ad quod faciendum, perpendat imprimis an propoſitiones ſive ſententiæ quibus enunciatur ſint omnes aptæ quæ terminis algebraicis deſignari poſſint, haud ſecus quam conceptus noſtri characteribus græcis vel latinis. Et ſi ita, (ut ſolet in quæſtionibus quæ circa numeros vel abſtractas quantitates verſantur) deinde nomina quantitatibus ignotis atque etiam notis, ſi opus fuerit, imponat[a]; & ſenſum quæſtionis, ſermone, ut ita loquar,

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[a] Algebriſtæ antiquiores ſolas quantitates incognitas per literas deſignarunt, notas autem per ipſos numeros, puta, A — 10 = 24. Uſum literarum alphabeti vice ſignorum antea adhibitorum primus introduxiſſe fertur *Buteo*, qui Tractatum Algebraicum edidit anno 1559. Recentiores vero poſt *Vietam* fere omnes deſignant quantitates tam datas quam quæſitas per literas aut ſpecies, cum magno artis

analytico deſignet. Et conditiones ejus ad algebraicos terminos ſic tranſlatæ, tot dabunt æquationes quot ei ſolvendæ ſufficiunt.

4. Quemadmodum *ſi quærantur duo numeri quorum ſumma ſit* 140, *et productum* 3136. Deſignantibus x et y, numeros quæſitos, quæſtio deducitur ad æquationes $x + y = 140$, et $xy = 3136$. Quarum ope per regulas mox tradendas inveſtigandi ſunt x et y.

5. Aut, *ſi quærantur tres numeri continue proportionales, quorum ſumma ſit* $20 = a$, *et quadratorum ſumma* $140 = b$. Sint numeri quæſiti x, y et z, et ex quæſtione prodeunt æquationes $xz = y^2$, $x + y + z = a$, et $x^2 + y^2 + z^2 = b$. Quarum reductione eruendi ſunt x, y et z.

6. Cæterum notandum eſt, ſolutiones quæſtionum eo magis expeditas et artificioſas utplurimum evadere quo pauciores incognitæ quantitates ſub initio ponuntur. Aliquando autem harum aliquam ex quæſtionum conditione aliqua deſignare licebit, dum ex conditionibus reliquis elicientur æquationes quæ in ſolutionem poſtulantur.

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analyticæ emolumento, ſive laboris inter operandum minuendi, ſive theorematum generalium eliciendorum gratiâ. Hinc diſtinctionis ergo, forma vetuſtior appellatur Algebra *numeroſa*, recentior *ſpecioſa*. At vero *Vieta*, eumque ſecuti *Oughtredus* aliique, uſi ſunt literis majuſculis; hodierni fere omnes partim brevitatis et compendii causâ, partim ad evitandam confuſionem quæ aliter in praxi non raro extaret, ex eo quod literæ majusculæ deſignandis figurarum geometricarum lineis tribuuntur, adhibent minuſculas. Quantitates datas diſtinxit *Carteſius* per literas alphabeti primas, a, b, c, &c. quæſitas autem per ultimas z, y, x, &c. Quantitatum poteſtates primus quoque deſignavit indicibus integris a^2, a^3, a^4, &c. Quantitatum radices deſignaſſe fertur *Albertus Girardus* indicibus fractis, $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, &c. pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c:a^5}$, &c. Quantitates reciprocas indicibus negativis notavit *Wallisius*, pro $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$, &c. ſcribendo a^{-1}, a^{-2}, a^{-3}, &c. qui et uſum indicum indefinitorum et ſurdorum introduxit. Cæterum ex quo invaluit indicum numeralium uſus, ut evitetur confuſio, coefficientes numerales literis præfixæ ſunt, quas affigebant vetuſtiores ſcribendo nimirum $x3$ pro $3x$.

Sic in quæſtione priore, ex una conditione quod ſumma quantitatum incognitarum ſit 140, unam deſignante x, alteram deſignabit $140 - x$; ex altera conditione autem elicietur æquatio $x \times \overline{140 - x} = 3136$, cujus reductio dabit x. Atque in quæſtione altera, ex prima conditione quod numeri quæſiti ſint continue proportionales, exponentibus x et y, primum et ſecundum, exponet $\frac{y^2}{x}$, tertium; ex reliquis autem conditionibus prodeunt duæ æquationes, $x + y + \frac{y^2}{x} = a$ et, $x^2 + y^2 + \frac{y^4}{x^2} = b$. Quarum reductione x, et y determinandi ſunt. Interdum etiam plures vel omnes quantitates incognitas ex conditionibus quæſtionum deſignare fas erit. Ut ex ſequentibus patebit.

7. Etſi deſideratas quantitates utplurimum immediate quærere ſolemus, ſi quando tamen duæ obvenerint ambiguæ, hoc eſt, quæ conditionibus omnino ſimilibus præditæ ſunt (ut quatuor numerorum continue proportionalium duo medii et duo extremi) præſtat alias quantitates non ambiguas quærere, quas inter et haſce intercedit nota quædam relatio, quemadmodum harum ſummam vel differentiam, vel rectangulum.

8. Nonnumquam quæſtiones geometricæ eadem facilitate iiſdemque legibus ad æquationes redigi poſſunt ac quæ de abſtractis quantitatibus proponuntur, ubi ſcilicet ſolæ quantitatum proportiones abſque poſitionibus linearum conſiderandæ veniunt. Ut *ſi data quædam recta in media et extrema proportione ſecanda ſit;* deſignante a rectam datam, et x ſegmentum majus, erit $a - x$ ſegmentum minus, et $x^2 = a \times \overline{a - x}$; æquatio quæ per reductionem dabit x.

9. Quod ſi ſermo quocum ſtatus quæſtionis exprimitur ineptus videatur qui in algebraicum poſſit verti, paucis mutationibus adhibitis, et ad ſenſum potius quam verborum

ſonos attendendo expedietur verſio. Cæterum methodum redigendi hujuſmodi problemata ad æquationes certis legibus vix inſtituere fas eſt, cum ex proprio cujuſque analyſtæ ingenio plurimum pendeat; exemplis ſane et exercitatione facilius quam præceptis illuſtranda et addiſcenda eſt. Placet igitur jam paucas inſtantias adjungere.

10. *Mercator quidam nummos ejus triente quotannis adauget, demptis £100 quas annuatim impendit in familiam, et poſt tres annos fit duplo ditior; quæruntur nummi?*

Plures latent in hac quæſtione propoſitiones, quæ omnes ſic eruuntur et enunciantur.

Mercator habet nummos quoſdam - - - - - -	x
Ex his anno primo expendit £100 - - -	$x - 100$
Et reliquum adauget triente - - - - - -	$x - 100 + \frac{x - 100}{3}$ ſive $\frac{4x - 400}{3}$
Annoque ſecundo expendit £100 - - -	$\frac{4x - 400}{3} - 100$ ſive $\frac{4x - 700}{3}$
Et reliquum adauget triente - - -	$\frac{4x - 700}{3} + \frac{4x - 700}{9}$ ſive $\frac{16x - 2800}{9}$
Et ſic anno tertio expendit £100 -	$\frac{16x - 2800}{9} - 100$ ſive $\frac{16x - 3700}{9}$
Et reliquum adauget triente -	$\frac{16x - 3700}{9} + \frac{16x - 3700}{27}$ ſive $\frac{64x - 14800}{27}$
Fitque duplo ditior quam ſub initio -	$\frac{64x + 14800}{27} = x$.

Quæſtio itaque ad æquationem $\frac{64x - 14800}{27} = x$ perducitur, cujus reductione eruendus eſt x.

11. Ex epitaphio quodam Arithmetico quæritur ætas *Diophanti* celeberrimi analyſtæ *Alexandrini* qui Anno 365 circiter floruit, et primus extat Algebræ ſcriptor apud Græcos.

Οὗτος τοι Διόφαντον ἔχει τάφος, ἆ μέγα θαῦμα.
Καὶ τάφος ἐκ τεχνῆς μέτρα βίοιο λέγει.
Εκτην κουρίζειν βιότου Θεὸς ὤπασε μόιρην,
Δωδεκάτη δ' ἐπιθεὶς μῆλα πόρεν χλοάειν.

Τῇ δ' ἄρ ἐπ' ἑβδομάτῃ τὺ γαμήλιον ἥψατο φέγγος,
Ἐκ δὲ γάμων πέμπτῳ παῖδ' ἐπένευσεν ἔτει.
Αἲ αἲ τηλύγετον δειλὸν τέκος, ἥμισυ πατρὸς,
Του δὲ καὶ ἡ κρυερὸς μέτρον ἑλὼν βιότου.
Πένθος δ' αὖ πισύρεσσι παρηγορέων ἐνιαυτοῖς
Τῇ δὲ πόσου σοφίῃ τέρμ' ἐπέρησε βίου;

Indicat utique epitaphium, *Diophantum* ſextam ætatis partem egiſſe puerum; partem inde duodecimam, puberem; completa ætatis parte ſeptima poſt, uxorem duxiſſe; interjecto autem quinquennio, filium genuiſſe; qui ſemiſſem paternæ ætatis quum attigiſſet, diem obiit ſupremum; patrem vero quadriennio poſt, fato quoque ceſſiſſe. Quæritur igitur *Diophanti* ætas: ſive in genere, quæritur numerus cujus partes ſexta, duodecima, ſeptima adjuncto quinario, et ſemiſſis adjuncto quaternario, ſimul conficiant totum. Sit numerus quæſitus x, et habebitur æquatio $\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4 = x$. Cujus reductione eruendus eſt x.

12. Quod ſi quæſtio ipſa omnes conditiones quæ in ſolutionem poſtulantur non ſuppeditet, evolvendæ ſunt quantitatum tum datarum tum quæſitarum relationes, quoquo modo compertæ, ſi quæ aptæ ſint equationibus rite pariendis. Hoc pacto etenim problemata, quæ primo conſpectu inſolubilia videntur, præſertim in *phyſicis* vel *mixtâ matheſi*, haud raro ſolutionem ſubire coguntur. Sic opificis fraudem, qui furatus eſt partem aliquam auri quod coronæ votivæ conflandæ gratiâ ei commiſerat *Hiero*, et argento ſubſtituto coronam juſti ponderis fabricavit, neutiquam detexiſſet *Archimedes*, rege jubente, ex datis tantum coronæ ipſius pondere et mole, ni pondera ſpecifica auri et argenti inſuper in computum adſciverat[b]. Uti poſthac oſtendetur in reductione problematum indeterminatorum.

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[b] Vide *Vitruvium*, lib. 9, c. 3.

13. Si ex quæſtione utcunque eliciendæ ſunt tot æquationes quot habet incognitas quantitates, determinata erit, et ſolutionem præbebit. Quod ſi conditionibus deficientibus, pauciores dentur æquationes quam incognitæ quantitates, quæſtio indeterminata erit, vel quodam reſpectu vel abſolute; et modò complures, modò innumeras ſortietur ſolutiones. Ut *ſi quærantur duo numeri* x *et* y *quorum ſumma ſit* 10. Deſignante x, 9; erit $y = 1$, deſignante x, 8; erit $y = 2$, &c. et inde novem prodeunt ſolutiones, ſi de numeris integris et poſitivis agatur; innumeræ autem, ſi de fractis vel negativis. Contra, ſi redundantibus conditionibus, plures habentur æquationes quam incognitæ quantitates, quæſtio quaſi pluſquam determinata evadet, et concluſio erit abſurda ſi inter ſe diſcordent conditiones, aut inepta ſi non. Sic *quæſitis duobus numeris quorum ſumma ſit* 8, *differentia* 2, *et productum* 12: Sint numeri x, et $8 - x$; eritque $x - 2 = 8 - x$. Adeoque $x = 5$, et alter $8 - x = 3$. Sed per tertiam conditionem productum 5×3 ſeu $15 = 12$ quod abſurdum. Et ſi hæc fuiſſet conditio ut *productum ſit* 15; inutilis evaderet, quippe quæ ex cæteris pendet, et æquationem identicam ſeu ineptam $15 = 15$ parit. Atque hoc pacto ſane detegentur quæſtionum conditiones aut ab invicem pendentes, aut inter ſe diſcordantes; ubi ſcilicet, opere rite inſtituto, ad æquationem identicam vel abſurdam devenitur.

De Dimensionibus Æquationum.

14. Dimenſiones æquationum ad quas in problematis ſolvendis deventum eſt, ex maxima dimenſione quantitatis incognitæ æſtimantur, nullo reſpectu ad quantitates cognitas habito, nec ad terminos intermedios. Sic æquatio $\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4 = x$ eſt ſimplex ſeu primi gradus, $140x - x^2 = 3136$ quadratica ſeu ſecundi gradus, & ſic deinceps. Et ſi æquatio plures quantitates incognitas ſortiatur, gradus hujus perinde quoque æſtimantur ex

maxima dimenſione ejuſdem aut plurium quantitatum incognitarum. Sic $xy = 3136$, et $x^4 + y^4 + \frac{y^4}{x} = b$ ſunt quadraticæ, $xyz + x^2 = a$ cubica, &c. Problema inſequens ad æquationes quotcunque dimenſionum deducet.

15. *Vir quidam nummos* a *in mercaturam impendit, quam vero hâc non ſuccederet, ſe expedire cupit ſub fine anni primi; at tali occaſione non oblatâ, ſolvi nequit niſi ſub fine anni ſecundi, aut tertii, aut denique anni* n*: quo tempore, ex nummis poſt primi anni jacturam redeuntibus deperit* b. *Quæritur damnum ejus pro ſingulis* £100 *quotannis?*

Sit x damnum quæſitum, et inſtituendo analogiam, $(100 : 100 - x :: a :)\ a \times \frac{100 - x}{100}$ ſive $a \times \overline{1 - \frac{x}{100}}|$ exponet mercatoris nummos poſt primi anni jacturam; et continuando analogiam, $(100 : 100 - x :: a \times \frac{100 - x}{100} :)$ $a \times \overline{\frac{100 - x}{100}}|^2$ ſive $a \times \overline{1 - \frac{x}{100}}|^2$ nummos poſt ſecundi anni jacturam; ſimiliter, $a \times \overline{1 - \frac{x}{100}}|^3$ nummos poſt anni tertii jacturam; et in genere, $a \times \overline{1 - \frac{x}{100}}|^n$ nummos poſt anni n jacturam. Exponet item $a \times \overline{1 - \frac{x}{100}} - b$ mercatoris nummos ſub fine anni cujuſque poſt primum. Poſito igitur illum a mercatura ſolvi ſub fine anni ſecundi, æquatio quadratica $a \times \overline{1 - \frac{x}{100}}|^2 = a \times \overline{1 - \frac{x}{100}} - b$, per reductionem dabit x; ſub fine anni tertii, æquatio cubica $a \times \overline{1 - \frac{x}{100}}|^3 = a \times \overline{1 - \frac{x}{100}} - b$; et ſub fine anni n, æquatio n^t gradus $a \times \overline{1 - \frac{x}{100}}|^n = a \times \overline{1 - \frac{x}{100}} - b$, per reſolutionem dabit x.

16. His præfatis de natura et dimenſionibus æquationum ad quas problemata deduci debent, inſtituenda eſt earum analyſis. Trademus itaque modos,

I. Concinnandi æquationes quotcunque dimenſionum; et reducendi quadraticas, aliaſque ordini quadraticarum adſcribendas.

II. Ex mediis pluribus in ſe collatis eliciendi finales æquationes, et reducendi indeterminatas primi et ſecundi gradus.

III. Reſolvendi finales æquationes quotcunque dimenſionum.

ANALYSIS ÆQUATIONUM.

PARS PRIMA.

DE CONCINNANDA ÆQUATIONE SOLITARIA.

17. Quoniam quantitates incognitæ quas referunt æquationes cognitis involvuntur per varios modos additionis vel ſubductionis, multiplicationis vel diviſionis, involutionis vel evolutionis; neceſſe eſt, ut per operationes iis contrarias quibus cum cognitis implicitæ fuerunt, extricentur et expediantur. De ordinatione & reductione igitur æquationis ſolitariæ in ſequentibus regulis agetur; quæ hoc pendunt principio, quod ſi quantitatibus quibuſlibet ſibi mutuo æqualibus addantur vel auferantur æquales, multiplicentur vel dividantur per æquales, involvantur vel evolvantur æqualiter, manebit adhuc quantitatum per has operationes reſultantium æqualitas.

18. Reg. I. *Termini æquationum ſecundum dimenſiones incognitæ quantitatis in ordinem ſemper redigendi ſunt.* Ita ſcilicet ut primum occupent locum in quibus incognita quantitas eſt plurimarum dimenſionum, et ſecundum locum in quibus ea eſt una dimenſione minor, et ſic deinceps; ad normam ſequentium: in quibus x deſignat quantitatem quæſitam; p, q, r, s, alias quaſcunque quantitates ex quibus determinatis et cognitis etiam x determinatur, et per regulas inſequentes erui poteſt valor ejus ſi ſit unius aut duarum dimenſionum, ſive maximæ poteſtatis ejus ſi plurium.

$x = p;$ vel, $x - p = 0,$
$x^2 = px + q,$ $x^2 - px - q = 0;$
$x^3 = px^2 + qx + r;$ $x^3 - px^2 - qx - r = 0;$
$x^4 = px^3 + qx^2 + rx + s,$ $x^4 - px^3 - qx^2 - rx - s = 0,$
&c. &c.

19. REG. II. *Siquæ ſunt quantitates quæ ſe mutuo deſtruere, vel per additionem aut ſubductionem coaleſcere poſſunt, termini perinde minuendi ſunt.*

Sic § 11, æquatio $x = \frac{1}{2}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{4}x + 4;$ addendo partes x et numeros integros reſpective, prodit $x = \frac{75x}{84} + 9,$ et ſubducendo $\frac{75x}{84}$ utrobique, $\frac{9x}{84} = 9.$ Sic $a^2 + x = a^2 + b$, deletâ a^2 per ſubductionem, dat $x = b.$ Sic quoque $\sqrt{48} = x - \sqrt{75}$, dat $x = \sqrt{48} + \sqrt{75} = 4\sqrt{3} + 5\sqrt{3} = 9\sqrt{3}.$ Itemque $\sqrt{48a^2bc} - x = \sqrt{\frac{16}{27}a^2bc}$ dat $x = \sqrt{48a^2bc} - \sqrt{\frac{48}{81}a^2bc} = 4a\sqrt{3bc} - \frac{4}{9}a\sqrt{3bc} = \frac{32}{9}a\sqrt{3bc}.$ Ad hanc igitur regulam referri debet ordinatio terminorum æquationis quæ fieri ſolet per tranſlationem ad partes contrarias cum ſigno mutato.

20. REG. III. *Siqua compareat quantitas per quam omnes æquationis termini multiplicantur, debent omnes per illam dividi; vel ſi per eandem quantitatem omnes dividantur, debent omnes per illam multiplicari.*

Sic $15b^2 = 24ab + 3bx,$ dividendo per $3b$ et tranſponendo, prodit $x = 5b - 8a.$ Itemque, $\frac{x}{c} - \sqrt{\frac{ab^3}{c^2}} = \frac{1}{2c}\sqrt{a^3b - 4a^2b^2 + 4ab^3},$ multiplicando per c, et tranſponendo, dat $x = \sqrt{ab^3} + \frac{1}{2}\sqrt{a^3b - 4a^2b^2 + 4ab^3} = b\sqrt{ab} + \frac{1}{2} \times \overline{a - 2b}\sqrt{ab} = \frac{1}{2}a\sqrt{ab}.$

21. *Aliquando reductio inſtitui poteſt dividendo æquationem per compoſitam aliquam quantitatem.* Sic $y^3 = \overline{b - 2c}\, y^2 + 3bcy - b^2c$, transferendo omnes terminos ad eandem partem, et dividendo per $y - b$, dat $yy + 2cy - bc = 0$. De inventione hujuſmodi diviſorum perutili poſthac agemus.

22. *Hoc pacto tolli poſſunt fractiones ex æquationibus.* Multiplicando ſcil; omnes terminos per uniuſcujuſque denominatorem. Sic habitâ § 10, æquatione $\frac{64x - 14800}{27} = 2x$, multiplicando per 27, et tranſponendo, prodit $10x = 14800$.

23. Reg. IV. *Si maxima poteſtas quantitatis incognitæ multiplicetur vel dividatur per cognitam quamlibet quantitatem, debet tota æquatio per eandem dividi vel multiplicari.*

Sic æquatio noviſſima $10x = 14800$, dividendo per 10, dat $x = 1480$, qui utique exponet *Mercatoris* nummos § 10, ut et lucrum. Itemque æquatio § 19, $\frac{9x}{84} = 9$, dividendo per $\frac{9}{84}$ fit $x = 84$. Vixit igitur *Diophantus* 84 annos. Et ſi habeatur $\sqrt{18ab^3}\, x = 6ab^2$, concinnando prodit $x = \sqrt{\frac{36a^2b^4}{18ab^3}} = \sqrt{2ab}$. Sic § 17, æquatio quadratica $a \times \left(1 - \frac{x}{100}\right)^2 = a \times \overline{1 - \frac{x}{100}} - b$, quadrando $1 - \frac{x}{100}$, et ordinando, prodit $x^2 - 100x = -10000\frac{b}{a}$. Et æquatio cubica $a \times \overline{1 - \frac{x}{100}}^3 = a \times \overline{1 - \frac{x}{100}} - b$, concinnando prodit $x^3 - 300x^2 + 20000x = 1000000\frac{b}{a}$, et ſic deinceps.

24. Reg. V. *Siqua ſit fractio irreducibilis in cujus denominatore reperiatur incognita quantitas, omnes æquationis termini per iſtum denominatorem, aut per aliquem diviſorem ejus multiplicandi ſunt.*

Sic æquatio $\frac{ax}{a-x}+b=x$, multiplicando per $a-x$ et tranſponendo prodit $x^2=bx-ab$. Et $\frac{a^2b^2}{cx^2}=\frac{x^2}{a+b-x}$ multiplicando primo per x^2, deinde per $a+b-x$ evadit $\frac{a^3b^2+a^2b^3-a^2b^2x}{c}=x^4$.

25. Reg. VI. *Sicui ſurdæ quantitati irreducibili involvatur quantitas incognita, cæteri omnes termini ad partes contrarias transferendi ſunt, et utraque æquationis pars in ſe* ſemel *multiplicanda ſi radix* quadratica *ſit, vel* bis *ſi ſit* cubica, &c.

Sic $\sqrt{a^2-ax}+a=x$ tranſlata a ad alteram partem fit $\sqrt{a^2-ax}=x-a$, et quadratis partibus, $a^2-ax=x^2-2ax+a^2$, et concinnando, $x=a$. Sic etiam $\sqrt{3:a^2x+2ax^2-x^3}=a-x$, partibus cubice multiplicatis evadit $a^2x+2ax^2-x^3=a^3-3a^2x+3ax^2-x^3$, ſeu $x^2=4ax-a^2$.

26. Reg. VII. *Aliquando reductio per extractionem radicis ex utraque æquationis parte inſtituitur.*

Quemadmodum ſi habeatur æquatio pura $x^2=\frac{1}{4}a^2-b^2$, extracta utrobique radice prodit $x=\sqrt{\frac{1}{4}a^2-b^2}$. Quod ſi habeatur $x^2+a^2=2ax+b^2$, transfer $2ax$, et exurget quadratum ex una parte perfectum $x^2-2ax+a^2=b^2$, extractiſque partium radicibus, $x-a=b$, ſeu $x=a+b$. Sic etiam $x^3-12x^2+48x=98$, demendo utrinque 64, exhibet cubum ex altera parte perfectum, ſcil; $x^3-12x^2+48-64=34$, extractiſque radicibus cubicis partium, prodit $x-4=\sqrt[3]{34}$ ſive $x=4+\sqrt[3]{34}$.

27. Cæterum æquationes ad quas in problematis ſolvendis deventum eſt, raro puras aut perfectas poteſtates exhibent ex iſta æquationis parte ad quam transferendi ſunt termini qui quantitatem incognitam aut ejus dimenſiones continent, quales jam poſuimus. Formam utplurimum imperfectam induunt, deficientibus nimirum uno pluribuſve terminis qui quadrato, aut cubo, &c. complendo inſerviunt, ut § 23, æquatio quadratica $x^2 - 100x = -10000\frac{a}{b}$, et cubica $x^3 - 300x^2 + 20000x = 1000000\frac{a}{b}$. Methodum generalem quidem reducendi æquationes hujuſmodi quadraticas, quas *adfectas* vocant, (quia præter quadratum incognitæ quantitatis, ſimplici quoque ejus poteſtate adficiuntur) jam docebimus; verùm methodus generalis reſolvendi æquationes altiorum dimenſionum per evolutionem, adhuc deſideratur.

DE REDUCTIONE ÆQUATIONUM QUADRATICARUM.

28. Si habeatur æquatio pura $x^2 = q$, (ubi q deſignat quantitatem quamcumque cognitam, ſive ſimplex fuerit ſive compoſita) extracta utrinque radice prodit $\pm x = \pm\sqrt{q}$, ſive, ſi explicetur hæc forma compendiaria, $+x = +\sqrt{q}$, $+x = -\sqrt{q}$, $-x = +\sqrt{q}$, vel $-x = -\sqrt{q}$. At ex his quatuor valoribus x, tertius tranſponendo revocatur ad ſecundum, et quartus ad primum; ſufficit igitur ſi affirmative tantum ponatur x, et bini ſtatuantur valores ejus, ſcil; $+\sqrt{q}$ vel $-\sqrt{q}$. Quod in ſequentibus quoque obſervandum. Incognitam vero quantitatem x, extracta radice quadraticâ duplicem ſortiri valorem, æquum eſt; quoniam enim quantitatis ſive affirmativæ ſive negativæ quadratum eſt ſemper affirmativum, e converſo, quadrati affirmativi radix poteſt eſſe

affirmativa vel negativa. 2do. Sit $x^2 = -q$, uterque valor x, ſcil; $+\sqrt{-q}$, vel $-\sqrt{-q}$ prodit impoſſibilis; cum quadrati negativi geneſis ſit impoſſibilis.

Hinc in æquatione quadratica pura, binæ exiſtunt radices, ſeu bini valores quantitatis incognitæ, quæ evadent ſimul reales ſi q ſit affirmativa, et ſimul impoſſibiles ſi q ſit negativa.

29. Quod ſi habeatur æquatio adfecta $x^2 + px = q$, (deſignantibus p et q quantitates quaſcunque cognitas) conſtat, membrum $x^2 + px$, a parte ſiniſtra æquationis, imperfectum exhibere quadratum binomiæ $x + \frac{1}{2}p$; quæ utique quadrando parit $x^2 + px + \frac{1}{4}p^2$. Utrinque igitur, a parte ſiniſtra ad complendum quadratum, et a parte dextra ad ſervandam membrorum æqualitatem addatur $\frac{1}{4}p^2$; (quarta pars quadrati coefficientis ſecundi termini,) et prodibit $x^2 + px + \frac{1}{4}p^2 = \frac{1}{4}p^2 + q$, et extracta utrinque radice, $x + \frac{1}{2}p = \pm\sqrt{\frac{1}{4}p^2 + q}$, et tranſponendo, $x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + q}$. Et ſic univerſaliter: *Si ſit* $x^2 = . px . q$, *erit* $x = . \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 . q}$. *Ubi* $\frac{1}{2}$ p & q *iiſdem ſignis ac* p & q *in æquatione priore afficienda ſunt, ſed* $\frac{1}{4}$ p^2 *ſemper affirmative ponendum*. Eſtque hoc exemplum Regula ad cujus ſimilitudinem æquationes omnes quadraticæ adfectæ ad formam ſimplicium reduci poſſunt.

30. Hinc, variantibus ſignis p et q, quatuor prodeunt quadraticarum formulæ[b]. Scilicet,

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[a] Hæc methodus complendi quadratum in ſpeciebus ad quadraticarum reductionem, extat in opere primi ſcriptoris algebraici *Lucæ de Burgo*, cui titulus *Summa de Arithmetica et Geometria*, impreſſo. Anno 1494, atque iterum Anno 1523. Hic tres formulas priores inſequentibus verſibus ſemibarbaris expoſuit: in quibus *res* et *radix* deſignant px; *cenſus*, x^2; *numerus* et *drachma*, q.

I. $x^2+px=+q$ adeoque $x=-\frac{1}{2}p+\sqrt{\frac{1}{4}p^2+q}$ vel, $=-\frac{1}{2}p-\sqrt{\frac{1}{4}p^2+q}$

II. $x^2-px=+q$ - - $x=+\frac{1}{2}p+\sqrt{\frac{1}{4}p^2+q}$ - $=+\frac{1}{2}p-\sqrt{\frac{1}{4}p^2+q}$

III. $x^2-px=-q$ - - - $x=+\frac{1}{2}p+\sqrt{\frac{1}{4}p^2-q}$ - $=+\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-q}$

IV. $x^2-px=-q$ - - $x=-\frac{1}{2}p+\sqrt{\frac{1}{4}p^2-q}$ - $=-\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-q}$

In formulis I et II, ubi quantitas abſoluta q eſt affirmativa, erit $\sqrt{\frac{1}{4}p^2+q}$ major quam $(\sqrt{\frac{1}{4}p^2}=)\ \frac{1}{2}p$, et proinde radix prior in utraque erit affirmativa, et poſterior negativa.

In formulis III et IV, ubi q eſt negativa, ſi ſit *minor* quam $\frac{1}{4}p^2$, erit $\sqrt{\frac{1}{4}p^2-q}$ affirmativa, et minor quam $\frac{1}{2}p$; et proinde in III, evadet utraque radix affirmativa, in IV, utraque negativa. Si ſit *æqualis* $\frac{1}{4}p^2$, evaneſcet $\sqrt{\frac{1}{4}p^2-q}$, et binæ radices in utraque evadent æquales, ſeu coaleſcent in unam. Et ſi q ſit *major* quam $\frac{1}{4}p^2$, erit $\frac{1}{4}p^2-q$ negativa, et $\sqrt{\frac{1}{4}p^2-q}$ impoſſibilis; et proinde binæ radices in utraque formula evadent impoſſibiles, quoniam hæc quantitas impoſſibilis earum compoſitionem ingreditur.

1. *Si res & cenſus numero coequantur, a rebus* $px+x^2=q.$
Dimidio ſumpto, cenſum producere debes
Addereque numero, cujus a radice totiens
Tolle ſemis rerum, cenſus latuſque redibit.

2. *Et ſi cum rebus drachmæ quadrato pares ſunt,* $q+px=x^2.$
Adde, ſicut primo; numerum producto quadrato
E rebus mediis, hujuſque radice recepta,
Si rebus mediis addes cenſus patefiet.

3. *At ſi cum numero radices cenſus æquabit,* $x^2+q=px.$
Drachmas a quadrato deme rei medietatis,
Hujus quod ſuperit radicem adde trahеve:
A rebus mediis ſic cenſus coſtæ noteſcet.

Quartam formulam non admiſerunt vetuſtiores, utpote quæ ambos valores x parit negativos, ut mox videbitur, negativos autem valores pro falſis habuerunt.

Hinc quoque in omni æquatione quadraticâ affectâ, binæ exiſtunt radices ſeu valores quantitatis incognitæ, quæ evadent ſimul reales, ubi q ſit affirmativa, aut ubi ſit negativa et non major ſaltem quam $\frac{1}{4}p^2$; nam ſi major, evadent ſimul impoſſibiles.

DE RADICIBUS ÆQUATIONUM QUADRATICARUM.

31. Ex radicibus ſeu valoribus quantitatis incognitæ realibus, affirmativi deſignant in genere quantitates *addititias*, negativi *ablatitias*. In quæſtionibus profecto ſolvendis quæ circa abſtractas quantitates verſantur, ubi nulla ſit rerum contrarietas, negliguntur negativi valores quaſi inutiles, et affirmativis tantum locus conceditur. Atque hinc quidem ablegabant Algebriſtæ vetuſtiores (necnon ablegant etiamnum nonnulli hujuſce ævi) radices negativas ſecundæ et tertiæ formulæ, et quartam omnino formulam. Verùm in mixta Matheſi, ubi oppoſitio quædam inter quantitates de quibus agitur, incidere poteſt, velut in motu locali inter progreſſum et regreſſum, in rebus humanis inter bona et debita, in phyſicis inter attractionem et repulſionem, jure admittendæ ſunt, utpote quæ ad caſus quæſtionum datis contrarios reſpiciunt. Negativarum naturam primus perſpexiſſe videtur acutiſſimus *Carteſius*, qui primus harum uſum in analyſin & geometriam introduxit.

32. Ex binis radicibus affirmativis tertiæ formulæ, haud raro contingit quod una tantum quæſtioni unde elicitur ſolvendæ reſpondet; nam quadratum $x^2 - px + \frac{1}{4}p^2$ ambigue pariet radix $x - \frac{1}{2}p$ vel $\frac{1}{2}p - x$, prout x ſit major vel minor quam $\frac{1}{2}p$. Unde ex binis valoribus per extractionem prodeuntibus major, ſcilicet $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$ deſignabit x in priore caſu, minor $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$ in poſteriore.

Cæterum in quæſtione ad hujuſmodi æquationem perducente, ſæpe ex profeſſo vel tacite, definitur quanta ſit x reſpectu $\frac{1}{2}p$.

33. Radices impoſſibiles propriè nec ſunt affirmativæ neque negativæ, licet ſub utraque forma deliteſcunt. Indicant profecto, quæſtiones unde naſcuntur ſolutionem prorſus reſpuere. Et dari quidem radices ſæpe impoſſibiles æquum eſt, ne caſus problematum, qui ſæpe impoſſibiles ſunt, exhibeant poſſibiles.

Hæc omnia de quadraticis abſtracte jam poſita, ex problematum inſequentium ſolutione luculentius patebunt.

P R O B. I.

34. *Vir quidam nummos* a *in mercaturam impendit, quum vero hâc non ſuccederet, ſe expedire cupit ſub fine anni primi; at occaſione non oblatâ, ſolvi nequit niſi ſub fine anni ſecundi: quo tempore, ex nummis poſt primi anni jacturam redeuntibus deperit* b. *Quæritur damnum ejus pro ſingulis* £100 *quotannis.*

Deſignante x damnum quæſitum, deventum erat § 15, ad æquationem $a \times \overline{1 - \frac{x}{100}}\Big|^2 = a \times \overline{1 - \frac{x}{100}} - b$, quæ concinnando prodit $x^2 - 100x = -10000\frac{b}{a}$. Et complendo quadratum, ſecundum Regulam § 29, erit $x^2 - 100x + 2500 = 2500 - 10000\frac{b}{a} = 2500 \times \overline{1 - \frac{4b}{a}}$. et extracta radice quadratica, $x - 50 = \pm 50\sqrt{1 - \frac{4b}{a}}$. adeoque $x = 50 \pm 50\sqrt{1 - \frac{4b}{a}}$.

Exempl. 1^{mo}. Sit b ad a in ratione 1 ad $4\frac{1}{6}$ ſeu 6 ad 25; erit $\frac{b}{a} = \frac{6}{25}$ et $\sqrt{1-\frac{4b}{a}} = \sqrt{1-\frac{24}{25}} = \frac{1}{5}$. Adeoque $x = 50 \pm 50 \times \frac{1}{5} = 60$ vel 40. Ex his vero uterque quæſtionem ſolvet. Nam ſi ſumatur $x = 60$ *per cent.* poſito quod a ſit £100000, ſecundum proportionem allatam debet eſſe $b =$ £24000. Cæterum *mercatoris* damnum ſub fine anni primi prodit £60000, reſiduum igitur ſit £40000: ex quo b ſeu damnum ſub fine anni ſecundi, prodit £24000, ut æquum eſt. Et ſi $x = 40$ *per cent.* erit damnum ſub fine anni primi £40,000 et reſiduum £60,000: adeoque $b =$ £24,000, ut antea. 2^{do}. Sit b ad a in ratione 1 ad 4; evaneſcente jam $\pm 50 \sqrt{1-\frac{4b}{a}}$ prodit unicus valor x, ſcil; 50, qui quæſtioni ſolvendæ reſpondet; ut conſtat. 3^{tio}. Sit b ad a in ratione 1 ad 3; prodibit $x = 50 \pm 50 \sqrt{-\frac{1}{3}}$, uterque ſcil; valor x impoſſibilis. Quod indicio eſt hunc quæſtionis caſum neutiquam eſſe ſolvendum.

PROB. II.

35. *Lapide in puteum decidente, ex ſono lapidis fundum percutientis, altitudinem putei cognoſcere.*

Sit altitudo putei x, et ſi lapis (neglectâ aeris reſiſtentiâ) motu uniformiter accelerato deſcendat per ſpatium quodlibet datum a (puta 16,122 pedum,) in tempore dato b (unius ſcil; minuti ſecundi,) et ſonus, (cum percurrit 1142 pedes in tempore unius minuti ſecundi motu uniformi) tranſeat per idem datum ſpatium a in tempore dato d; lapis deſcendet per ſpatium x in tempore $\frac{b\sqrt{x}}{\sqrt{a}}$, (nam ob tempora deſcenſus gravium ut radices ſpatiorum, erit hoc tempus ad tempus datum b ut $\sqrt{x}$ ad $\sqrt{a}$,) ſonus autem qui fit a lapide in fundum putei impingente aſcendet per idem ſpatium x in tempore $\frac{dx}{a}$. (Erit enim hoc tempus ad tempus datum d, ob motum ſoni uniformem, ut ſpatium x ad ſpatium a.) Ex horum temporum ſumma, conflatur tempus a lapide demiſſo ad ſoni reditum. Hoc tempus ex obſervatione cognoſci poteſt: ſit ipſum t, et erit $\frac{b\sqrt{x}}{\sqrt{a}} + \frac{dx}{a} = t$. Ac $\frac{b\sqrt{x}}{\sqrt{a}} = t - \frac{dx}{a}$. Et partibus quadratis, $\frac{b^2x}{a} = t^2 - \frac{2tdx}{a} + \frac{d^2x^2}{a^2}$. Et concinnando $x^2 = \frac{2adt+ab^2}{d^2}x - \frac{a^2t^2}{d^2}$. Et inde ope formulæ III, conferendo $\frac{2adt+ab^2}{d^2}$ cum p, et $\frac{a^2t^2}{d^2}$ cum q; evadet $\frac{1}{2}p = \frac{2adt+ab^2}{2d^2}$. $\frac{1}{4}p^2 - q = \frac{4a^2d^2t^2+4a^2b^2dt+a^2b^4}{4d^4} - \frac{a^2t^2}{d^2} = \frac{4a^2d^2t^2+4a^2b^2dt+a^2b^4-4a^2d^2t^2}{4d^4} = \frac{4a^2b^2dt+a^2b^4}{4d^4} = \frac{a^2b^2}{4d^4} \times \overline{b^2+4dt}$. Et $\sqrt{\frac{1}{4}p^2-q} = \frac{ab}{2d^2}\sqrt{b^2+4dt}$. Adeoque $x = \frac{adt+\frac{1}{2}ab^2}{d^2} \pm \frac{ab}{2d^2}\sqrt{b^2+4dt}$. Cæterum altitudo putei x duplex eſſe nequit, hanc igitur exponet minor radix ſeu valor x, ſcilicet $\frac{adt+\frac{1}{2}ab^2}{d^2} - \frac{ab}{2d^2}\sqrt{b^2+4dt}$. Nam tempora $\frac{b\sqrt{x}}{\sqrt{a}}$ et $\frac{dx}{a}$ ex hypotheſi ſimul conficiunt t tempus obſervatione compertum, erit igitur tempus $\frac{dx}{a}$ ſeorſim minus quam t, ſive x minor quam $\frac{at}{d}$; quod in hoc caſu ita ſe habet. Liquet enim $\sqrt{b^2+4dt}$ eſſe majorem quam b; finge æqualem $b+c$, eritque $x = \frac{adt+\frac{1}{2}ab^2}{d^2} - \frac{\frac{1}{2}ab}{d^2}\,\overline{b+c} = \frac{adt+\frac{1}{2}ab^2-\frac{1}{2}ab^2-\frac{1}{2}abc}{d^2} = \frac{adt-\frac{1}{2}abc}{d^2}$ et proinde x minor quam $\frac{adt}{d^2} = \frac{at}{d}$.

36. **Exempl.** Sit tempus obſervatione compertum a lapide demiſſo ad ſoni reditum, quinque minutorum ſecundorum, ſeu $t = 5''$. Adeoque, cum $a =$ ped. 16,122. $b = 1''$. et d propter analogiam $= ,014117''$. (nam ped. 1142 : ped. 16,122 :: $1''$: $,014117''$) evadet $adt = 16,122 \times 5 \times ,014117 = 1,13797137$. $\frac{1}{2}ab^2 = 8,061$. et $\frac{adt+\frac{1}{2}ab^2}{d^2} = 46158,79$. Erit quoque $b^2+4dt = 1,28234$. $\sqrt{b^2+4dt} = 1,1324$. $\frac{ab}{2d^2} = 40448,6556$. $\frac{ab}{2d^2}\sqrt{b^2+4dt} = 45804,057$. Et proinde, $\frac{adt+\frac{1}{2}ab^2}{d^2} - \frac{ab}{2d^2}\sqrt{b^2+4dt} = 354,733$ ped.

Altitudinem putei recte ſtatui ped. 354,733. ſic probare licet. Tempora $b\sqrt{\frac{x}{a}}$ et $\frac{dx}{a}$ per hypotheſin ſimul æquari debent t ſive quinque minutis ſecundis. Sed $x = 354,733$. $\frac{x}{a} = 22,0030$. et $\sqrt{\frac{x}{a}}$ id eſt, $b\sqrt{\frac{x}{a}} = 4,69''$. Eſt quoque $dx = 5,00776$. et $\frac{dx}{a} = ,31''$. Et proinde $b\sqrt{\frac{x}{a}} + \frac{dx}{a} = 4,69''. + ,31''. = 5''$.

37. Alter valor x, ſcilicet $\frac{adt+\frac{1}{2}ab^2}{d^2} + \frac{ab}{2d^2}\sqrt{b^2+4dt}$ alio ſatisfacit problematis caſui; cujus ea eſt conditio ut tempus obſervatione compertum ſit exceſſus temporis quo aſcendit ſonus ſupra tempus quo decidit lapis, ſeu $\frac{dx}{a} - b\sqrt{\frac{x}{a}} = t$. nam hæc æquatio, tranſponendo et quadrando dabit $\frac{b^2x}{a} = t^2 - \frac{2dtx}{a} + \frac{d^2x^2}{a^2}$ ut antea. Adeoque per reductionem $x = \frac{adt+\frac{1}{2}ab^2}{d^2} \pm \frac{ab}{2d^2}\sqrt{b^2+4dt}$.

38. In problematum phyſicorum reſolutione obſervandum eſt, quod concluſionum algebraicè elicitarum auctoritas, pendet ex principiorum phyſicorum quibus innituntur certitudine et accuratione, tum ex ipſorum experimentorum fide. Sic in hac ſolutione minus accurata prodit concluſio, 1mo. ex eo quod negligitur aeris reſiſtentia, quâ gravia cadendo notabiliter retardantur; 2do. quod valores a et d vix forſan accurate determinandi ſunt; et 3^{tio}. quod in ipſum t parvus error, unius ſemiſecundi ſaltem, irrepere poteſt; verùm hoc errorem haud contemnendum in concluſionem inducet.

P R O B. III.

39. *Invenire punctum æqualis attractionis inter duos planetas ſitum, quorum innoteſcit diſtantia ab invicem, et ratio maſſarum.*

Sit d diſtantia lunæ a tellure, x diſtantia puncti quæſiti a luna, erit hujus diſtantia a tellure $d-x$. Et quoniam harum attractiones quibuſcunque diſtantiis ſunt ut maſſæ t et l directe, et quadrata diſtantiarum inverſe; erit attractio telluris ut $\frac{t}{(d-x)^2}$, lunæ autem ut $\frac{l}{x^2}$. Adeoque hìc poſita attractionum æqualitate, erit $\frac{t}{(d-x)^2}=\frac{l}{x^2}$. et concinnando, $x^2+\frac{2dl}{t-l}x=\frac{d^2l}{t-l}$. Et collatis hujus terminis cum formula I. erit $\frac{1}{2}p=\frac{dl}{t-l}$. $\frac{1}{4}p^2+q=\frac{d^2l^2}{(t-l)^2}+\frac{d^2l}{t-l}$ $=\frac{d^2lt}{(t-l)^2}=\frac{d^2}{(t-l)^2}\times lt$. Et $\sqrt{\frac{1}{4}p^2+q}=\frac{d}{t-l}\sqrt{lt}$. Adeoque $x=\frac{-dl\pm d\sqrt{lt}}{t-l}$.

Horum valorum x, affirmativus $\frac{-dl+d\sqrt{lt}}{t-l}$ quæſtionem ſolvet: negativus vero $\frac{-dl-d\sqrt{lt}}{t-l}$ ad alterum quoddam punctum æqualis attractionis ultra lunam ſitum reſpicit; ut ex eo conſtat, quod ſi quæratur alterum punctum iſtud, diſtantiis jam deſignatis per x et $d+x$, habebitur æquatio $\frac{t}{(d+x)^2}=\frac{l}{x^2}$. Et per reductionem, $x=\frac{dl\pm d\sqrt{lt}}{t-l}$. idem ſcilicet geminus valor x ut antea, at contrariis ſignis affectus: quorum affirmativus hunc caſum ſolvet, negativus ad priorem refertur.

40. Et quidem binos valores x in utroque caſu diverſa ſortiri ſigna æquum eſt, quum duo hæc puncta æqualis attractionis ad plagas *lunæ oppoſitas* ſiſtantur. Verùm poſito quod x ſit diſtantia puncti quæſiti a tellure, erit $d-x$ diſtantia ejuſdem a luna. Adeoque $\frac{t}{x^2}=\frac{l}{(d-x)^2}$ unde per reductionem, prodit $x=\frac{dt\pm d\sqrt{lt}}{t-l}$. uterque jam valor x affirmativus, ut oportet; quum ad *eandem* plagam, reſpectu *telluris*, bina attractionis æqualis puncta ſiſtantur. Hinc liquet, quod quantitates affirmativæ et negativæ, conceptibus noſtris tantum diſcrepantes, in rerum naturâ æque reales ſunt habendæ. Ex his autem valoribus, minor $\frac{dt-d\sqrt{lt}}{t-l}$ quæſtionem ſolvet. Sit enim $d=60$ ſemidiametris telluris, et t ad l in ratione 40 ad 1 circiter; evadet diſtantia puncti quæſiti a tellure, $=\frac{2400-379{,}464}{39}=51{,}8$ ſeu 52 ſemidiam. telluris fere, et diſtantia alterius puncti prodibit $\frac{2400+379{,}464}{39}=72$ ſemidiam. fere.

41. Reductio æquationis $\frac{t}{x^2} = \frac{l}{(d-x)^2}$ aliter inſtitui poteſt. Nam extracta radice quadratica erit $\frac{\sqrt{t}}{x} = \frac{\pm\sqrt{l}}{d-x}$. Et concinnando, $x = \frac{\pm d\sqrt{t}}{\sqrt{t}+\sqrt{l}}$. ſeu multiplicando numeratorem et denominatorem per $\sqrt{t} - \sqrt{l}$, prodit x ut antea $= \frac{dt \pm d\sqrt{lt}}{t-l}$.

P R O B. IV.

42. *Rectam datam in extrema et media proportione ſecare.*

Sit rectæ datæ a ſegmentum majus x, erit ſegmentum minus $a - x$, adeoque per quæſtionem, $x^2 = a \times \overline{a-x} = -ax + a^2$. Et per reductionem, $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$. Alter valor x qui prodit negativus, ſcil; $-\frac{1}{2}a - \sqrt{\frac{5}{4}a^2}$ diverſo problematis caſui reſpondet, nempe, *Rectam datam in extrema et media proportione producere.* Hoc eſt, ita ut quadratum augmenti x ſit æquale rectangulo ſub a et $a + x$ contento, ſive $x^2 = ax + a^2$. Quæ per reductionem dabit eoſdem valores x, at contrariis ſignis affectos: ex his autem, $\frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$ hunc quæſtionis caſum ſolvet.

P R O B. V.

43. *Mercatores* A *et* B *pro mercibus* per annum *venumdatis, eandam ſummam* £2112 *recipiunt; verùm* A *lucrando* 2 per cent. *plus quam* B, *nummos ſuos adauget incremento* £100 per annum *plus quam* B. *Quæritur utriuſque lucrum* per cent. *et totum utriuſque lucrum* per annum.

Sit x lucrum *per cent.* mercatoris *A*, eritque propter analogiam, (ut $100+x$, reditus pro £100 impenſis, ad x; ita 2112, reditus totus *per annum* ad) $\frac{2112x}{100+x}$ totum lucrum mercatoris *A per annum*. Et ſimiliter, ($98+x : x-2 :: 2112$:) $\frac{2112x-4224}{98+x}$ totum lucrum *B per annum*. Adeoque per quæſtionem, $\frac{2112x}{100+x} - \frac{2112x-4224}{98+x} = 100$. Et concinnando, $x^2 = -198x - 5576$. Et per reductionem, $x = -99 \pm 65$. Prodeunte igitur utroque valore negativo, colligere licet x in hoc caſu deſignare non lucrum ſed damnum, et quæſtionem aliter eſſe efferendam. Scilicet,

44. *Mercatores* A *et* B *pro mercibus, &c. eandem ſummam* £2112 *recipiunt. Verùm* A, *amittendo* 2 per cent. *minus quam* B, *amittit* £100 per annum *minus quam* B. *Quæritur utriuſque damnum* per cent. *et totum utriuſque damnum* per annum.

Deſignante jam x damnum *per cent.* mercatoris *A*, erit ($100-x : x :: 2112$:) $\frac{2112x}{100-x}$ totum damnum *A per annum*. Et ſimiliter, ($98-x : x+2 :: 2112$:) $\frac{2112x+4224}{98-x}$ totum damnum *B per annum*. Sed per quæſtionem, $\frac{2112x+4224}{98-x} - \frac{2112x}{100-x} = 100$. Et concinnando, $x^2 = 198x - 5576$. Et per reductionem, $x = 99 \pm 65$. Horum valorum x affirmative jam prodeuntium, minor ſcil; $99 - 65 = 34$, quæſtionem ſolvet. Erit enim damnum *A*, £34 *per cent.* et totum damnum ejus *per annum* $\frac{2112 \times 34}{66} = £1088$. Eritque damnum *B*, £36 *per cent.* et totum ejus damnum *per annum* $\frac{2112 \times 34 + 4224}{64} = £1188$.

Alter valor x, ſcil; $99 + 65 = 164$, utpote major quam 100, damnum pro ſingulis £100 impenſis deſignare nequit, verùm caſui inſequenti ſolvendo reſpondet.

45. *Mercatorum* A *et* B *lucrum* per annum *nummos ab utroque in mercaturam impenſos excedit ſummâ* £2112. *At* A *quum lucratus ſit* 2 per cent. *minus quam* B, *lucratur* £100 per annum *plus quam* B. *Quæritur utriuſque lucrum* per cent. *et totum utriuſque lucrum* per annum.

Deſignante x lucrum *per cent.* mercatoris *A*, erit, ($x - 100 : x :: 2112 :$) $\frac{2112x}{x-100}$ totum lucrum *A per annum*.

Et ſimiliter, $x - 98 : x + 2 :: 2112 :$) $\frac{2112x+4224}{x-98}$ totum lucrum *B per annum*. Sed per quæſtionem, $\frac{2112x}{x-100} - \frac{2112x+4224}{x-98} = 100$. Et concinnando, $x^2 = 198x - 5576$, ut antea. Adeoque per reductionem, $x = 164$. Erit igitur lucrum *A per cent.* £164, et totum lucrum ejus *per annum* $\frac{2112 \times 164}{64} =$ £5412. Et lucrum *B per cent.* £166, et totum lucrum ejus *per annum* $\frac{2112 \times 164 + 4224}{66} =$ £5312.

P R O B. VI.

46. *Numerum* 10 *ita in duas partes dividere, ut productum ex partibus in ſe ductis conficiat* 30.

Sit x una pars, alteram deſignabit $10 - x$, eritque $x \times \overline{10 - x} = 30$, et per reductionem, $x = 5 \pm \sqrt{-5}$. Uterque igitur prodit valor x impoſſibilis: et jure quidem, nam conditio ut factum ſub partibus conficiat 30, eſt impoſſibilis, cum factum maximum (ubi ſcilicet partes inter ſe æquantur) ſit 25.

Deſignante enim in genere d dimidium quantitatis datæ, ſit una pars $d + y$, alteram deſignabit $d - y$. Sed harum productum $\overline{d+y} \times \overline{d-y} = d^2 - y^2$ ſit maximum, ut conſtat, ubi evaneſcente y^2 deſignatur tantum per d^2. Quod ex *Euclid. lib.* 2, *prop.* 5, quoque conſtat.

47. Hoc problema (et ſimilia) verſione in ſermonem Algebraicum paulum immutatâ, ad puram quadraticam facile redigere licet. Sit enim pars major $5 + y$, minorem deſignabit $5 - y$. Adeoque erit $\overline{5+y} \times \overline{5-y} = 30$. Sive, $25 - y^2 = 30$. Adeoque $y^2 = -5$. Seu $y = \pm\sqrt{-5}$ uterque valor impoſſibilis ut antea.

De reductione æquationum quæ ordini Quadraticarum adſcribendæ sunt.

48. Æquationes parium dimenſionum formæ $x^{2n} = . px^n . q$. (ubi duos tantum terminos ingreditur incognita quantitas, et index termini altiſſimi ſit duplus ſecundi) juxta legem quadraticarum, ad puras dimenſionis dimidiatæ ſunt deprimendæ; quas poſtea, ſi index n ſit integer per evolutionem, ſi fractus per involutionem reſolvere licet. Nam $x^{2n} = . px^n . q$ extracta radice quadratica, prodit $x^n = . \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 . q}$ adeoque, $x = \sqrt[n]{. \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 . q}}$: Problemata ſequentia hujuſmodi æquationes pariunt.

P R O B. I.

49. *Invenire duos numeros quorum productum ſit* 100, *et differentia radicum quadraticarum* 3.

Sit minor x, majorem deſignabit $\frac{100}{x}$. Adeoque erit $\frac{10}{\sqrt{x}} - \sqrt{x} = 3$. ſive concinnando, $x = -3\sqrt{x} + 10$. et per reductionem, $\sqrt{x} = \frac{-3 \pm 7}{2} = 2$ vel -5. Et per involutionem, $x = 4$ vel 25. Erit itaque x in hoc caſu 4, et alter numerus 25: aut vice verſa, ſi ſtatuatur x major numerus.

P R O B. II.

50. *Invenire duos numeros quorum productum ſit* 1, *et ſumma quadratorum* 6.

Deſignante x unum numerum, alterum deſignabit $\frac{1}{x}$. Adeoque erit $x^2 + \frac{1}{x^2} = 6$. Seu concinnando, $x^4 = 6x^2 - 1$. Et per reductionem, $x^2 = 3 \pm \sqrt{8}$. Et radice extractâ, $x = \pm\sqrt{3 \pm \sqrt{8}}$. Quadruplex nimirum prodit valor x, ſcil. $+\sqrt{3+\sqrt{8}}$, $+\sqrt{3-\sqrt{8}}$, $-\sqrt{3+\sqrt{8}}$, vel $-\sqrt{3-\sqrt{8}}$ ex quibus duo poſteriores, cum ſint iidem ac priores ſed contrariis ſignis affecti, utplurimum negliguntur. Quod in ſequentibus quoque obſervandum. Ex valoribus autem receptis uterque ſolvet problema. Nam $\sqrt{3+\sqrt{8}} = \sqrt{5{,}828} = 2{,}414$. Adeoque, erit $x^2 = 5{,}828$. $\frac{1}{x^2} = 0{,}171$. Et horum ſumma prodit 5,999, ſeu 6 quamproxime. Sic quoque alter valor x, $\sqrt{3-\sqrt{8}}$ problema ſolvet quamproxime.

P R O B. III.

51. *Invenire duos numeros, quorum quadrata ſimul conficiant* $8\sqrt{2}$, *et quadratorum productum ſit* 8.

Sit x^2 quadratum unius numeri, erit $8\sqrt{2} - x^2$ quadratum alterius, adeoque $x^2 \times \overline{8\sqrt{2} - x^2} = 8$. Seu concinnando, $x^4 = 8\sqrt{2}\,x^2 - 8$. Et per reductionem, $x^2 = \sqrt{32} \mp \sqrt{24}$. Et radice iterum extracta, $x = \sqrt{\sqrt{32} \pm \sqrt{24}}$.

Haud ſecus in ſpeciebus, $x^4 = (2gb + 4f^2)\,x^2 - g^2b^2$ per reductionem, prodit $x^2 = gb + 2f^2 \pm \sqrt{4f^2gb + 4f^4}$. Et extracta radice, $x = \sqrt{gb + 2f^2 \pm 2f\sqrt{gb+f^2}}$.

Aliquando radices quadraticæ hujuſmodi quantitatum ex radicalibus aut ex integris et radicalibus ad invicem incommenſurabilibus compoſitarum, accurate deſignari poſſunt per ſimpliciores quantitates radicales; ope regulæ inſequentis, quam ex *Bombellio* expreſſit *Newtonus*.

52. *Deſignet* A *quantitatis alicujus partem majorem*, B *partem minorem: et erit* $\frac{A+\sqrt{A^2-B^2}}{2}$ *quadratum majoris partis radicis;* & $\frac{A-\sqrt{A^2-B^2}}{2}$ *quadratum partis minoris, quæ quidem majori adnectenda eſt cum ſigno ipſius* B. *Adeoque* $\sqrt{\frac{A+\sqrt{A^2-B^2}}{2}} \pm \sqrt{\frac{A-\sqrt{A^2-B^2}}{2}}$ *radix quæſita.*

Sit enim $y \pm 2\sqrt{yz} + z$ quantitas aliqua cujus radix $\sqrt{y} \pm \sqrt{z}$ extrahenda eſt. Deſignet A partem ejus *majorem* $y + z$, et B partem *minorem* $2\sqrt{yz}$[d], quo $\sqrt{A^2 - B^2}$ ſit poſſibilis. Quadrando, erit $y^2 + 2yz + z^2 = A^2$, et $4yz = B^2$. Et ſubducendo poſteriorem, $y^2 - 2yz + z^2 = A^2 - B^2$. Et extracta radice, $y - z = \sqrt{A^2 - B^2}$. Adeoque, $\overline{y+z} + \overline{y-z}$ ſeu $2y = A + \sqrt{A^2 - B^2}$ et $y = \frac{A + \sqrt{A^2 - B^2}}{2}$. Itemque $\overline{y+z} - \overline{y-z}$ ſeu $2z = A - \sqrt{A^2 - B^2}$ et $z = \frac{A - \sqrt{A^2 - B^2}}{2}$. Adeoque $\sqrt{y} \pm \sqrt{z} = \sqrt{\frac{A + \sqrt{A^2 - B^2}}{2}} \pm \sqrt{\frac{A - \sqrt{A^2 - B^2}}{2}}$.

Ut ſi quantitatis $3 \pm \sqrt{8}$ radix extrahenda eſt, ſcribe 3 pro A et $\sqrt{8}$ pro B, eritque $\sqrt{A^2 - B^2} = 1$. Et quadrata partium radicis ſimul, ſcil. $\frac{A + \sqrt{A^2 - B^2}}{2} \pm \frac{A - \sqrt{A^2 - B^2}}{2} = \frac{3+1}{2} \pm \frac{3-1}{2} = 2 \pm 1$. Ergo radix eſt $1 \pm \sqrt{2}$. Hæc enim quadrando dabit $1 + 2 \pm 2\sqrt{2} = 3 \pm \sqrt{8}$. Eodem modo ſi de $2f^2 + gh \pm 2f\sqrt{f^2 + gh}$ radix extrahi debet, ſcribendo $2f^2 + gh$ pro A, et $2f\sqrt{f^2 + gh}$ pro B, erit $\sqrt{A^2 - B^2} = gh$; adeoque $\frac{A + \sqrt{A^2 - B^2}}{2} = f^2 + gh$ et $\frac{A - \sqrt{A^2 - B^2}}{2} = f^2$. Ergo radix ipſa $f \pm \sqrt{f^2 + gh}$. Haud ſecus, ſi de quantitate impoſſibili $1 \pm 4\sqrt{-3}$ radix ſit extrahenda, poſito $A = 1$, $B = 4\sqrt{-3}$, et $B^2 = -48$, erit $\sqrt{A^2 - B^2} = \sqrt{1 + 48} = 7$.

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[d] Sit quantitatis alicujus dimidium d, et pars major $d + x$, erit pars minor $d - x$. Sed rectangulum ſub partibus $\overline{d+x} \times \overline{d-x} = d^2 - x^2$. et bis rectangulum $2d^2 - 2x^2$, fit *minus* quam $\overline{d+x}^2 + \overline{d-x}^2 = 2d^2 + 2x^2$ quadrata partium ſimul. Quod ex *Euclid. Lib.* 2, *prop.* 4 *et* 5 quoque colligere licet.

Et quadrata partium radicis, $= \frac{1+7}{2} \pm \frac{1-7}{2} = 4 + -3$. Ergo, radix ipſa $2 \pm \sqrt{-3}$. Rurſus, ſi habeatur $2\sqrt{-1}$, poſito $A = 0$. $B = 2\sqrt{-1}$, $B^2 = -4$, erit $\sqrt{A^2-B^2} = \sqrt{0+4} = 0+2$. Adeoque radix $= \frac{\sqrt{0+0+2}}{2} + \frac{\sqrt{0-0-2}}{2} = 1 + \sqrt{-1}$. Porro, ſi ex $\sqrt{32} \pm \sqrt{24}$, ex radicalibus ſolis conflatâ, radix extrahenda ſit; ponendo $\sqrt{32}$ pro A, et $\sqrt{24}$ pro B, erit $\sqrt{A^2-B^2} = \sqrt{8}$, et inde quadratum majoris partis radicis $\frac{\sqrt{32}+\sqrt{8}}{2} = \frac{2\sqrt{8}+\sqrt{8}}{2} = \frac{3\sqrt{8}}{2} = 3\sqrt{2} = \sqrt{18}$. Et quadratum minoris, $\frac{\sqrt{32}-\sqrt{8}}{2} = \sqrt{2}$. Radix itaque eſt $\sqrt[4]{18} \pm \sqrt[4]{2}$. Denique, ſi habeatur $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$, ponendo $6+\sqrt{8} = A$ et $-\sqrt{12}-\sqrt{24} = B$, fiet $A^2 = 44 + 12\sqrt{8}$, $B^2 = 36 + 2\sqrt{288} = 36 + 12\sqrt{8}$, et $\sqrt{A^2-B^2} = \sqrt{8}$, unde quadratum majoris partis radicis erit $\frac{6+2\sqrt{8}}{2} = 3 + \sqrt{8}$, atque illud minoris $= 3$, adeoque radicis pars major $\sqrt{3+\sqrt{8}}$, hoc eſt, ut ſupra $1 + \sqrt{2}$; et pars minor $\sqrt{3}$. Et proinde radix ipſa $1 + \sqrt{2} - \sqrt{3}$.

53. Cæterum ubi plures ſunt hujuſmodi termini radicales ad invicem incommenſurabiles, excogitavit ſagaciſſimus *Newtonus* methodum citius inveniendi partes radicis; *dividendo ſcilicet factum quarumvis duarum radicalium per tertiam aliquam radicalem quæ producit quotum rationalem et integrum, donec tot eruantur hujuſmodi quoti diverſi quot ſunt partes radicis quæſitæ. Nam dimidii quoti cujuſque radix erit pars radicis.*

Nam ſi tres habeantur hujuſmodi radicales, ſit radix quæſita, $\sqrt{x} \pm \sqrt{y} \pm \sqrt{z}$, quadrando erit $x + y + z \pm 2$

$\sqrt{xy} \pm 2\sqrt{xz} \pm 2\sqrt{yz} = A \pm B$. At trium rectangulorum unoquoque duas radicis partes continente, ſi dividatur factum duorum quorumvis, tres utique radicis partes comprehendentium, per tertium, duas itidem radicis partes continens, quotiens unam radicis partem involvet: ſcil. $\frac{2\sqrt{xy} \times 2\sqrt{xz}}{2\sqrt{yz}} = 2x$, $\frac{2\sqrt{xy} \times 2\sqrt{yz}}{2\sqrt{xz}} = 2y$, et $\frac{2\sqrt{xz} \times 2\sqrt{yz}}{2\sqrt{xy}} = 2z$. Ergo radix dimidii cujuſque quoti erit pars radicis quæſitæ. Sic, in exemplo noviſſimo $\frac{\sqrt{8} \times \sqrt{12}}{\sqrt{24}} = 2$, $\frac{\sqrt{8} \times \sqrt{24}}{\sqrt{12}} = 4$, $\frac{\sqrt{12} \times \sqrt{24}}{\sqrt{8}} = 6$. Ergo partes radicis ſunt 1, $\sqrt{2}$, $\sqrt{3}$. ut ſupra.

Ad ſigna partium quod attinet, quoniam factum partium duarum ſignis ſimilibus utentium erit affirmativum, ſecus vero negativum: erunt partes radicis 1 et $\sqrt{2}$ quas continet prima radicalis $+ \sqrt{8}$ $(= 2\sqrt{xy})$ ambæ affirmativæ, vel ambæ negativæ; ſecundæ autem radicalis $- \sqrt{12}$ $(= -2\sqrt{xz})$ partes, ſcil. 1 et $\sqrt{3}$ contrariis affici debent ſignis. Radix igitur erit vel $1 + \sqrt{2} - \sqrt{3}$, vel $-1 - \sqrt{2} + \sqrt{3}$.

54. Si quatuor ſint hujuſmodi radicales, ſit radix $\sqrt{v} \pm \sqrt{x} + \sqrt{y} \pm \sqrt{z}$. quadrando erit $v + x + y + z \pm 2\sqrt{vx} \pm 2\sqrt{vy} \pm 2\sqrt{vz}$; $\pm 2\sqrt{xy} \pm 2\sqrt{xz}$; $\pm 2\sqrt{yz} = A \pm B$. ex 15 autem diverſis horum rectangulorum combinationibus quas inſtituere licet, tripliciter erui poteſt quæque radicis pars, unde duodecim prodibunt quoti rationales et integri: ſcil. $\frac{2\sqrt{vx} \times 2\sqrt{vy}}{2\sqrt{xy}}$, vel $\frac{2\sqrt{vz} \times 2\sqrt{vx}}{2\sqrt{xz}}$, vel $\frac{2\sqrt{vy} \times 2\sqrt{vz}}{2\sqrt{yz}} = 2v$; $\frac{2\sqrt{vy} \times 2\sqrt{xy}}{2\sqrt{vx}}$ vel &c. $= 2x$; et ſic deinceps. Tres autem qui ſuperſunt quoti evadent irrationales, quum facta jam reſtantia, ſcil.; $2\sqrt{vx} \times 2\sqrt{yz}$, $2\sqrt{vy} \times 2\sqrt{xz}$, et $2\sqrt{vz} \times 2\sqrt{xy}$,

utpote quæ omnes radicis partes continent, diviſorem inter reliqua rectangula nullum ſortientur qui quotum rationalem & integrum pariet.

55. Ex dictis colligere licet, quod radices hujuſmodi quantitatum vel ex integris et radicalibus, vel ex radicalibus ſolis compoſitarum ope regulæ præfatæ quidem extrahere fas erit, ubi $\sqrt{A^2-B^2}$ exhibeat quantitatem rationalem et integram, vel ſi non, ubi radicales quas exhibent A et $\sqrt{A^2-B^2}$ extrahendo quicquid eſt rationale, ex parte radicali conveniant.

Huic parti, *Appendicem* ſubjungam de celeberrimo *Theoremate Binomio*, cujus uſus tam in involutione et evolutione, quam in omni fere Analyſeos parte egregius.

De Theoremate Binomio.

56. Præclarum hoc theorema, quod tentando & inductione extrudit *Newtonus*, non niſi anno 1676 epiſtolis ad *Oldenburg* miſſis, cum *Leibnitio* communicandis, patefecit; licet hoc unà cum præcipuis ſuis inventis analyticis, ſaltem ante peſtem *Londinenſem* anno 1665 ingruentem, ei innotuiſſe ex iſtis conſtat. Suſpicabatur nimirum vir egregius, haud minus modeſtiâ quam ingenio inſignis, inventa hæcce ſua, aut aliis perſpecta jam eſſe, aut ſaltem fore, prius quam ipſe ætatem ſcribendo parem attigerit [e]. Natus eſt *Newtonus* anno 1642.

57. Theorema Binomium ita profertur,

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} \times \left(1 + \frac{m}{n}Q + \frac{m}{n}\cdot\frac{m-n}{2n}Q^2 + \frac{m}{n}\cdot\frac{m-n}{2n}\cdot\frac{m-2n}{3n}Q^3 \text{ \&c.}\right)$$

Sive, ſecundum formulam *Newtonianam*,

$$= P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ \text{ \&c.}$$

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[e] —— "About that time the plague breaking out, obliged me to get "hence [from *Cambridge*] and think of ſomething elſe. Notwithſtanding, I "added to my other diſcoveries at that time, a certain conſtruction of Logarithms deduced from the Area of the Hyperbola. —— I am aſhamed to "tell to how many places of figures I carried theſe computations, having no "other buſineſs at that time; for I pleaſed myſelf too much with theſe diſcoveries. But as ſoon as that ingenious performance of *Nicolaus Mercator* appeared [in the year 1668] I mean his *Logarithmotechnia*, (who, I doubt not, "firſt diſcovered the things he publiſhed,) I began to cool a little upon thoſe "things, ſuſpecting that either he underſtood the extraction of roots as well as "diviſion of fractions; or at leaſt, that others, upon the diſcovery of diviſion, "would find out the reſt, before I could be of a ſufficient age for writing." See his Letters to *Oldenburg*, *Commerc. Epiſtol.* N° 48 and 55; or *Stewart's Quadrature of Curves*, &c. p. 357.

Ubi $P + PQ$ deſignat quantitatem quamlibet compoſitam, ſive binomiam ſive polynomiam, cujus radix quævis, dimenſio quævis, vel radix dimenſionis inveſtiganda eſt; P, primum ejus terminum; Q, quotientem reliquorum terminorum per primum diviſorum, et proinde PQ, ſummam reliquorum ſive affirmativè ſive negativè ponendam; $\frac{m}{n}$, numeralem indicem radicis vel dimenſionis $P + PQ$, ſive integer ſit, ſive fractus, ſive affirmativus ſive negativus: et in formula poſteriore, A, B, C, &c. terminos inter operandum inventos; nempe A, primum terminum $P^{\frac{m}{n}}$; B, ſecundum $\frac{m}{n} AQ$; C, tertium $\frac{m-n}{2n} BQ$, &c. Porro, eodem redeunt hæ formulæ, uti conferendo inter ſe terminos utriuſque homologos manifeſtum eſt. Nam $P^{\frac{m}{n}} = P^{\frac{m}{n}} \times 1$; $\frac{m}{n} AQ = P^{\frac{m}{n}} \times \frac{m}{n} Q$; et ſic deinceps. Utramque vero poſuimus, ex eo quod prior legem progreſſionis planius exhibendo, terminis ſeriei continuandis, poſterior hiſce addendis aptior, ubi pro A, B, C, &c. ſubſtituuntur valores numerales quos ſortiuntur reſpectivè.

58. Theorematis Binomii complures dantur aut dari ponuntur demonſtrationes. Miſſis autem demonſtrationibus ex fluxionali computo hauſtis, ab opere elementario nimirum ablegandis; vel ex permutationum ſeu combinationum doctrinâ, non niſi prolixe elicitis: inveſtigationem ejus breviter inſtituere libet, ex principiis algebræ propriis, ad mentem *Johannis Landen*, *Treatiſe on the Reſidual Analyſis*. Cæterum neceſſe eſt ut præmittatur Theorema, Lemmatis inſtar, a *Landeno* abſque demonſtratione poſitum.

LEMMA.

Deſignantibus *m* et *n*, numeros quoſlibet integros, erit,

$$\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q} = p^{\frac{m}{n}-1} \times \frac{1 + \frac{q}{p} + \left.\overline{\frac{q}{p}}\right|^{2} + \left.\overline{\frac{q}{p}}\right|^{3} \&c. \ldots m}{1 + \left.\overline{\frac{q}{p}}\right|^{\frac{m}{n}} + \left.\overline{\frac{q}{p}}\right|^{\frac{2m}{n}} + \left.\overline{\frac{q}{p}}\right|^{\frac{3m}{n}} \&c. \ldots\ldots n}$$

Nam reſolvendo in factores ſuos, fiet,

$$\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q} = \frac{p^{\frac{m}{n}}}{p} \times \frac{1 - \left.\overline{\frac{q}{p}}\right|^{\frac{m}{n}}}{1 - \frac{q}{p}}$$

$$= p^{\frac{m}{n}-1} \times \frac{1 - \left.\overline{\frac{q}{p}}\right|^{\frac{m}{n}}}{1 - \frac{q}{p}}$$ ſive, multiplicando numeratorem et denominatorem eadem quantitate $1 - \left.\overline{\frac{q}{p}}\right|^{m}$, cujus index *m* indicis utriuſque $\frac{m}{n}$ et 1, debet eſſe multiplus,

$$= p^{\frac{m}{n}-1} \times \frac{1 - \left.\overline{\frac{q}{p}}\right|^{\frac{m}{n}}}{1 - \frac{q}{p}} \times \frac{1 - \left.\overline{\frac{q}{p}}\right|^{m}}{1 - \left.\overline{\frac{q}{p}}\right|^{m}}$$ ſive,

$$= p^{\frac{m}{n}-1} \times \frac{1 - \left.\overline{\frac{q}{p}}\right|^{m}}{1 - \frac{q}{p}} \times \frac{1 - \left.\overline{\frac{q}{p}}\right|^{\frac{m}{n}}}{1 - \left.\overline{\frac{q}{p}}\right|^{m}}$$

At ſi $\overline{1-\frac{q}{p}}\Big|^{m}$ dividatur per $1-\frac{q}{p}$ et $\overline{1-\frac{q}{p}}\Big|^{\frac{m}{n}}$ reſpective;

Fiet,

$$\frac{\overline{1-\frac{q}{p}}\Big|^{m}}{1-\frac{q}{p}} = \frac{1+\frac{q}{p}+\overline{\frac{q}{p}}\Big|^{2}+\overline{\frac{q}{p}}\Big|^{3}\ \&c.\,.\,\overline{\frac{q}{p}}\Big|^{m-1}}{1}\text{, uſque ad } m \text{ terminos.}$$

Itemque,

$$\frac{\overline{1-\frac{q}{p}}\Big|^{\frac{m}{n}}}{\overline{1-\frac{q}{p}}\Big|^{m}} = \frac{1}{1+\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{2m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{3m}{n}}\&c.\,.\,\overline{\frac{q}{p}}\Big|^{\overline{n-1}\times\frac{m}{n}}}\text{. uſque ad } n \text{ terminos.}$$

Adeoque,

$$\frac{\overline{1-\frac{q}{p}}\Big|^{m}}{1-\frac{q}{p}}\times\frac{\overline{1-\frac{q}{p}}\Big|^{\frac{m}{n}}}{\overline{1-\frac{q}{p}}\Big|^{m}} = \frac{1+\frac{q}{p}\ \ +\overline{\frac{q}{p}}\Big|^{2}\ \ +\overline{\frac{q}{p}}\Big|^{3}\ \ldots m}{1+\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{2m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{3m}{n}}\ \ldots\ldots n.}$$

Ergo,

$$\frac{p^{\frac{m}{n}}-q^{\frac{m}{n}}}{p-q} = p^{\frac{m}{n}-1}\times\frac{1+\frac{q}{p}\ \&c.\ \ldots\ldots m}{1+\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}\ \&c.\ \ldots\ldots n.}$$

Hoc Theoremate demonſtrato, ipſius Theorematis Binomii inveſtigationem expedire jam licet.

THEOREMATIS BINOMII INVESTIGATIO.

59. Quoniam $(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}\times(1+Q)^{\frac{m}{n}}$. inveſtiganda eſt ſeries quantitati $(1+Q)^{\frac{m}{n}}$ æqualis. Aſſumatur igitur huic æquipollere ſeriem $1+aQ+bQ^2+cQ^3+dQ^4$, &c. Nec obſtat quo minus per hujuſmodi ſeriem, terminis nimirum integris indicibus perpetuo creſcentibus, rite deſignetur $(1+Q)^{\frac{m}{n}}$, cum termini ſucceſſivi intra juſtum valorem coerceri poſſint, ſi modo coefficientes a, b, c, ſortiantur fractos. Compendii gratiâ, itaque, ſubſtituto x pro Q,

Ponatur $(1+x)^{\frac{m}{n}}=1+ax+bx^2+cx^3+dx^4$, &c.

Et ſimiliter, aſſumptâ, ad libitum, aliâ quantitate y,

Erit $(1+y)^{\frac{m}{n}} \quad =1+ay+by^2+cy^3+dy^4$, &c.

Et ſubducendo poſteriorem a priore,

$$(1+x)^{\frac{m}{n}}-(1+y)^{\frac{m}{n}}=(ax-ay)+(bx^2-by^2)+(cx^3-cy^3) \text{ \&c.}$$

Et dividendo per $(1+x)-(1+y)=x-y$,

$$\frac{(1+x)^{\frac{m}{n}}-(1+y)^{\frac{m}{n}}}{(1+x)-(1+y)}=\frac{a(x-y)+b(x^2-y^2)+c(x^3-y^3), \text{ \&c.}}{x-y}.$$

Sive, per *Lemma* [f], ſubſtituendo $1+x$ pro p; et $1+y$ pro q;

---

[f] *Lemma* præcedens abſque demonſtratione protulit *Landen*, in Tractatu cui titulus, *A Diſcourſe concerning the Reſidual Analyſis*, *Lond.* 1758. ſpeciminis inſtar, antegreſſo opus ipſum *The Reſidual Analyſis*, anno 1764 impreſſum; ubi inveſtigationem ejus aliter inſtituit auctor, pag. 6. Verùm hujus mihi antehac copia non fuit.

$$(1+x)^{\frac{m}{n}} - 1 \times \frac{1+\frac{1+y}{1+x}+\left(\frac{1+y}{1+x}\right)^2+\left(\frac{1+y}{1+x}\right)^3 \ldots m}{1+\left(\frac{1+y}{1+x}\right)^{\frac{m}{n}}+\left(\frac{1+y}{1+x}\right)^{\frac{2m}{n}}+\left(\frac{1+y}{1+x}\right)^{\frac{3m}{n}} \ldots\ldots n}$$

$$= a + b(x+y) + c(x^2+xy+y^2), \text{ \&c.}$$

Sed ſalva æquatione, variari poteſt y ad libitum; fiat igitur $y = x$, et ſingulis terminis tam numeratoris $1 + \frac{1+y}{1+x}$ &c. quam denominatoris $1 + \left(\frac{1+y}{1+x}\right)^{\frac{m}{n}}$ &c. per unitatem jam deſignandis, exponet m ſummam unitatum in numeratore, et n ſummam unitatum in denominatore, et proinde $\frac{m}{n}$ totius ſeriei valorem. Hinc erit,

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^4 \text{ \&c.}$$

Et multiplicando per $1 + x$, erit,

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}} \text{ ſive,}$$

$$\frac{m}{n} \times (1 + ax + bx^2 \text{ \&c. ſive;}$$

$$\frac{m}{n} + \frac{m}{n}ax + \frac{m}{n}bx^2 \text{ \&c.} = a + \frac{2b}{a}x + \frac{3c}{2b}x^2 + \frac{4d}{3c}x^3 \text{ \&c.}$$

Collatis igitur terminis hinc inde homologis, ſcil. $\frac{m}{n} = a$; $\frac{m}{n}ax = (2b+a)x$; &c. eruentur coefficientes,

$$a = \frac{m}{n}.$$

$$b = a \times \frac{m-n}{2n} = \frac{m}{n} \cdot \frac{m-n}{2n}.$$

$$c = b \times \frac{m-2n}{3n} = \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n}.$$

&c. *&c.*

Reſtituto jam Q pro x; et ſubſtitutis his valoribus a, b, c, &c. in ſeriem $1 + aQ + bQ^2$ &c. emerget,

$$(1+Q)^{\frac{m}{n}} = 1 + \frac{m}{n}Q + \frac{m}{n}\cdot\frac{m-n}{2n}Q^2 \text{ \&c.}$$

Et proinde,

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} \times (1 + Q)^{\frac{m}{n}} = P^{\frac{m}{n}} \times (1 + \frac{m}{n}Q + \frac{m}{n}\cdot\frac{m-n}{2n}Q^2 \text{ \&c. ſive, } = P^{\frac{m}{n}} \times (1 + \frac{m}{n}Q + \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}}{2}Q^2 \text{ \&c.}$$

Hinc quoque, erit,

$$(P-PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} \times (1 - \frac{m}{n}Q + \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}}{2}Q^2 - \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}\cdot\overline{\frac{m}{n}-2}}{2\cdot 3}Q^3 + \text{\&c.}$$

mutatis ſcilicet ſignis terminorum alternorum a ſecundo incipientium, quos ingrediuntur impares dimenſiones quantitatis jam negativæ Q.

Cor. I. Creſcente indice Q communi differentiâ 1 in terminis ſucceſſivis, pariter decreſcit index P, ita ut ſumma indicum utriuſque ſingulis in terminis evadat $\frac{m}{n}$; cum Q revera ſit fractio cujus denominator eſt P. Et quo melius forſan conſtet ratio decrementi indicis P, formulâ inſequente theorema utplurimum efferunt recentiores,

$$(p+q)^{\frac{m}{n}} = p^{\frac{m}{n}} + \frac{m}{n}p^{\frac{m}{n}-1}q + \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}}{2}p^{\frac{m}{n}-2}q^2 \text{ \&c.}$$

Ubi deſignat p, primum terminum; q ſummam reliquorum.

Nam $(p+q)^{\frac{m}{n}} = p^{\frac{m}{n}} \times \left(1 + \frac{q}{p}\right)^{\frac{m}{n}}$

$$= p^{\frac{m}{n}} \times \left(1 + \frac{m}{n}\,\frac{q}{p} + \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}}{2}\,\left.\overline{\frac{q}{p}}\right|^{2} \text{ \&c.}\right.$$

$$= p^{\frac{m}{n}} + \frac{m}{n}\,p^{\overline{\frac{m}{n}-1}}\,q + \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}}{2}\,p^{\frac{m}{n}-2}\,q^{2} \text{ \&c.}$$

COR. II. Si index $\frac{m}{n}$ ſit integer et affirmativus, terminabitur ſeries dimenſionem quæſitam deſignans; ſi non, excurret in infinitum.

Nam ſi $\frac{m}{n}$ ſit integer et poſitivus, ſcil. $\frac{m}{1} = m$; (evadente jam $n = 1$.) progreſſio numeratorum m, $m-1$, $m-2$, $m-3$, &c. terminis continuatis per $m - m = 0$, neceſſario tranſibit, qui idcirco iſtum terminum cujus compoſitionem ingreditur, deſtruet; at hoc termino evaneſcente, evaneſcent quoque ſequentes inde conflati, adeoque rumpetur ſeries. Numerus autem terminorum in hoc caſu erit $m + 1$, cum progreſſio m, $m - 1$, &c. non niſi a ſecundo ſeriei termino auſpicatur, exiſtente unitate primi termini coefficiente. Si $\frac{m}{n}$ ſit integer et negativus, ſcil. $\frac{-m}{1} = -m$; erit $-m-1$ magis negativus, et adhuc magis $-m-2$; adeoque longius a nihilo recedentibus terminis ſequentibus, progreſſio $-m$, $-m-1$, $-m-2$, &c. per nihilum tranſire nequit, adeoque in infinitum excurret ſeries. Hoc quoque ita ſe habebit, ſi $\frac{m}{n}$ ſit fractus; nam ſi affirmativè ponatur, *propè* nihilum ſed non *per* nihilum tranſibit progreſſio $\frac{m}{n}$, $\frac{m}{n} - 1$, &c. Ex. gr. ſit $\frac{m}{n} = \frac{1}{2}$; erit $\frac{m}{n} - 1 = -\frac{1}{2}$; $\frac{m}{n} - 2 = -\frac{3}{2}$, &c. ſi negativè, longius ut antea, a nihilo recedent termini ſucceſſivi.

60. Hujus theorematis ope, quantitatis cujuſlibet compoſitæ ſimplices poteſtates eliciuntur. Ut ſi deſideretur quadratum $a+b$ ſive $\overline{a+b}^2$ erit $P=a$; $Q=\frac{b}{a}$; $m=2$; $n=1$. adeoque $A\left(=P^{\frac{m}{n}}\right)=a^2$; $B\left(=\frac{m}{n}AQ\right)=2ab$; $C\left(=\frac{\frac{m}{n}-1}{2}BQ\right)=b^2$; $D\left(=\frac{\frac{m}{n}-2}{3}CQ\right)=0$, cum factor $\frac{m}{n}-2=2-2=0$. Ergo termino tertio terminabitur ſeries, eritque $\overline{a+b}^2=a^2+2ab+b^2$. Eſt quoque $\overline{a+b}^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$. Nam $P=a$; $Q=\frac{b}{a}$; $m=5$; $n=1$. Adeoque, $A\left(=P^{\frac{m}{n}}\right)=a^5$; $B\left(=\frac{m}{n}AQ\right)=5a^4b$; $C=10a^3b^2$; $D=10a^2b^3$; $E=5ab^4$; $F=b^5$ et $G\left(=\frac{\frac{m}{n}-5}{6}FQ\right)=0$.

61. Eodem modo extrahuntur radices aut quantitatis ipſius aut quarumlibet ejus dimenſionum. Ut ſi quæratur radix quadratica $a^2+2ab+b^2$ ſive $\overline{a^2+2ab+b^2}^{\frac{1}{2}}$, erit $P=a^2$; $Q=\frac{2ab+b^2}{a^2}$; $m=1$; $n=2$; adeoque,

$$
\begin{array}{lrrrrr}
A\left(=P^{\frac{m}{n}}\right) & =a & & & & \\
B\left(=\frac{m}{n}AQ\right) & =+b+\frac{1b^2}{2a} & & & & \\
C\left(=\frac{m-n}{2n}BQ\right)= & & -\frac{1b^2}{2a} & -\frac{1b^3}{2a^2} & -\frac{1b^4}{8a^3} & \\
D\left(=\frac{m-2n}{3n}CQ\right)= & & & +\frac{1b^3}{2a^2} & +\frac{3b^4}{4a^3} & +\frac{3b^5}{8a^4}+\&c. \\
E\left(=\frac{m-3n}{4n}DQ\right)= & & & & -\frac{5b^4}{8a^3} & -\frac{5b^5}{4a^4}-\&c. \\
\&c. & & & & & \\
\hline
(a^2+2ab+b^2)^{\frac{1}{2}} & =a+b & * & * & * & \&c.
\end{array}
$$

Erit igitur $a + b$ radix, quæſita, reliquis terminis ſeſe deſtruentibus, ut æquum eſt, ſi ſeries in infinitum continuatur. Atque hac methodo, radices omnium quantitatum perfectas poteſtates referentium extrahendæ ſunt accurate, aliarum vero quamproxime. Sic radix quadrato cubica $c^5 + c^4 x - x^5$, ſive $(c^5 + c^4 x - x^5)^{\frac{1}{5}} = c + \frac{c^4 x - x^5}{5c^4} - \frac{2c^8 x^2 + 4c^4 x^6 - 2x^{10}}{25c^9} +$ &c. ut patebit ſubſtituendo c^5 pro P; $\frac{c^4 x - x^5}{c^5}$ pro Q; 1 pro m; et 5 pro n. fiet enim $A = c$. $B = \frac{c^4 x - x^5}{5c^4}$ &c. Potuit quoque ſeries ordine inverſo inſtitui; et tunc foret $= -x + \frac{c^4 x + c^5}{5x^4} + \frac{2c^8 x^2 + 4c^9 x + 2c^{10}}{25x^9} +$ &c. Prior modus eligendus eſt, ſi c ſit major quam x; poſterior ſi contra, ut convergat ſeries: et quo minor Q quam P, eò citius converget ſeries, ſive paucioribus terminis adhibitis exponetur tota quamproxime. Poſito enim quod Q evaneſcat, primus ſeriei terminus A totam deſignabit; evaneſcentibus ſecundò $\frac{m}{n}$ AQ, et reliquis inde genitis; et proinde quò minor ſit Q, eò minor primo termino evadet ſecundus; et ſecundo tertius, &c. Itemque radix cubica ex bi-quadrato $a + b$ ſive $(a + b)^{\frac{4}{3}}$ evadit $a^{\frac{4}{3}} + \frac{4a^{\frac{1}{3}}b}{3} + \frac{2b^2}{9a^{\frac{2}{3}}} - \frac{4b^3}{81a^{\frac{5}{3}}} +$ &c. Nam $P = a$. $Q = \frac{b}{a}$. $m = 4$. $n = 3$. Adeoque, $A \left(= P^{\frac{m}{n}}\right) = a^{\frac{4}{3}}$ &c.

62. Quinetiam diviſio ſive ſimplex fuerit, ſive per poteſtates aut quantitates radicales, ope hujus theorematis perficitur. Ut ſi $\frac{1}{d+e}$ (hoc eſt $\overline{d+e}^{\,-1}$ ſive $\overline{d+e}^{\,-\frac{1}{1}}$) in ſeriem ſimplicium terminorum reſolvenda ſit; erit $P = d$, $Q = \frac{e}{d}$. $m = -1$. $n = 1$. Adeoque, $A\,(= P^{\frac{m}{n}}) = d^{-1}$ ſeu $\frac{1}{d}$. $B\,(= \frac{m}{n}AQ) = -1 \times \frac{1}{d} \times \frac{e}{d} = -\frac{e}{d^2}$. Et ſic $C = \frac{e^2}{d^3}$. $D = -\frac{e^3}{d^4}$, &c. Hoc eſt, $\frac{1}{d+e} = \frac{1}{d} - \frac{e}{d^2} + \frac{e^2}{d^3} - \frac{e^5}{d^4} +$ &c.

Sic et $\frac{N}{(d+e)^3}$ hoc eſt, N diviſa ter per $d+e$, vel ſemel per cubum ejus, vel $N \times \overline{d+e}^{\,-3} = N \times \frac{1}{d^3} - \frac{3e}{d^4} + \frac{6e^2}{d^5} - \frac{10e^3}{d^6} +$ &c.

Sic $\frac{N}{(y^3-a^2y)^{\frac{1}{3}}}$ hoc eſt, $N \times \overline{y^3-a^2y}^{\,-\frac{1}{3}} = N \times \frac{1}{y} + \frac{a^2}{3y^3} + \frac{2a^4}{9y^5} + \frac{14a^6}{81y^7}$ &c. Nam $P = y^3$. $Q = -\frac{a^2}{y^2}$. $m = -1$. $n = 3$. Adeoque, $A\,(= P^{\frac{m}{n}} = y^{3\times-\frac{1}{3}}) = y^{-1}$ ſeu $\frac{1}{y}$. $B\,(= \frac{m}{n}AQ) = -\frac{1}{3} \times \frac{1}{y} \times -\frac{a^2}{y^2} = \frac{a^2}{3y^3}$ &c.

Itemque, $\frac{\sqrt{c^4 \frac{-cc}{+rr}x^2}}{\sqrt{c^4-ccx^2}} = \overline{c^4 \frac{-cc}{+rr}x^2}\Big|^{\frac{1}{2}} \times \overline{c^4-ccx^2}\Big|^{-\frac{1}{2}}$

ſive compendii gratia, ſubſtituto a pro c^4 et b pro $-cc+rr$ evadit factor prior,

$$(a^2+bx^2)^{\frac{1}{2}} = a + \frac{bx^2}{2a} - \frac{b^2x^4}{8a^3} + \frac{b^3x^6}{16a^5} - \&c.$$

Et poſterior,

$$(a^2-ax^2)^{-\frac{1}{2}} = \frac{1}{a} + \frac{x^2}{2a^2} + \frac{3x^4}{8a^3} + \frac{5x^6}{16a^4} + \&c.$$

Adeoque,

$$(a^2+bx^2)^{\frac{1}{2}} \times (a^2-ax^2)^{-\frac{1}{2}} = \left\{ \begin{array}{rrrrr} 1+\frac{bx^2}{2a^2} & -\frac{b^2x^4}{8a^4} & +\frac{b^3x^6}{16a^6} & -\&c. \\ \frac{x^2}{2a} & +\frac{bx^4}{4a^3} & -\frac{b^2x^6}{16a^5} & +\&c. \\ & +\frac{3x^4}{8a^2} & +\frac{3bx^6}{16a^4} & -\&c. \\ & & +\frac{5x^6}{16a^3} & +\&c. \\ \hline \end{array} \right.$$

$$= 1 + \frac{a+b}{2a^2}x^2 + \frac{3a^2+2ab-b^2}{8a^4}x^4 + \&c.$$

Adeoque reſtituendo c^2 pro a et r^2-c^2 pro b, evadit,

$$\left(c^4 \begin{smallmatrix}-cc \\ +rr\end{smallmatrix} x^2\right)^{\frac{1}{2}} \times \left(c^4-ccx^2\right)^{-\frac{1}{2}} = 1 + \frac{r^2}{2c^4}x^2 + \frac{4r^2c^2-r^4}{8c^8}x^4$$

$+ \&c.$

63. Per eandem regulam, in numeris perinde ac in ſpeciebus geneſis poteſtatum, diviſio per poteſtates aut per quantitates radicales, et præſertim extractio radicum altiorum commode inſtituuntur. Nam quiſque numerus ex partibus componi concipiatur, adeoque in binomium æquipollentem reſolvi poteſt.

64. Ut ſi deſideretur quinta poteſtas numeri 6, reſolvatur 6 in partes 1 et 5, eritque $\overline{1+5}^5 = 7776$. Sit enim $P = 1$, $Q = 5$, $\frac{m}{n} \left(= \frac{5}{1}\right) = 5$ erit,

$$
\begin{aligned}
A &= P^{\frac{m}{n}} &&= 1 \\
B &= \tfrac{m}{n} AQ &&= 25 \\
C &= \frac{\frac{m}{n} - 1}{2} BQ &&= 250 \\
D &= \frac{\frac{m}{n} - 2}{3} CQ &&= 1250 \\
E &= \frac{\frac{m}{n} - 3}{4} DQ &&= 3125 \\
F &= \frac{\frac{m}{n} - 4}{5} EQ &&= \underline{3125} \\
& && \ 7776
\end{aligned}
$$

Potuit ſeries ordine inverſo inſtitui, ponendo $P = 5$ et $Q = \frac{1}{5}$; et idem, ut æquum eſt, prodiret valor. Potuit quoque 6 in alias partes reſolvi, 4 et 2; 3 et 3; 11 et — 5, &c. cæterum facillima prodit involutio ubi P ſit unitas.

65. At ſi quæratur radix quadratica 6, reſolvendus eſt 6 in partes 4 et 2; prior enim debet eſſe numerus quadratus quo rationalis evadet primus ſeriei terminus, alterâ autem

major quo convergat ſeries. Sit itaque $P = 4$, $Q = \frac{1}{2}$, $m = 1$, $n = 2$, eritque, $A \left(= P^{\frac{m}{n}}\right) = 2$, $B \left(= \frac{m}{n} AQ\right) = \frac{1}{2}$, $C \left(= \frac{m-n}{2n} BQ\right) = -\frac{1}{16}$, &c. Cæterum aggregatum duorum terminorum $2 + \frac{1}{2}$ radicem fere deſignabit; nam quadratum hujus ſeu $\frac{5}{2}$ eſt $\frac{25}{4} = 6\frac{1}{4}$, trium vero $2 + \frac{1}{2} - \frac{1}{16} = \frac{39}{16}$, propius; nam hujus quadratum $= \frac{1521}{256} = 6 - \frac{15}{256}$. Et ſic plurium terminorum additione adhuc propius radicem eruere liceret. Præſtat autem de novo ſeriem citius convergentem inſtituere, reſolvendo 6 in partes $\frac{25}{4}$ et $-\frac{1}{4}$. Deſignante enim P $\frac{25}{4}$, Q $-\frac{1}{25}$, erit $A = \frac{5}{2}$, $B = -\frac{1}{20}$, &c. at $\frac{5}{2} - \frac{1}{20} = \frac{49}{20}$ cujus quadratum $\frac{2401}{400} = 6 + \frac{1}{400}$. Et ſi adhuc propius radicem eruere quæramus, inſtitui poteſt ſeries citiſſime convergens, reſolvendo 6 in partes $\frac{2401}{400}$ et $-\frac{1}{400}$ his enim pro P et PQ ſubſtitutis, evadet $A = \frac{49}{20}$, $B = -\frac{1}{1960}$, at $\frac{49}{20} - \frac{1}{1960} = \frac{4801}{1960}$ cujus quadratum prodit $\frac{23049601}{3841600} = 6 + \frac{1}{3841600}$.

Si deſideretur radix cubica 3, reſolvendus eſt 3 in partes duas quarum prior debet eſſe numerus cubicus, ſcil. 8 et -5, deſignante enim P, 8; Q, $-\frac{5}{8}$; m, 1; n, 3, erit

$A\ (= P^{\frac{m}{n}}) = 2$, $B\ (= \frac{m}{n} AQ) = -\frac{5}{12}$, $C\ (= \frac{m-n}{2n} BQ) = -\frac{25}{288}$, &c. Cæterum $2 - \frac{5}{12} - \frac{25}{288} = \frac{431}{288}$ radicem aliquantulum ſuperabit, nam hujus cubus fit $3 + \frac{8399375}{23887872}$. Satius igitur foret ſeriem ab initio quæ citius convergeret condere, quærendo radicem cubicam numeri $8 \times 3 = 24$, quæ duplo major prodibit radice cubica ternarii. Reſolvatur itaque 24 in partes 27, et -3, et ponendo $P = 27$, $Q = \frac{-1}{9}$, erit $A = 3$, $B = -\frac{1}{9}$, &c. Cæterum erit $3 - \frac{1}{9} = \frac{26}{9}$ radix cubica numeri 24 fere, adeoque hujus ſemiſſis $\frac{13}{9} = \sqrt[3]{3}$ fere, nam hujus cubus fit $\frac{2197}{729} = 3 + \frac{10}{729}$. Quamproxime autem eruetur radix ope ſeriei citiſſime convergentis, iterum reſolvendo 3 in partes $\frac{2197}{729}$ et $-\frac{10}{729}$, erit enim $A = \frac{13}{9}$, $B = \frac{-10}{4563}$, &c. Sed radicis $\frac{13}{9} - \frac{10}{4563} = \frac{6581}{4563}$ cubus prodit $3 + \frac{1976300}{95006081547}$.

Si deſideretur radix quadrato-cubica numeri 33, reſolvendus eſt 33 in partes quarum prior debet eſſe numerus quadrato-cubicus ſcilicet 32 et 1, deſignante enim P, 32; Q, $\frac{1}{32}$; *m*, 1; et *n*, 5. evadet $A = 2$, $B = \frac{1}{80}$. radicis autem $2 + \frac{1}{80} = \frac{161}{80}$ quinta dimenſio eſt $33 + \frac{41216801}{3276800000}$.

ANALYSIS ÆQUATIONUM.

PARS SECUNDA.

DE ÆQUATIONE FINALI EX DUABUS PLURIBUSVE MEDIIS ELICIENDA.

66. Cum in alicujus problematis ſolutionem plures habentur æquationes ſtatum quæſtionis comprehendentes, quarum unicuique plures etiam quantitates incognitæ involvuntur; æquationes iſtæ, (duæ per vices ſi modo ſint plures duabus) ſunt *ita connectendæ*, ut una ex quantitatibus incognitis per ſingulas operationes tollatur, et tandem emergat æquatio in qua unica manebit quantitas incognita, ſi tot ſunt quantitates incognitæ quot habentur æquationes; ut in problematis determinatis ſolet. Sin quantitates incognitæ ſint unâ plures quam æquationes habentur, tum in æquatione ultimo reſultante duæ manebunt quantitates incognitæ; ſi duabus plures, &c. manebunt tres; et ſic deinceps: ut in problematis indeterminatis. Nam ſciendum eſt, quod per quamlibet æquationem una quantitas incognita tollatur.

67. Quemadmodum *Si quærantur duo numeri* x *et* y *quorum ſumma ſit* a *et differentia* d. Erit $x + y = a$ et $x - y = d$. Adeoque addendo æqualia æqualibus $2x = a + d$ et $x = \frac{1}{2}a + \frac{1}{2}d$. Sive, demendo æqualia ab æqualibus $2y = a - d$ et $y = \frac{a-d}{2}$. Sic quoque, habitis æquationibus $ax = \frac{b^2x - b^3}{z}$ et $x = \frac{az}{x-b}$, multiplicando æqualia æqualibus, prodibit $ax^2 = \frac{ab^2xz - ab^3z}{xz - bz} = ab^2$ ſive $x = b$.

Cæterum raro ita contexuntur æquationes ut exterminationem tam facilem et expeditam patiantur quantitates incognitæ quas involvunt, præſertim ſi ſint altiorum dimenſionum. Poſſunt etiam duæ vel plures quantitates incognitæ per duas tantum æquationes *fortaſſe* tolli. Ut ſi ſit $ax - by = ab - az$, et $bx + by = b^2 + az$; tum æqualibus ad æqualia additis prodibit $ax + bx = ab + b^2$, exterminatis utriſque y et z: adeoque $x = b$. Sed ejuſmodi caſus vel arguunt vitium aliquod in ſtatu quæſtionis latere, vel calculum erroneum eſſe aut non ſatis artificioſum. Modus autem quo una quantitas incognita per ſingulas æquationes tollatur ex ſequentibus patebit.

DE UNICA INCOGNITA QUANTITATE E DUABUS ÆQUATIONIBUS TOLLENDA.

68. REG. I. *Cum quantitas tollenda eſt unius tantum dimenſionis in utraque æquatione; valor ejus uterque, per regulas jam ante traditas quærendus eſt, & alter valor ſtatuendus alteri æqualis.*

Hoc idem quoque perficitur (et plerumque facilius) *ſubducendo alterutrum valorem quantitatis incognitæ ab altero, et ponendo reſiduum æquale nihilo.*

69. REG. II. *Cum quantitas tollenda unius exiſtit dimenſionis in una æquatione, et duarum pluriumve in altera; quærendus eſt valor ejus in priore, & pro ſe in alteram ſubſtituendus.*

70. REG. III. *Cum quantitas tollenda plurium in utraque æquatione dimenſionum exiſtit; valor maximæ poteſtatis ejus in utraque æquatione quærendus eſt, deinde, ſi poteſtates iſtæ non ſint eædem (et æquatio poteſtatis majoris deprimi nequeat, adeo ut evadat ejuſdem poteſtatis cum æquatione minoris) æquatio poteſtatis*

minoris multiplicanda eſt per tollendam quantitatem, aut per ejus quadratum aut cubum, &c. ut ea evadat ejuſdem poteſtatis cum æquatione altera. Tum valores illarum poteſtatum ponendæ ſunt æquales, et æquatio nova prodibit ubi maxima poteſtas ſive dimenſio tollendæ quantitatis diminuitur. Et hanc operationem iterando quantitas illa tandem auferetur.

71. Harum regularum generalia quædam exempla trademus, quæ tanquam *Formulas* ad minuendum laborem quantitatis multarum dimenſionum e duabus æquationibus tollendæ, abſque ſolutione commendat *Newtonus.* Analogiæ autem gratiâ a caſu ſimpliciſſimo ordiri libet, quo melius conſtet creſcente quantitatis incognitæ dimenſione, quantum creſcit exterminandi labor. Porro, notandum eſt, literis initialibus a, b, c, d, &c. in his formulis deſignari tum quantitates cognitas, tum quaſcunque poteſtates alterius cujuſdam quantitatis incognitæ.

FORMULA I.

$$ax + b = 0.$$
$$fx + g = 0.$$

Per REG. I. Erit, $\frac{b}{a} (= - x) = \frac{g}{f}$ adeoque, $bf - ag = 0.$

FORMULA II.

$$ax^2 + bx + c = 0.$$
$$fx + g = 0.$$

Per REG. II. Subſtituto $-\frac{g}{f}$ pro x in æquationem priorem,

Erit, $\frac{ag^2}{f^2} - \frac{bg}{f} + c = 0.$

Adeoq. $ag^2 - bfg + cf^2 = 0.$

Sive, $(bf - ag)(g) - cf^2 = 0.$

FORMULA III.

$$ax^2 + bx + c = 0.$$
$$fx^2 + gx + h = 0.$$

Per REG. III. Erit, $\frac{bx + c}{a} (= - x^2) = \frac{gx + h}{f}$

Adeoque, $x = \frac{ah - cf}{bf - ag}.$

Et, $- x^2 = \frac{(- ah + cf)x}{bf - ag}.$

Adeoque, $\frac{bx + c}{a} (= - x^2) = \frac{(- ah + cf)x}{bf - ag}.$

Et, $x = \frac{- bcf + acg}{b^2f - abg + a^2h - acf}$

Adeoque, $\frac{ah - cf}{bf - ag} (= x) = \frac{- bcf + acg}{b^2f - abg + a^2h - acf}.$

Unde concinnando, et ordinando erit

$$a^2h^2 - abgh - 2acfh : + b^2fh - bcfg : + acg^2 + c^2f^2 = 0.$$

Et reſolvendo in factores *Newtonianos*,

$$(ah - bg - 2cf)(ah) : + (bh - cg)(bf) : + (ag^2 + cf^2)(c) = 0.$$

Idem quoque perfici potuit ſubſtituendo per REG. II. unum ex valoribus x, ſcilicet $\frac{ah - cf}{bf - ag}$ in alterutram æquationum propoſitarum, puta priorem, nam ex $a \times \left(\frac{ah - cf}{bf - ag}\right)^2 + b \times \frac{ah - cf}{bf - ag} + c = 0.$ concinnando et ordinando eadem prodibit concluſio.

72. Simili ratione, at calculo maximè laborioſo, quantitatem tollendam altiorum dimenſionum eliminare poſſimus. Verùm longe concinnius expedietur exterminatio ope regulæ inſequentis.

73. **Reg. IV.** *Multiplicetur viciſſim utraque æquatio eundem referens aut ad eundem reducta gradum, per coefficientem termini altiſſimi in altera, tum ſubducendo alterutram æquationem ab altera, tolletur poteſtas maxima quantitatis incognitæ, et emerget nova et depreſſior æquatio. Multiplicetur iterum viciſſim utraque æquatio per alterius terminum abſolutum ſeu ultimum, tum ſubducendo, &c. tolletur utriuſque terminus ultimus, et dividendo inſuper per incognitam quantitatem alia emerget æquatio ad eandem dimenſionem depreſſa. Deinde, collatis his novis æquationibus, eandem operationem iterando tandem tolletur* x.

Ad normam hujus jam expediamus ſolutionem Formulæ tertiæ, eritque

$$afx^2 + bfx + cf = 0 \qquad \text{et } abx^2 + bbx + cb = 0.$$

$$\underline{afx^2 + agx + ah = 0} \qquad \underline{cfx^2 + cgx + ch = 0.}$$

$$* \ (bf - ag)x + cf - ah = 0 \qquad (cf - ah)x + (cg - bh * = 0.$$

$$(bf - ag)(cf - ah)\,x + (cf - ah)^2 \qquad = 0.$$

$$\underline{(bf - ag)(cf - ah)\,x + (bf - ag)(cg - bh) = 0.}$$

$$* \ (cf - ah)^2 - (bf - ag)(cg - bh) = 0.$$

Sive, $c^2f^2 - 2acfh + a^2h^2 - bcfg + b^2fh + acg^2 - abgh = 0.$

Formula IV.

$$ax^3 + bx^2 + cx + d = 0.$$

$$fx^2 + gx + h = 0.$$

$$afx^3 + bfx^2 + cfx + df \qquad = 0.$$

$$\underline{afx^3 + agx^2 + ahx \qquad = 0.}$$

$* \ (bf - ag)x^2 + (cf - ah)x + df = 0.$ Sive abbreviandi gratiâ,

$$Ax^2 + Bx + df \qquad = 0.$$

$$\begin{array}{ll} Afx^2 + Bfx + df^2 = 0. & Ahx^2 + Bhx + dfh = 0. \\ Afx^2 + Agx + Ah = 0. & df^2x^2 + dfgx + dfh = 0. \\ \hline {}^*(Bf - Ag)x + df^2 - Ah = 0. & (df^2 - Ah)x + (dfg - Bh\, {}^* = 0. \end{array}$$

$$\begin{array}{l} (Bf - Ag)(df^2 - Ah)x + (df^2 - Ah)^2 = 0. \\ (Bf - Ag)(df^2 - Ah)x + (Bf - Ag)(dfg - Bh) = 0. \\ \hline {}^*\ (df^2 - Ah)^2 - (Bf - Ag)(dfg - Bh) = 0. \end{array}$$

ſive,

$$d^2f^4 - 2Adf^2h + A^2h^2 - Bdf^2g + B^2fh + Adfg^2 - ABgh = 0;$$

$$= d^2f^3 - 2Adfh - Bdfg + B^2h + Adg^2 + \frac{A^2h^2 - ABgh}{f} = 0.$$

Reſtitutis jam $bf - ag$ pro A, et $cf - ah$ pro B, prodit,

$$\left.\begin{array}{l} d^2f^3 - 2bdf^2h + 3adfgh - cdf^2g + bdfg^2 - adg^3 + c^2f^2h \\ - 2acfh^2 + a^2h^3 + b^2fh^2 - abgh^2 - bcfgh + acg^2h \ \ldots \end{array}\right\} = 0.$$

Sive ſecundum formulam *Newtonianam*,

$$\left.\begin{array}{l} (ah - bg - 2cf)\, ah^2 : + (bh - cg - 2df)\, bfh : \\ + (ch - dg)(ag^2 + cf^2) : + (3agh + bg^2 + df^2)\, df \end{array}\right\} = 0.$$

FORMULA V.

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$
$$fx^2 + gx + h = 0.$$

$$\begin{array}{l} afx^4 + bfx^3 + cfx^2 + dfx + ef = 0. \\ afx^4 + agx^3 + ahx^2 = 0. \\ \hline {}^*\ (bf - ag)x^3 + (cf - ah)x^2 + dfx + ef = 0. \end{array}$$

Sive, $Ax^3 + Bx^2 + dfx + ef = 0.$

$$\begin{aligned} &Afx^3 + Bfx^2 + df^2x + ef^2 &&= 0. \\ &Afx^3 + Agx^2 + Ahx &&= 0. \\ \hline * \; &(Bf - Ag)\,x^2 + (df^2 - Ah)\,x + ef^2 &&= 0. \\ \text{Sive,} \quad &A'x^2 + B'x + ef^2 &&= 0. \end{aligned}$$

$$\begin{aligned} &A'fx^2 + B'fx + ef^3 = 0. & &A'hx^2 + B'hx + ef^2h = 0. \\ &A'fx^2 + A'gx + A'h = 0. & &ef^3x^2 + ef^2gx + ef^2h = 0. \\ \hline &*(B'f - A'g)x + ef^3 - A'h = 0. & &(ef^3 - A'h)\,x + ef^2g - B'h \; * = 0. \end{aligned}$$

$$\begin{aligned} &(B'f - A'g)\,(ef^3 - A'h)\,x + (ef^3 - A'h)^2 &&= 0. \\ &(B'f - A'g)\,(ef^3 - A'h)\,x + (B'f - A'g)\,(ef^2g - B'h) &&= 0. \\ \hline * \; &(ef^3 - A'h)^2 - (B'f - A'g)\,(ef^2g - B'h) &&= 0. \end{aligned}$$

$$e^2f^6 - 2A'ef^3h + A'^2h^2 - B'ef^3g + A'ef^2g^2 + B'^2fh - A'B'gh = 0.$$

$$\text{ſive}\; e^2f^5 - 2A'ef^2h - B'ef^2g + A'efg^2 + B'^2h + \frac{A'^2h^2 - A'B'gh}{f} = 0.$$

Reſtitutis valoribus A' et B' ſcilicet $Bf - Ag$ et $df^2 - Ah$ erit,

$$\left.\begin{aligned} &e^2f^5 - 2Bef^3h + 3Aef^2gh - def^4g + Bef^2g^2 - Aefg^3 \\ &+ d^2f^4h - 2Adf^2h^2 + A^2h^3 + B^2fh^2 - ABgh^2 \\ &- Bdf^2gh + Adfg^2h \; - \; - \; - \; - \; - \end{aligned}\right\} = 0.$$

Reſtitutis iterum valoribus A et B, ordinando et dividendo per f, erit,

$$\left.\begin{aligned} &a^2h^4 - abgh^3 - 2acfh^3 + b^2fh^3 - bcfgh^2 - 2bdf^2h^2 \\ &+ acg^2h^2 - adg^3h + aeg^4 - 4aefg^2h + c^2f^2h^2 - cdf^2gh \\ &+ cef^2g^2 - 2cef^3h + 3adfgh^2 + bdfg^2h + d^2f^3h \\ &+ 2aef^2h^2 + 3bef^2gh - def^3g + e^2f^4 - befg^3 \end{aligned}\right\} = 0.$$

Unde facile elicitur formula *Newtoniana*,

$$\left.\begin{array}{l}(ab - bg - 2cf)\,ab^3 : + (bb - cg - 2df)\,bfb^2 : + (ag^2 \\ + cf^2)\,(cb^2 - dgb + eg^2 - 2efb) : + (3agb + bg^2 \\ + df^2)\,dfb : + (2ab^2 + 3bgb - dfg + ef^2)ef^2 : - (bg \\ - 2ab)\,efg^2 \quad - \quad - \quad - \quad - \quad - \quad - \quad - \end{array}\right\} = 0.$$

Cæterum obſervandum eſt terminum $- 4aefg^2b$ ſupra memoratum a *Newtono* in binos reſolvi, ſcilicet $ag^2 \times - 2efb$ & $- 2ab \times efg^2$.

FORMULA VI.

$$ax^3 + bx^2 + cx + d = 0.$$
$$fx^3 + gx^2 + bx + k = 0.$$

$$\begin{array}{ll} afx^3 + bfx^2 + cfx + df = 0. & akx^3 + bkx^2 + ckx + dk = 0. \\ \underline{afx^3 + agx^2 + abx + ak = 0.} & \underline{dfx^3 + dgx^2 + dbx + dk = 0.} \\ {}^*(bf - ag)x^2 + (cf - ab)x + df - ak = 0. & (df - ak)x^2 + (dg - bk)x + db - ck^* = 0. \\ \text{Sive,} \quad Ax^2 + Bx + C = 0. & Cx^2 + Dx + F = 0. \end{array}$$

$$\begin{array}{ll} ACx^2 + BCx + C^2 = 0. & AFx^2 + BFx + CF = 0. \\ \underline{ACx^2 + ADx + AF = 0.} & \underline{C^2x^2 + CDx + CF = 0.} \\ {}^*(BC - AD)\,x + C^2 - AF = 0. & (C^2 - AF)\,x + CD - BF^* = 0. \end{array}$$

$$(BC - AD)\,(C^2 - AF)\,x + (C^2 - AF)^2 = 0.$$
$$\underline{(BC - AD)\,(C^2 - AF)\,x + (BC - AD)\,(CD - BF) = 0.}$$
$${}^*(C^2 - AF)^2 - (BC - AD)\,(CD - BF) = 0.$$

$$\text{Sive,}\ C^3 - 2ACF - BCD + B^2F + AD^2 + \frac{A^2F^2 - ABDF}{C} = 0.$$

Sive reſtitutis valoribus A, B, C, D, F, erit,

$$\left.\begin{array}{l} d^3f^3 - 3ad^2f^2k + 3a^2dfk^2 - a^3k^3 - 2bd^2f^2h + 3ad^2fgh \\ + abdfhk - 3a^2dghk + 3bcdf^2k - acdfgk - 3abcfk^2 \\ + 2a^2cgk^2 - cd^2f^2g + a^2bhk^2 + c^2df^2h - 2acdfh^2 + a^2dh^3 \\ - c^3f^2k + 2ac^2fhk - a^2ch^2k + bd^2fg^2 - 2b^2dfgk + b^3fk^2 \\ - ad^2g^3 + 2abdg^2k - ab^2gk^2 + b^2dfh^2 - abdgh^2 - b^2cfhk \\ + abcghk - bcdfgh + acdg^2h + bc^2fgk - ac^2g^2k \end{array}\right\} = 0$$

Unde ſi ordine debito diſponantur termini elicietur formula *Newtoniana*,

$$\left.\begin{array}{l} (ah - bg - 2cf)\,(adh^2 - achk) : + (ak + bh - cg - 2df) \\ bdfh : (- ak + bh + 2cg + 3df)\,a^2k^2 : + (cdh - d^2g \\ - c^2k + 2bdk)\,(ag^2 + cf^2) : + (3agh + bg^2 + df^2 - 3afk) \\ d^2f : (- 3ak - bh + cg + df)\,bcfk : + (bk - 2dg)\,b^2fk : \\ (- b^2k - 3adh - cdf)\,agk \quad - \quad - \quad - \quad . \end{array}\right\} = 0$$

Cæterum notandum eſt terminum $3bcdf^2k$ a *Newtono* in binos reſolvi, ſcilicet $2bdk \times cf^2$ & $df \times bcfk$.

FORMULA VII.

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$
$$fx^4 + gx^3 + hx^2 + kx + l = 0.$$

$$\begin{array}{ll} afx^4 + bfx^3 + cfx^2 + dfx + ef = 0. & alx^4 + blx^3 + clx^2 + dlx + el = 0 \\ afx^4 + agx^3 + ahx^2 + akx + al = 0. & efx^4 + egx^3 + ehx^2 + ekx + el = 0 \\ \hline (bf - ag)x^3 + (cf - ah)x^2 + (df - ak)x + ef - al = 0. & (ef - al)x^3 + (eg - bl)x^2 + (eh - cl)x + ek - dl = 0. \end{array}$$

ſive, $Ax^3 + Bx^2 + Cx + D = 0.$ $Dx^3 + Fx^2 + Gx + H = 0.$

$$ADx^3 + BDx^2 + CDx + D^2 = 0. \qquad AHx^3 + BHx^2 + CHx + DH = 0.$$
$$ADx^3 + AFx^2 + AGx + AH = 0. \qquad D^2x^3 + DFx^2 + DGx + DH = 0.$$
$$* (BD - AF)x^2 + (CD - AG)x + D^2 - AH = 0. \qquad (D^2 - AH)x^2 + (DF - BH)x + DG - CH^* = 0$$

$$\text{ſive,} \quad A'x^2 + B'x + C' = 0. \qquad C'x^2 + D'x + F' = 0.$$

$$A'C'x^2 + B'C'x + C'^2 = 0. \qquad A'F'x^2 + B'F'x + C'F' = 0.$$
$$A'C'x^2 + A'D'x + A'F' = 0. \qquad C'^2x^2 + C'D'x + C'F' = 0.$$
$$* (B'C' - A'D')\,x + C'^2 - A'F' = 0. \qquad (C'^2 - A'F')x + C'D' - B'F'^* = 0.$$

$$(B'C' - A'D')\,(C'^2 - A'F')\,x + (C'^2 - A'F')^2 = 0.$$
$$(B'C' - A'D')\,(C'^2 - A'F')\,x + (B'C' - A'D')\,(C'D' - B'F) = 0.$$
$$* (C'^2 - A'F')^2 - (B'C' - A'D')\,(C'D' - B'F') = 0.$$

Sive,

$$C'^4 - 2A'C'^2F' + A'^2F'^2 - B'C'^2D' + B'^2C'F' + A'C'D'^2 - A'B'D'F' = 0.$$

Sive,

$$C'^2 - 2A'F' - B'D + \frac{B'^2C'F' + A'C'D'^2 + A'^2F'^2 - A'B'D'F'}{C'^2} = 0.$$

75. Hujus reductione, ſubſtitutis nimirum valoribus A', B', C', D' et F' tandem eruere licet æquationem 8 dimenſionum quæ formulam quæſitam præ ſe feret. Simili ratione inſtituenda eſt eliminatio quantitatis x ex æquationibus altiorum dimenſionum. In genere autem notandum eſt, quod dimenſio æquationis ultimo reſultantis tranſcendere non debet ſummam dimenſionum æquationum propoſitarum. Huc etenim redit hæc eliminatio, ut inveniatur æquatio nova cujus radix quælibet ſeu valor quantitatis x in alterutram æquationem datam ſubſtituta eam evaneſcere faciet; at ſi dimenſio æquationis finalis hac methodo comparandæ hanc ſummam excedat, ex radicibus ejus nonnullæ finem quæſitum non aſſequentur, radicibus in-

inutilibus unà cum utilibus, admiſtis. Inutiles igitur radices, deprimendo æquationem ad dimenſionem debitam ope diviſoris alicujus ſemper in æquatione finali latentis, reciduntur. Sic in formula VI aſcendit ad octavum gradum æquatio finalis $C^4 - 2AC^2F$, &c. $= 0$, dividendo autem per quadraticam C ſive $df - ak$, $= 0$, ad ſextum deprimitur. In noviſſima item formula, æquatio $C'^4 - 2A'C'^2F'$, &c. $= 0$. 16 dimenſionum, ad æquationem 8 dimenſionum deprimenda eſt ope diviſoris C'^2 ſive $\overline{(cf - al)}^2 - \overline{(bf - ag)}\,\overline{(ek - dl)}^2 = 0$. Et quo altius aſſurgunt dimenſiones æquationum propoſitarum eo grandior adhibendus eſt diviſor, et proinde eo moleſtior exterminandi labor. Hujus igitur minuendi gratiâ varias excogitarunt exterminandi methodos Viri Clari, *Cramer*, *Introduction a l'Analyſe des lignes Courbes Algebriques*; *Euler*, *Acad. Berlin*, *anno* 1764; *Bezout*, *Memoirs de l'Acad. Paris*, *anno* 1764; et *La Grange*, *Acad. Berlin* 1769.

Formulis his *Newtonianis* jam inveſtigatis, (exemplis quidem in exercitationem tyronum probe accommodatis) quorundam problematum ſolutiones adjungere libet, quo melius conſtet regularum necnon formularum uſus.

PROB. I.

76. *Invenire duos numeros* x *et* y, *quorum ſumma ſit* s *et productum* p.

Habitis duabus æquationibus $x + y = s$, et $xy = p$, ſubſtituatur valor x ex priore reſultans $s - y$ in poſteriorem, et prodibit $\overline{s - y} \times y = p$. Adeoque, $y^2 = sy - p$. Et per reductionem, $y = \frac{1}{2}s \pm \sqrt{\frac{1}{4}s^2 - p}$. Ut ſi s ſit 140; et p, 3136; ſubſtituendo prodibit $y = 70 \pm 42 = 28$ vel 112, quorum alterutrum deſignante y alterum deſignabit x.

Exterminari potuit x ex æquationibus $x+y-s=0$ et $xy-p=0$, ope Formulæ I. ſubſtituendo pro a, b; f, g; 1, $y-s$; y et $-p$ reſpective: prodit enim $(\overline{y-s}\times y)-(1\times -p)=0$. ſeu $y^2=sy-p$ ut ſupra.

77. Hinc liquet, quod propoſitis duabus quantitatibus incognitis methodus deſignandi alteram ex aliqua quæſtionis conditione, in ſolvendis problematis antecedentibus adhibita, eodem redit ac exterminatio ſpeciei hanc quantitatem incognitam referentis, eliciendo valorem ejus ex una æquatione eundemque in alteram ſubſtituendo.

P R O B. II.

78. *Invenire ſummam quadratorum, cuborum, biquadratorum, &c. duorum numerorum quorum ſumma ſit* s *et productum* p.

Deſignantibus x et y numeros quæſitos, erit,

$x+y=s$. Adeoque quadrando, $x^2+2xy+y^2=s^2$

et $xy=p$. Subduc - - $2xy=2p$.

Erit *ſumma quadratorum* $x^2+y^2=s^2-2p$.

Multiplicando hanc æquationem per $x+y=s$,

Prodit - $x^3+(x+y)\,xy+y^3=s^3-2sp$

Subduc - $(x+y)\,xy=sp$

Eritque *ſumma cuborum* $x^3+y^3=s^3-3sp$.

Multiplicando iterum per $x+y=s$,

Prodit $x^4+(x^2+y^2)\,xy+y^4=s^4-3s^2p$

Subduc $(x^2+y^2)\,xy=(s^2-2p)p$

Eritque *ſumma biquad.* $x^4+y^4=s^4-4s^2p+2p^2$.

Hinc innoteſcit lex progreſſionis, ſi quæratur ſumma poteſtatum altiorum; ſemper enim invenietur multiplicando ſummam poteſtatum ultimo inventarum per s, et ſubducendo a producto ſummam penultimarum in p ductarum. Et ſic univerſaliter: erit ſumma poteſtatum $n =$

$$s^{n} - ns^{n-2}p + n.\frac{n-3}{2}.s^{n-4}p^{2} - .\frac{n-4}{2}.\frac{n-5}{3}.s^{n-6}p^{3} + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}.s^{n-8}p^{4}, \&c.$$

P R O B. III.

79. *Invenire ſummam quadratorum, cuborum, biquadratorum, &c. ex ſingulis quantitatibus* a, b, c, d, *&c. ordinis cujuſcunque, quarum ſumma* a + b + c + d, *&c. ſit* P; *ſumma rectangulorum ſub ſingulis binis,* ab + ac + ad, *&c.* bc + bd, *&c.* + cd, *&c. ſit* Q; *ſumma contentorum ſub ſingulis ternis,* abc + abd + abe, *&c.* + acd + ace, *&c.* + bcd, *&c. ſit* R, *&c. &c.*

Deſignantibus

p, ſummam quantitatum b, c, d, &c. poſt primam a.
q, ſummam rectangulorum $bc + bd$, &c. $+ cd$, &c. poſt pa.
r, ſummam contentorum bcd, &c. $+ cde$, &c. poſt qa.
&c. &c.

Erit

$P = a + p$. Adeoque, quadrando, - - $P^{2} = a^{2} + 2pa + p^{2}$
$Q = pa + q$. Subduc - - - $\underline{2Q = 2pa + 2q.}$
$R = qa + r$. Eritque *ſumma quadrat.* - $P^{2} - 2Q = a^{2} + p^{2} - 2q$
$S = ra + s$. Multiplicando hanc æquationem per $P = a + p$.
&c. &c. Erit $P^{3} - 2PQ = a^{3} + p^{3} + pa^{2} + p^{2}a - 2qa - 2pq$
Subduc $PQ = pa^{2} + p^{2}a + qa + pq$

$P^{3} - 3PQ = a^{3} + p^{3} - 3qa - 3pq$
Adde $3R = + 3qa + 3r$

Eritque *Summa cuborum* $P^{3} - 3PQ + 3R = a^{3} + p^{3} - 3pq + 3r$.

Multiplicando hanc per $P = a + p$,

Q

Erit $P^4-3P^2Q+3PR=a^4+p^4+pa^3+p^3a-3pqa+3ra-3p^2q+3pr$
Subduc $P^2Q-2Q^2=pa^3+p^3a-2pqa+qa^2+p^2q-2q^2$

$P^4-4P^2Q+3PR+2Q^2=a^4+p^4-qa^2-pqa+3ra-4p^2q+3pr+2q^2$
Adde $PR=qa^2+pqa+ra+pr$

$P^4-4P^2Q+4PR+2Q^2=a^4+p^4+4ra-4p^2q+4pr+2q^2$
Subduc $4S=4ra+4s$

Eritque *ſumma biquadratorum*
$P^4-4P^2Q+4PR+2Q^2-4S=a^4+p^4-4p^2q+4pr+2q^2-4s$.

Similiter, præcedentes operationes continuando donec omnes exterminentur dimenſiones a præter quæſitam, prodibit *ſumma quintarum poteſtatum*, $P^5-5P^3Q+5P^2R+5PQ^2-5PS-5QR+5T$. Et ſic deinceps. Cæterum combinando valores inter operandum inventos, contractiùs exponere licet ſummas poteſtatum quæſitarum; Ut eleganter *Newtonus*, qui haſce formulis inſequentibus primus abſque demonſtratione propoſuit. Scilicet

P - - - - - - - - $=A$
P^2-2Q - - - - - - $=PA-2Q=B$
$P^3-3PQ+3R$ - - - $=PB-QA+3R=C$.
$P^4-4P^2Q+4PR+2Q^2-4S=PC-QB+RA-4S=D$.
P^5-5P^3Q+ &c. - - $=PD-QC+RB-SA+5T=E$.
P^6-6P^4Q+ &c. - - $=PE-QD+RC-SB+TA-6V=F$.

Et ſic in infinitum, obſervatâ lege progreſſionis. Neceſſe autem eſt ut probe obſerventur ſigna quantitatum a, b, c, &c. ſi modo diverſis afficiantur ſignis. Has omnes affirmativè poſuimus, hoc tantum innuentes, eas, ſecundum ſigna ſingulis propria eſſe addendas ſeu aggregandas, ſive hæc ſint affirmativa ſive negativa.

Concluſiones præcedentes verè exponere ſummam quadratorum, cuborum, &c. ita per inductionem conſtabit.

$$P^2 - 2Q = a^2 + b^2 + c^2 + d^2, \text{ \&c.}$$

Caſ. 1. Datis tribus quantitatibus a, b, c, erit $p = b + c$. Sed per *Prob.* II. $p^2 - 2q = b^2 + c^2$. Ergo, $P^2 - 2Q$ $(= a^2 + p^2 - 2q) = a^2 + b^2 + c^2$.

Caſ. 2. Datis quatuor, erit $p = b + c + d$. Sed per *Caſ.* 1. $p^2 - 2q = b^2 + c^2 + d^2$. Ergo, $P^2 - 2Q$ $(= a^2 + p^2 - 2q) = a^2 + b^2 + c^2 + d^2$.

Caſ. 3. Datis quinque, erit $p = b + c + d + e$. Sed per *Caſ.* 2. $p^2 - 2q = b^2 + c^2 + d^2 + e^2$. Ergo, &c. Et ſic deinceps.

$$P^3 - 3PQ + 3R = a^3 + b^3 + c^3 + d^3, \text{ \&c.}$$

Caſ. 1. Datis tribus quantitatibus, erit $p = b + c$. Sed per *Prob.* II. $p^3 - 3pq = b^3 + c^3$. Ergo, $P^3 - 3PQ$ $(= a^3 + p^3 - 3pq) = a^3 + b^3 + c^3$. Ab hoc caſu nimirum ablegatis $3R$ et $3r$.

Caſ. 2. Datis quatuor, erit $p = b + c + d$. Sed,

$$\begin{array}{rl}
p^3 & = b^3 + c^3 + d^3 + 3b^2c + 3bc^2 + 3b^2d + 3c^2d + 3bd^2 + 3cd^2 + 6bcd. \\
-3pq & = \hphantom{b^3 + c^3 + d^3} - 3b^2c - 3bc^2 - 3b^2d - 3c^2d - 3bd^2 - 3cd^2 - 9bcd. \\
+3r & = \hphantom{b^3 + c^3 + d^3 + 3b^2c + 3bc^2 + 3b^2d + 3c^2d + 3bd^2 + 3cd^2} + 3bcd. \\
\hline
\end{array}$$

Sive, $p^3 - 3pq + 3r = b^3 + c^3 + d^3$.
Ergo, $P^3 - 3PQ + 3R = a^3 + b^3 + c^3 + d^3$.

Caſ. 3. Datis quinque, erit $p = b + c + d + e$. Sed per *Caſ.* 2. $p^3 - 3pq + 3r = b^3 + c^3 + d^3 + e^3$. Ergo, &c. Et ſic deinceps. Simili quoque argumentatione in poteſtatibus altioribus conſtabit.

80. Brevius quidem et elegantius, cæterum argumento paulo ſubtiliore idem colligit *Simpſon*, *Algebra*. *p*. 129. Quoniam enim ſingulæ quantitates a, b, c, d, &c. pariter verſantur in æquationibus originalibus ſeu datis $P = a + b + c$, &c. $Q = abc$, &c. neceſſe eſt, ut pariter quoque verſentur in concluſionibus inde elicitis; adeoque, ſi valores p et q actu ſubſtituantur in æquationem $P^2 - 2Q = a^2 + p^2 - 2q$, liquet nullas reſtare poſſe dimenſiones quantitatum a, b, c, &c. præter quadrata, ex eo quod nulla dimenſio quantitatis a præter quadratum in hâc æquatione occurrit. Et ſimili argumento colligi poteſt nullas dimenſiones quantitatum a, b, c, &c. præter cubos in æquatione $P^3 - 3PQ + 3R = a^3 + p^3 - 3pq + 3r$ verſari. Atque ita deinceps.

P R O B. IV.

81. *Invenire tres numeros continue proportionales, quorum ſumma ſit* $20 = a$, *et quadratorum ſumma* $140 = b$.

Habitis æquationibus $x + y + \frac{y^2}{x} = 20$, et $x^2 + y^2 + \frac{y^4}{x^2} = 140$, ſubſtituatur in poſteriorem pro $\frac{y^4}{x^2}$ valor ejus ex priore comparandus, ſcilicet $(20 - x - y)^2$, et concinnando prodibit $x^2 + y^2 + \overline{y - 20}\,x - 20y = -130$. Subducatur hæc æquatio ab $x^2 + y^2 + \overline{y - 20}\,x = 0$, ad quam utique revocanda eſt prior, et manebit $20y = 130$ ſeu $y = 6\frac{1}{2}$.

Invento jam y, ut inveniatur x, ſubſtituatur $6\frac{1}{2}$ pro y in æquationem $x + y + \frac{y^2}{x} = 20$. Et concinnando prodibit

$x^2 = 13\frac{1}{2}\,x - 42\frac{1}{4}$. Et per reductionem $x = 6\frac{3}{4} \pm \sqrt{45\frac{9}{16} - 42\frac{1}{4}} = 6\frac{3}{4} \pm \sqrt{3\frac{5}{16}}$. Nempe $6\frac{3}{4} + \sqrt{3\frac{5}{16}}$ eſt maximus quæſitorum trium numerorum, et $6\frac{3}{4} - \sqrt{3\frac{5}{1}}$ minimus. Nam x alterutrum extremorum numerorum ambique deſignat, indeque gemini prodeunt valores, quorum alteruter poteſt eſſe x, exiſtente altero $\frac{y^2}{x}$; cum in ſerie proportionalium vel creſcente vel decreſcente conſtituetur x, indeque in priore caſu deſignabit maximum proportionalium, in poſteriore minimum.

82. Exterminari potuit x ex æquationibus $x^2 + \overline{y - 20}\,x + y^2 = 0$, et $x^4 + \overline{y^2 - 140}\,x^2 + y^4 = 0$, ad quas redigendæ ſunt præcedentes, ſubſtituendo in *Formulam* V, pro a, b, c, d, e; f, g, & h: 1, 0, $y^2 - 20$, 0, y^4; 1, $y - 20$, & y^2, reſpective. Et, propter defectum termini ſecundi et quarti in æquatione poſteriore, evaneſcentibus b & d necnon formulæ terminis quorum factores ſunt b & d, emerget $(-y^2 + 280)y^6 : * + (2y^2 - 40y + 260)(260y^4 - 40y^5) : * + (3y^4)y^4 : -2y^2(y^6 - 40y^5 + 400y^4) = 0$. Unde multiplicando et delendo ſuperflua, prodibit $1600y^6 - 20800y^5 - 67600y^4 = 0$, ac reducendo, $4y^2 - 52y + 169 = 0$. Sive radice extractâ, $2y - 13 = 0$, ſeu $y = 6\frac{1}{2}$. Ut ſupra. Veruntamen exterminatio quantitatis incognitæ plurium dimenſionum plerunque operoſior evadit formularum ope, quam ſi ab ipſis regulis auſpicemur.

Sunt et alii modi quibus hæc eadem abſolvi poſſunt; idque ſæpenumero contractius.

83. Ut ſi generaliter enuncietur quæſtio, habitis æquationibus $x + y + \frac{y^2}{x} = a$, et $x^2 + y^2 + \frac{y^4}{x^2} = b$, prior tranſponendo dat $a - y = x + \frac{y^2}{x}$. Et partibus quadratis $a^2 - 2ay$

$+ y^2 = x^2 + 2y^2 + \frac{y^4}{x^2}$. Et ſubducendo utrinque y^2, erit $a^2 - 2ay \left(= x^2 + y^2 + \frac{y^4}{x^2}\right) = b$. Adeoque $y = \frac{a - \frac{b}{a}}{2}$. Quo ſubſtituto pro y in æquationem $x^2 = \overline{a - y}\, x - y^2$, erit $x^2 = \frac{a + \frac{b}{a}}{2}\, x - \frac{\overline{a^2 - 2b + \frac{b^2}{a^2}}}{4}$. Et per reductionem,

$$x = \frac{a + \frac{b}{a} \pm \sqrt{-3a^2 + 10b - \frac{3b^2}{a^2}}}{4}.$$

Aliter.

84. Deſignantibus jam x et y proportionales extremos, medium deſignabit $\sqrt{xy}$. Eritque, $x + \sqrt{xy} + y = a$, & $x^2 + xy + y^2 = b$. Dividatur ſecunda æquatio per primam et prodibit $x - \sqrt{xy} + y = \frac{b}{a}$. Subducatur hæc a prima, eritque $2\sqrt{xy} = a - \frac{b}{a}$; et medius $\sqrt{xy} = \frac{a}{2} - \frac{b}{2a}$. Ut ſupra. Unde cognoſcatur ſumma et productum extremorum x et y, adeoque & ipſi extremi per *Prob.* I.

PROB. V.

85. *Invenire quatuor numeros continue proportionales, quorum datur ſumma* a, *&* *ſumma quadratorum* b.

Deſignantibus x et y medios numeros quæſitos, deſignabunt, ex natura continuè proportionalium, $\frac{x^2}{y}$ et $\frac{y^2}{x}$ extremos. Habentur itaque æquationes $\frac{x^2}{y} + x + y + \frac{y^2}{x} = a$, et $\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b$. Ponamus jam $x + y = s$ et $xy = p$; eritque $\frac{x^2}{y} + \frac{y^2}{x} = a - s$, itemque $\frac{x^2y^2}{xy} = p$ propter proportionalitatem. Sed per *Prob.* II.

Erit ſumma quadrat. $x^2 + y^2 = s^2 - 2p$
Et ſumma quadrat. $\frac{x^4}{y^2} + \frac{y^4}{x^2} = (a - s)^2 - 2p$.

Adeoque his additis $\frac{x^4}{y^2} + x^2$ &c. $= (a - s)^2 + s^2 - 4p = b$
Sive, ſubſtit. $\frac{s^3}{2s + a}$ pro p * - - $(a - s)^2 + s^2 - \frac{4s^3}{2s + a} = b$
Sive, concinnando, - - - - $s^2 = -\frac{b}{a}s + \frac{1}{2}a^2 - \frac{1}{2}b$
Et per reductionem, - - - - $s = -\frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} + \frac{1}{2}a^2 - \frac{1}{2}b}$.
Invento s, innoteſcit $p = \frac{s^3}{2s + a}$. Itemque ob $s - y$ $(= x) = \frac{p}{y}$ prodit $sy - y^2 = p$, et $y = \frac{1}{2}s + \sqrt{\frac{1}{4}s^2 - p}$. Invento autem y innoteſcit x, et proinde quatuor proportionales quæſiti.

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* $\frac{s^3}{2s + a} = p$. Nam - - $\frac{x^2}{y} + \frac{y^2}{x} = a - s$.
Adeoque, - - $x^3 + y^3 (= \overline{a - s}\; xy) = (a - s)p$
Itemque, per *Prob.* II. $x^3 + y^3 = s^3 - 3sp$.
Adeoque, - - - - - - - $s^3 - 3sp = ap - sp$
Et - - - - - - - - - - - $s^3 = 2sp + ap$.
Ergo, - - - - - - - $\frac{s^3}{2s + a} = p$.

In hac ſolutione ſecuti ſumus *Simpſon*. Eundem valorem s, at operoſius quidem, contra morem ſuum, exquirit *Newtonus*. Aliam quoque et elegantem hujus problematis (necnon inſequentis) ſolutionem a *De Moivre* inventam proferre jam libet. Vide *Sanderſon*. *Algebra*, Vol. 1, p. 263.

86. Deſignantibus x^3, x^2y, xy^2, y^3 quatuor proportionales quæſitos, erit

$x^3 + x^2y + xy^2 + y^3 = a$. Adeoque, $(x+y)\,(x^2+y^2) = a$.
$x^6 + x^4y^2 + x^2y^4 + y^6 = b$. Adeoque, $(x^2+y^2)\,(x^4+y^4) = b$.
Abbreviandi gratiâ, finge $x + y = z$, et $x^2 + y^2 = v$.

Eritque - - - - - - $zv = a$, et $v = \frac{a}{z}$

Itemque - - $v \times (x^4 + y^4) = b$, et $x^4 + y^4 = \frac{b}{v}$.

Cæterum ob $x + y = z$, erit $x^2 + y^2 + 2xy = z^2$
Sive, ſubſtituendo, - - - $v + 2xy = z^2$
Adeoque, - - - - - - - $2xy = z^2 - v$.
Et quadrando, - - - - - - $4x^2y^2 = z^4 - 2vz^2 + v^2$.
Porro, ob $x^2 + y^2 = v$, erit $x^4 + y^4 + 2x^2y^2 = v^2$
Sive, ſubſtituendo, - - - $\frac{b}{v} + 2x^2y^2 = v^2$

Adeoque, - - - - - - - $2x^2y^2 = v^2 - \frac{b}{v}$

Et, - - - - - - - - - $4x^2y^2 = 2v^2 - \frac{2b}{v}$

Erit igitur, - $z^4 - 2vz^2 + v^2 \,(= 4x^2y^2) = 2v^2 - \frac{2b}{v}$

Adeoque, - - - $z^4 - 2vz^2 = v^2 - \frac{2b}{v}$

Sive, ob $v = \frac{a}{z}$, - - $z^4 - 2az = \frac{a^2}{z^2} - \frac{2bz}{a}$

Adeoque, - - - $z^6 + \left(\frac{2b}{a} - 2a\right) z^3 = a^2$

Unde per reductionem § 48, elicietur z et exinde v. His autem cognitis innoteſcent x et y. Erat enim,

$x^2+y^2+2xy=z^2$
Subduc - - - $x^2+y^2 \quad = v$
Eritque - - - $* \quad 2xy = z^2 - v$
Atque harum differentia $x^2+y^2-2xy = v-(z^2-v) = 2v-z^2$
Et extracta radice - $x - y = \sqrt{2v-z^2}$
Huic adde - - - $x + y = z$
Eritque - - - $2x \quad * \quad = z + \sqrt{2v-z^2}$
Et - - - $x = \frac{1}{2}z + \frac{1}{2}\sqrt{2v-z^2}$
Erit quoque - - - $2y \quad * \quad = z - \sqrt{2v-z^2}$
Et - - - - - - $y = \frac{1}{2}z - \frac{1}{2}\sqrt{2v-z^2}$

Cognitis igitur x et y, innoteſcent quatuor proportionales quæſiti.

Exemp. Sit $a = 30$, et $b = 340$; erat, per *præced.*

$$z^6 = \frac{2a^2-2b}{a}z^3 + a^2$$

Sive, poſito $\frac{a^2-b}{a} = d$, $z^6 = 2dz^3 + a^2$

Adeoque - - - $z^3 = d + \sqrt{d^2+a^2}$

Cæterum $d = \frac{a^2-b}{a} = \frac{900-340}{30} = \frac{56}{3}$

Adeoque $d^2+a^2 = \frac{3136}{9} + 900 = \frac{11236}{9}$

Et - $\sqrt{d^2+a^2} = \frac{106}{3}$.

Adeoq. $z^3 = d + \sqrt{d^2+a^2} = 54 = 27 \times 2$. Sit $c^3 = 2$,

Eritq. $z^3 = 27c^3$, et proinde $z = 3c$. Sed $v = \frac{a}{z} = \frac{30}{3c} = \frac{10}{c} = \frac{10c^2}{c^3} = 5c^2$.

Itemque $2v - z^2 = 10c^2 - 9c^2 = c^2$, et $\sqrt{2v-z^2} = c$.

Ergo $x = \frac{z+\sqrt{2v-z^2}}{2} = \frac{3c+c}{2} = 2c$. Et $y = \frac{z-\sqrt{2v-z^2}}{2} = c$.

Adeoque proportionales $x^3,\ x^2y,\ xy^2,\ y^3$
Evadent - - - - $8c^3,\ 4c^3,\ 2c^3,\ 1c^3$
Sive - - - - - $16,\ 8,\ 4,\ 2.$

PROB. VI.

87. *Invenire quinque numeros continue proportionales, quorum datur ſumma* a, *et quadratorum ſumma* b.

Deſignantibus x, y & z medios numeros quæſitos, deſignabunt $\frac{x^2}{y}$ et $\frac{z^2}{y}$ extremos. Habentur itaque æquationes $\frac{x^2}{y} + x + y + z + \frac{z^2}{y} = a$, et $\frac{x^4}{y^2} + x^2 + y^2 + z^2 + \frac{z^4}{y^2} = b$.

Ponamus $x + z = u$, eritque $\frac{x^2}{y} + \frac{z^2}{y} = a - u - y$; et propter proportionalitatem, tam $\frac{x^2z^2}{y^2}$ quam $xz = y^2$.

Adeoque per *Prob.* II. $\frac{x^4}{y^2} + \frac{z^4}{y^2} = (a - u - y)^2 - 2y^2$.

Et ſimiliter, $x^2 + z^2 = u^2 - 2y^2$. Seu dividendo per y, $\frac{x^2}{y} + \frac{z^2}{y} = \frac{u^2 - 2y^2}{y}$. Adeoque, ſubſtituendo erit

Summa proport. - - - $\frac{u^2 - 2y^2}{y} : + u : + y = a$

Et ſum. quad. $(a - u - y)^2 - 2y^2 : + u^2 - 2y^2 : + y^2 = b$

Quibus reductis, erit

$$a^2 - 2au - 2ay + 2u^2 + 2uy - 2y^2 = b$$

Et - - - - - - - $ay - u^2 - uy + y^2 = 0$

Harum priori addatur duplum poſterioris,

Eritque - - - $a^2 - 2au = b$; ſive $u = \frac{1}{2}a - \frac{\frac{1}{2}b}{a}$

Cæterum tranſponendo, fuit $y^2 + ay - uy = u^2$

Adeoque - - - - - - $y = \frac{u - a}{2} + \sqrt{\frac{(u-a)^2}{4} + u^2}$.

Cognitis autem u et y, innoteſcent x et z, et proinde quinque proportionales quæſiti.

Aliter.

88. Habitis jam ſecundum notationem *Moivreanam*, æquationibus

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = a, \text{ adeoque } \frac{x^5 - y^5}{x - y} = a$$

$$x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8 = b, \text{ adeoque } \frac{x^{10} - y^{10}}{x^2 - y^2} = b.$$

Dividendo quartam æquationem per tertiam, erit

$\frac{x^5 + y^5}{x + y}$ ſeu $x^4 - x^3y + x^2y^2 - xy^3 + y^4 = \frac{b}{a}$

Adde - $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = a$

Eritque - $2x^4 + 2x^2y^2 + 2y^4 = a + \frac{b}{a}$

Adeoque $x^4 + x^2y^2 + y^4 = \frac{1}{2}a + \frac{\frac{1}{2}b}{a} =$ puta s.

Subduc $x^4 - x^3y$, &c.
ab $x^4 + x^3y$, &c. eritque, $2x^3y + 2xy^3 = a - \frac{b}{a}$

Adeoque - - - - $x^3y + xy^3 = \frac{1}{2}a - \frac{\frac{1}{2}b}{a} =$ puta d.

Sive - - - - - - $(x^2 + y^2)\,xy = d$.

Sive, poſitis $xy = z$,
Et $x^2 + y^2 = v$, - - - - - $zv = d$, adeoq. $z = \frac{d}{v}$.

Cæterum erat - - - $x^4 + x^2y^2 + y^4 = s$

Et $x^2 + y^2 = v$, quadr. dat, $x^4 + 2x^2y^2 + y^4 = v^2$

Et ſubducendo - - - - - $x^2y^2 = v^2 - s$

Adeoque - - $\frac{d^2}{v^2}\ (= z^2 = x^2y^2) = v^2 - s$

Sive - - $v^4 = sv^2 + d^2$.

Unde elicietur v et exinde z. Hiſce vero inventis innoteſcent x et y.

Nam ob $xy = z$, erit - $2xy = 2z$
Cæterum fuit - $x^2 + y^2 = v$
Et his additis, $x^2 + 2xy + y^2 = v + 2z$
Et radice extracta, $x + y = \sqrt{v + 2z}$
Erit quoque - $x - y = \sqrt{v - 2z}$
Et his additis, - $2x = \sqrt{v + 2z} + \sqrt{v - 2z}$
Sive - - $x = \dfrac{\sqrt{v + 2z} + \sqrt{v - 2z}}{2}$
Et - - - $y = \dfrac{\sqrt{v + 2z} - \sqrt{v - 2z}}{2}$

Exemp. Sit $a = 62$, $b = 1364$. Erit $s = \frac{1}{2}a + \dfrac{\frac{1}{2}b}{a} = 42$, et $d = \frac{1}{2}a - \dfrac{\frac{1}{2}b}{a} = 20$. Adeoque $v^2 = \frac{1}{2}s + \sqrt{\frac{1}{4}s^2 + d^2} = 50 = 25 \times 2$. Sit $c^2 = 2$, eritque $v^2 = 25c^2$, et $v = 5c$. Adeoque $\sqrt{v + 2z} = \sqrt{5c + 4c} = \sqrt{9c} = 3\sqrt{c}$. Et $\sqrt{v - 2z} = \sqrt{c}$.
Erit igitur $x = \dfrac{\sqrt{v + 2z} + \sqrt{v - 2z}}{2} = \dfrac{3\sqrt{c} + \sqrt{c}}{2} = 2\sqrt{c}$
Et - $y = \sqrt{c}$.
Et proinde $x^2 = 4c$, $xy = 2c$, $y^2 = c$.
Et proportionales x^4, x^3y, x^2y^2, xy^3, y^4
Evadent $16c^2$, $8c^2$, $4c^2$, $2c^2$, $1c^2$
Sive, - 32, 16, 8, 4, 2.

89. His conſtare poteſt quanta ſit in ſolutionum inventione varietas; et obiter, quod alii modi ſint aliis multo concinniores. Quapropter, ſi in primas de ſolutione problematis alicujus cogitationes modus computationi male accommodatus inciderit, relationes quantitatum iterum evolvendæ ſunt, donec modum quam poteris idoneum et elegantem machinatus fueris.

DE EXTERMINATIONE PLURIUM QUANTITATUM INCOGNITARUM E PLURIBUS ÆQUATIONIBUS.

90. Si duæ quantitates incognitæ è tribus æquationibus tollendæ ſunt, aut plures è pluribus; conferendo inter ſe harum quaſque binas, per vices, exterminandi opus per regulas jam ante traditas, gradatim peragetur.

91. Ex tribus pluribuſve æquationibus inter ſe collatis erui quoque poſſint *formulæ*, ad inſtar præcedentium, pro exterminanda *x*, ubi ſit unius, duarum, pluriumve dimenſionum. Cæterum hæ formulæ ad gradus plane enormes aſſurgunt, ubi altiores dimenſiones *x* tollendæ ſunt. Sic exterminato *x* ex tribus quadraticis, prodit æquatio 16mi gradûs; ex tribus cubicis, 81mi gradus; ex tribus biquadraticis, 256mi gradus; et ſic deinceps. Nec huic incommodo ſatis medetur methodus exterminationis quam invenit *Bezout*, (*ſupra cit.* § 75) quippe quæ in caſibus jam memoratis, ad æquationes 12mi, 49mi, et 156mi gradus reſpective, deducit.

Solutiones problematum quorundam plures quantitates incognitas parientium afferre jam libet.

PROB. I.

92. *Diſſimiles duarum pluriumve rerum miſturas ita componere, ut res illæ commiſtæ datam inter ſe rationem acquirant.*

Deſignent A, B, C, &c. res miſtas, et *d*, *e*, *f*, *g*, *h*, &c. proportiones earundem in miſturis. Sit unius miſturæ data quantitas $dA + eB + fC$, alterius eadem quantitas $gA + hB + kC$, et eadem tertiæ $lA + mB + nC$. Sintque hæ miſturæ ita componendæ ut compoſitionis eadem quantitas contineat $pA + qB + rC$. Finge x, y, et z, partes eſſe trium miſturarum, ſive numeros eſſe per quos ſi tres datæ miſturæ reſpective multiplicentur earum aggregatum evadet $pA + qB + rC$.

$$\text{Eſt itaque}\ \left.\begin{matrix} dxA + exB + fxC \\ + gyA + hyB + kyC \\ + lzA + mzB + nzC \end{matrix}\right\} = pA + qB + rC.$$

Adeoque, collatis terminis, ob $dxA + gyA + lzA + = pA$, atque ita deinceps,

$$\text{Erunt}\begin{cases} dx + gy + lz = p, \text{ adeoque } x = \dfrac{p - gy - lz}{d} \\ ex + hy + mz = q, \text{ adeoque } x = \dfrac{q - hy - mz}{e} \\ fx + ky + nz = r, \text{ adeoque } x = \dfrac{r - ky - nz}{f}. \end{cases}$$

$$\text{Et rurſus ſunt}\begin{cases} \dfrac{p - gy - lz}{d} = \dfrac{q - hy - mz}{e}, \text{ adeoq. } y = \dfrac{ep - dq + dmz - elz}{eg - dh} \\ \dfrac{q - hy - mz}{e} = \dfrac{r - ky - nz}{f}, \text{ adeoq. } y = \dfrac{fq - er + enz - fmz}{fh - ek} \end{cases}$$

Et denique, ſi abbrevientur hi valores y, ſcribendo α pro $ep - dq$, β pro $dm - el$, γ pro $eg - dh$, δ pro $fq - er$, ζ pro $en - fm$, et θ pro $fh - ek$; evadet $\dfrac{\alpha + \beta z}{\gamma} = \dfrac{\delta + \zeta z}{\theta}$, & per reductionem, $z = \dfrac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta}$. Invento z, pone $\dfrac{\alpha + \beta z}{\gamma} = y$, et $\dfrac{p - gy - lz}{d} = x$.

Exemp. Si tres ſint metallorum colliquefactorum miſturæ, quarum primæ pondo continet argenti uncias 12, æris 1, & ſtanni 3, ſecundæ pondo continet argenti unciam 1, æris 12, et ſtanni 3, et tertiæ pondo continet æris uncias 14, ſtanni 2, et argenti nihil; ſintque hæ miſturæ ita componendæ ut pondo compoſitionis contineat argenti uncias 4, æris 9, et ſtanni 3: Pro d, e, f; g, h, k; l, m, n; p, q, r, ſcribe 12, 1, 3; 1, 12, 3; 0, 14, 2; 4, 9, 3 reſpective, & erit α $(= ep - dq = 1 \times 4 - 12 \times 9)$ $= -104$, & β $(= dm - el = 12 \times 14 - 1 \times 0) = 168$, & ſic $\gamma = -143$, $\delta = 24$, $\zeta = -40$, & $\theta = 33$. Adeoque $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-3432 + 3432}{5720 - 5544}\right) = 0$, $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-104 + 0}{-143}\right) = \frac{8}{11}$, & $x \left(= \frac{p - gy - lz}{d} = \frac{4 - \frac{8}{11}}{12}\right) = \frac{3}{11}$. Quare ſi miſceantur $\frac{8}{11}$ partes pondo miſturæ ſecundæ, $\frac{3}{11}$ partes pondo primæ, et nihil tertiæ, aggregatum erit pondo continens quatuor uncias argenti, novem æris, et tres ſtanni. Uti ex eo conſtat, quod $\frac{3}{11}$ partes pondo primæ $= \frac{3}{11} \times \left(\frac{12 + 1 + 3}{16}\right) = \frac{36 + 3 + 9}{176}$. Et $\frac{8}{11}$ partes ſecundæ $= \frac{8}{11} \times \left(\frac{1 + 12 + 3}{16}\right) = \frac{8 + 96 + 24}{176}$, et proinde aggregatum $= \frac{44 + 99 + 33}{176} = \frac{4 + 9 + 3}{16}$.

93. Cæterum notandum eſt quod diſſimiles hujuſmodi miſturas ſecundum datam quamlibet rationem componere aliquando impoſſibile evadet, quando, ſcil. valores x, y vel z prodeunt negativi; tales enim in quæſtionis ſolutione valores affirmativos tantum poſtulantis, locum habere nequeunt. Ut ſi primæ pondo contineat argenti, æris et

ſtanni 3, 6, et 7 uncias reſpective, ſecundæ pondo, 1, 12, et 3 uncias, tertiæ pondo 4, 2 et 10 uncias, et deſideretur compoſitio cujus pondo contineat 4, 9 et 3 uncias: Pro d, e, f, &c. ſcriptis 3, 6, 7, &c. erit $\alpha = -3$, $\beta = -18$, $\gamma = -30$, $d = 45$, $\zeta = 46$, et $\theta = 66$. Adeoque

$$z = \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-198 + 1350}{-1380 + 1188} = \frac{1152}{-192} = -6,$$

$$y = \frac{\alpha + \beta z}{\gamma} = -\frac{7}{2}, \ \& \ x = \frac{p - gy - lz}{d} = \frac{4 + \frac{7}{2} + 24}{3} = \frac{63}{6} = \frac{21}{2}.$$

Reſpiciunt nimirum valores y et z jam negativi ad quæſtionis caſum in praxi impoſſibilem, nempe. *Diſſimiles rerum miſturas ita ab invicem ſecernere, ut reſiduum res illas datâ quadam ratione inter ſe commiſtas exhibeat.* Sic in hoc exemplo, ſi a 10½ pondo miſturæ primæ demantur 3½ pondo ſecundæ et 6 pondo tertiæ, reſiduum erit pondo continens argenti æris et ſtanni 4, 9 & 3 uncias reſpective.

PROB. II.

94. *Invenire valores* x, y *et* z *in æquationibus* $x^2 + xy = 10$; $y^2 + yz = 21$, *et* $z^2 + zx = 24$.

Subſtituatur nx pro y, et mx pro z, eritque

$$\left.\begin{array}{l} x^2 + nx^2 = 10, \\ n^2x^2 + nmx^2 = 21, \\ m^2x^2 + mx^2 = 24, \end{array}\right\} \text{ ſive } \left\{\begin{array}{l} x^2 = \dfrac{10}{1 + n} \\ x^2 = \dfrac{21}{n^2 + nm} \\ x^2 = \dfrac{24}{m^2 + m}. \end{array}\right.$$

Adeoque $\frac{24}{m^2+m} (= x^2) = \frac{10}{1+n}$, ergo $n = \frac{5m \cdot \overline{m+1} - 12}{12}$.

Itemq. $\frac{24}{m^2+m} (= x^2) = \frac{21}{n^2+nm}$, ergo $8n^2 + 8nm = 7m \cdot \overline{m+1}$.

Sed - - $n^2 = \frac{25m^2 \cdot \overline{m+1}^2 - 120m \cdot \overline{m+1} + 144}{144}$

Et - - $nm = \frac{5m^2 \cdot \overline{m+1} - 12m}{12}$.

Adeoq. $7m \cdot \overline{m+1} (= 8n^2 + 8nm) = \frac{25m^2 \cdot \overline{m+1}^2 - 120m \cdot \overline{m+1} + 144}{18}$

$$+ \frac{10m^2 \cdot \overline{m+1} - 24m}{3}.$$

Sive concinnando, $25m^4 + 110m^3 - 161m^2 - 390m + 144 = 0$.
Et divid. per $m - 2$, $25m^3 + 160m^2 + 159m - 72 = 0$.
Unde per reſolutionem, prodit $m = 2$.
Adeoque $n \left(= \frac{10 \cdot 3 - 12}{12}\right) = \frac{3}{2}$, $x \left(= \sqrt{\frac{10}{1+1\frac{1}{2}}}\right) = 2$, $y = 3$, et $z = 4$.

P R O B. III.

95. *Si boves* a *depaſcant pratum* b *in tempore* c; *& boves* d *depaſcant pratum æque bonum* e *in tempore* f, *& gramen uniformiter creſcat: Quæritur quot boves depaſcent pratum ſimile* g, *in tempore* h.

Diſtinguantur boves cujuſque prati binos in claſſes; ſint boves $a = r + s$, quorum boves r depaſcant gramen quod in prato b ante boves admiſſos creverat, boves autem s graminis incrementum poſt. Similiter, ſint boves $d = t + u$. Et boves quæſiti $x = y + z$. Sed boves prioris claſſis r, t, et y, dato tempore, ſunt ut prata depaſcenda, (quippe quæ per hypotheſin æque bona ſunt) et datis pratis, reciproce ut depaſcendi tempora, et proinde neutris datis, ut prata directe et depaſcendi tempora reciproce, hoc eſt, ut $\frac{b}{c}$, $\frac{e}{f}$, $\frac{g}{h}$ reſpective. Adeoque per analogiam, $r : t :: \frac{b}{c} : \frac{e}{f}$, et ductis mediis et extremis $\frac{tb}{c} = \frac{cr}{f}$, adeoque $t = \frac{cer}{bf}$. Et ſimiliter, $y = \frac{cgr}{bh}$. Sunt quoque boves poſterioris claſſis s, u, z & tempora depaſcendi conjunctim, ut prata b, e, g et tempora graminis in hiſce creſcendi (uniformiter per hypotheſin) conjunctim; ſive, æquipollentibus his temporibus, boves s, u, z ut prata b, e, g reſpective. Adeoque, per analogiam, boves $u = \frac{es}{b}$, et $z = \frac{gs}{b}$. Subſtituantur in æquationem ſecundam et tertiam valores t, y, u, et z jam inventi, et prodibunt

$a = r + s$, adeoque $r = a - s$.

$d = \frac{cer}{bf} + \frac{es}{b} = \frac{ce \times \overline{a - s}}{bf} + \frac{es}{b}$, adeoque $s = \frac{bdf - cea}{ef - ce}$.

$x = \frac{cgr}{bb} + \frac{gs}{b} = \frac{cg \times \overline{a - s}}{bb} + \frac{gs}{b}$, adeoque $s = \frac{bbx}{gb} - \frac{cga}{cg}$.

$$\frac{bbx - cga}{gb - cg} (= s) = \frac{bdf - cea}{ef - ce}, \text{ adeoque}$$

$$befbx - bcebx = bdfgb - acegb - bcdfg + acefg$$

Sive - - $x = \frac{bdfgb - acegb - bcdfg + acefg}{befb - bceb}$.

96. Longe elegantius, at ſubtilius quidem, ſolutionem expedit *Newtonus*. Primo enim, exquirit quot boves in tempore b depaſcerent graminis incrementum in prato e per tempus c; deinde, quot boves in eodem tempore b depaſcerent graminis incrementum in eodem prato e per tempus

$b - c$; nam ex horum ſumma conflatur totus numerus boum quibus per tempus b paſcendis ſufficeret totum graminis incrementum in hoc prato e per tempus b. Et inde per analogiam, colligit quot boves depaſcerent pratum g in eodem tempore b. Hoc præmiſſo, ipſius ſolutionem, notulis quibuſdam ſubjectis proferre jam libet.

Si boves a in tempore c depaſcant pratum b; tum per analogiam, boves $\frac{e}{b}a$ in eodem tempore c (a), vel boves $\frac{ec}{bf}a$ in tempore f (b), vel boves $\frac{ec}{bb}a$ in tempore b, depaſcent pratum e: puta ſi gramen poſt tempus c non creſceret. Sed cum propter graminis incrementum boves d in tempore f depaſcant ſolummodo pratum e, ideo graminis in prato e incrementum illud per tempus $f - c$ tantum erit quantum per ſe ſufficit paſcendis bobus $d - \frac{eca}{bf}$ per tempus f, hoc eſt quantum ſufficit paſcendis bobus $\frac{df}{b} - \frac{eca}{bb}$ per tempus b (c). Et in tempore $b - c$ per analogiam tantum erit incrementum quantum per ſe ſufficit paſcendis bobus $\frac{h-c}{f-c}$ in $\frac{df}{h} - \frac{eca}{bh}$, ſive $\frac{bdfh - ecah - bdcf + acc^2}{bfh - bch}$ (d) [per tempus h.] Hoc incrementum adjice bobus $\frac{acc}{bh}$ et prodibit $\frac{bdfh - ecah - bdcf + ecfa}{bfh - bch}$ numerus boum quibus paſcendis ſufficit pratum e per tempus h. Adeoque per analogiam, pratum g bobus $\frac{bdfgh - ecagh - bdcgf + ecfga}{befh - bceh}$ per idem tempus h paſcendis ſufficiet.

(a) $b : e :: a : \frac{ea}{b}$ ob boves ut prata, dato graminis incremento per tempus c.

(b) $\left.\begin{array}{l} \frac{1}{c} : \frac{1}{f} :: \frac{ea}{b} : \frac{cea}{bf} \\ \frac{1}{c} : \frac{1}{b} :: \frac{ea}{b} : \frac{cea}{bb} \end{array}\right\}$ ob boves reciproce ut tempora depaſcendi pratum datum e, dato graminis incremento per tempus c.

(c) $\frac{1}{f} : \frac{1}{b} :: d - \frac{cea}{bf} : \frac{fd}{b} - \frac{fcea}{bfb}$ ſeu $\frac{fd}{b} - \frac{cea}{bb}$ ob boves dato graminis incremento per tempus $f - c$, reciproce ut depaſcendi tempora.

◆

EXEMP. Si 12 boves depaſcant 3$\frac{1}{3}$ jugera prati in 4 ſeptimanis; & 21 boves depaſcant 10 jugera conſimilis prati in 9 ſeptimanis; quæritur quot boves depaſcant 24 jugera in 18 ſeptimanis? Reſp. 36. Iſte enim numerus invenietur ſubſtituendo in $\frac{bdfgh - ecagh - bcdgf + ecfga}{befh - bceh}$ numeros 12, 3$\frac{1}{3}$, 4, 21, 10, 9, 24, et 18, pro literis a, b, c, d, e, f, g et h reſpective.

Cæterum ſolutio forte haud minus expedita erit ſi è primis principiis ad formam ſolutionis præcedentis literalis eruatur. Utpote ſi 12 boves in 4 ſeptimanis depaſcant 3$\frac{1}{3}$ jugera, tum per analogiam 36 boves in 4 ſeptimanis vel 16 boves in 9 ſeptimanis vel 8 boves in 18 ſeptimanis depaſcent 10 jugera: puta ſi gramen non creſceret. Sed cum propter graminis incrementum 21 boves in 9 ſeptimanis depaſcant ſolummodo 10 jugera, illud graminis in 10 jugeris per poſteriores 5 ſeptimanis incrementum tantum erit quantum per ſe ſufficit exceſſui boum 21 ſupra 16, hoc eſt 5 bobus per 9 ſeptimanas, vel quod perinde eſt $\frac{5}{2}$ bobus per 18 ſeptimanas paſcendis. Et in 14 ſeptimanis (exceſſu 18 ſupra 4 primas) incrementum illud per analogiam tantum erit quantum ſufficiat 7 bobus per 18 ſeptimanas paſcendis; eſt enim 5 ſept. ad 14 ſeptimanas ut $\frac{5}{2}$ boves ad 7 boves. Quare 8 bobus quos 10 jugera ſine incremento graminis paſcere poſſunt per 18 ſeptimanas adde hoſce 7 boves quibus paſcendis ſolum incrementum graminis ſufficit, et ſumma erit 15 boves. Ac denique ſi 10 jugera 15 bobus per 18 ſeptimanas paſcendis ſufficiant, tum per analogiam 24 jugera per idem tempus ſufficient 36 bobus.

(d) $f - c : h - c :: \frac{df}{h} - \frac{eca}{bh}$ ſeu $\frac{bdf - eca}{bh} : \frac{bdfh - ecah - bdcf + eac^2}{bfh - bch}$ ob boves ut graminis incrementa in prato e per tempora $f - c$ et $h - c$.

In priore ſolutione literali ſeptem adſciſcuntur quantitates incognitæ, in poſteriore ne una quidem. Luculenter profectò evincit hoc exemplum non ſemper ad Algebram neceſſario recurrendum eſſe in problematum ſolutionem quæ primâ fronte difficillima videantur. Verùm *Newtoni* ſagacitas haud cuique Analyſtæ contingit.

DE EXTERMINATIONE QUANTITATUM SURDARUM EX ÆQUATIONIBUS.

97. Tollendæ ſunt ſurdæ quadraticæ ex æquatione qualibet haud plures quam quatuor terminos complectente, ſive hi omnes ſint irrationales ſive non, per involutionem iteratam, ope Regulæ VI, § 25. Sic æquatio $\sqrt{a} - \sqrt{b} - \sqrt{c} - \sqrt{d} = 0$, (ubi $\sqrt{a}$, $\sqrt{b}$, &c. quantitates quaſcun-

que ſurdas deſignant ſive ſimplices ſive complexas) tranſponendo, dat $\sqrt{a} - \sqrt{b} = \sqrt{c} + \sqrt{d}$; et quadrando, $a - 2\sqrt{ab} + b = c + 2\sqrt{cd} + d$; atque iterum tranſponendo, $2\sqrt{ab} + 2\sqrt{cd} = a + b - c - d =$ puta g; & partibus iterum quadratis, $4ab + 8\sqrt{abcd} + 4cd = g^2$; atque denuo tranſponendo, $\sqrt{abcd} = \frac{g^2 - 4ab - 4cd}{8} =$ puta b; partibus autem denuo quadratis emergit æquatio quatuor dimenſionum a ſurdis libera, $abcd = b^2$. Verùm ſi datæ æquationi ſubquadraticæ accedat terminus quintus, ſive hic ſit rationalis ſive non, amovendæ ex æquatione aſymmetriæ impar erit hæc regula utcunque diſtribuas terminos. Nam ex unâ æquationis parte neceſſario ſiſtentur tres, quatuor, aut quinque termini, adeoque ex alterâ duo, unus, aut nullus, reſpective; at ſurda quantitas binomia quadrando parit unum terminum irrationalem, (ut in exemplo allato) trinomia parit tres, quadrinomia ſex, et quinquinomia decem: ergo ex involutione partium quotieſcunque iteratâ provenient termini ſurdi numero plures quam quatuor, vel ſaltem non pauciores, in iſto caſu ubi quatuor terminis ſurdis accedit quintus terminus qui ſit rationalis. Minus itaque valet hæc regula ad ſurdarum quadraticarum in omni caſu exterminationem, licet harum genus ſit ſimpliciſſimum. Et *a fortiori*, intra limites adhuc arctiores concludetur ſurdarum cubicarum exterminatio.

98. Cæterum quantitates *quotcunque* ſurdas ex æquationibus tollere licet ope methodi *Fermatianæ*, eas nempe fingendo literis quibuſlibet æquales; deinde, ex æquationibus reſultantibus tollendo gradatim has literas, per regulas præcedentes, emerget tandem æquatio libera ab omni aſymmetria. Quemadmodum ſi ſit $\sqrt{ay} - \sqrt{a^2 - ay} = 2a + \sqrt[3]{ay^2}$, ſcribendo t pro $\sqrt{ay}$, v pro $\sqrt{a^2 - ay}$, et x pro $\sqrt[3]{ay^2}$, ha-

bentur quatuor æquationes $t - v = 2a + x$, $t^2 = ay$, $v^2 = a^2 - ay$, et $x^3 = ay^2$. Sed per primam æquationem,

Erit - - - - - - - - $2a + x + v = t$

Et - $4a^2 + 4ax + 4av + x^2 + 2xv + v^2 = (t^2 =)\ ay$

Adeoque, ſubſtituendo $a^2 - ay$ pro v^2, et tranſponendo,

$$x^2 + 4ax - 2ay + 5a^2 = -2xv - 4av$$

Adeoque - - - $\dfrac{x^2 + 4ax - 2ay + 5a^2}{2x + 4a} = -v.$

Et $\dfrac{x^4 + 8ax^3 - 4ayx^2 + 26a^2x^2 - 16a^2yx + 40a^3x + 4a^2y^2 - 20a^3y + 25a^4}{4x^2 + 16ax + 16a^2}$

$= (v^2 =)\ a^2 - ay$. Unde ſubſtituendo ay^2x pro x^4, $8a^2y^2$ pro $8ax^3$, et concinnando, evadit

$$22ax^2 + y^2x + 24a^2x + 12ay^2 - 4a^2y + 9a^3 = 0.$$

Sive, abbreviandi gratiâ ſcribendo

A pro $22a$, B pro $y^2 + 24a^2$, et C pro $12ay^2 - 4a^2y + 9a^3$,

Erit - - - - - - $Ax^2 + Bx + C = 0$

Adeoque - - - - $Ax^3 + Bx^2 + Cx = 0$

Sed $x^3 - ay^2 = 0$, $\therefore\ Ax^3 - Aay^2 = 0$

$$\overline{{}^{*}\,Bx^2 + Cx + Aay^2 = 0.}$$

$$\begin{array}{ll}
ABx^2 + B^2x + BC = 0 & A^2ay^2x^2 + ABay^2x + ACay^2 = 0 \\
ABx^2 + ACx + A^2ay^2 = 0 & BCx^2 + C^2x + ACay^2 = 0 \\
\hline
{}^{*}\,(B^2 - AC)x + BC - A^2ay^2 = 0 & (BC - A^2ay^2)\,x + C^2 - ABay^2\,{}^{*} = 0
\end{array}$$

$$\begin{array}{l}
(B^2 - AC)\,(BC - A^2ay^2)\,x + (BC - A^2ay^2)^2 = 0 \\
(B^2 - AC)\,(BC - A^2ay^2)\,x + (B^2 - AC)\,(C^2 - ABay^2) = 0 \\
\hline
{}^{*}\,(BC - A^2ay^2)^2 - (B^2 - AC)\,(C^2 - ABay^2) = 0
\end{array}$$

Sive $A^3a^2y^4 + B^3ay^2 + C^3 - 3ABCay^2 = 0.$

Unde ſubſtituendo valores A, B, et C, et dividendo per a, prodit æquatio a ſurdis expurgata

$$x^8 + 1008a^2y^6 - 1464a^3y^5 - 2762a^4y^4 + 3680a^5y^3 + 2916a^6y^2 - 972a^7y + 729a^8 = 0$$

DE PROBLEMATIS INDETERMINATIS.

99. Hactenus prolatæ ſunt quæſtiones quæ tot pariunt æquationes quot habent quantitates incognitas, et quæ idcirco, ad ſolutionem determinatam ſeu certam perducunt. Quæſtiones jam apponemus quæ conditionum inopia laborantes, pauciores præbent æquationes quam quantitates incognitas, atque ſolutionem indeterminatam ſeu incertam ſortiuntur.

PROB. I.

100. *Datis et miſturæ et miſtorum gravitatibus ſpecificis, invenire proportionem miſtorum inter ſe.*

Sint x et y miſtorum moles, a et b horum gravitates ſpecificæ, & e gravitas ſpecifica miſturæ, et cum gravitas abſoluta ſeu pondus componatur ex mole corporis et gravitate ſpecifica, erit ax pondus prioris, by pondus poſterioris, et $e \times \overline{x+y}$ pondus miſturæ; adeoque $ax + by = ex + ey$, et tranſponendo, $ax - ex = ey - by$, ſive $\overline{a-e} \times x = \overline{e-b} \times y$, indeque $x : y :: e - b : a - e$. Sed cognita jam proportione molium x et y inter ſe, et gravitatum ſpecificarum a et b, componendo innoteſcet proportio ponderum miſtorum inter ſe.

Exempl. Sit a auri gravitas experimentis hydroſtaticis comperta ut 19, b argenti gravitas ut $10\frac{1}{3}$, et e Coronæ *Hieronis* ut 17. Eritque moles auri in corona ad molem argenti, ut ($e - b : a - e :: 17 - 10\frac{1}{3} :: 19 - 17 :: 6\frac{2}{3} : 2 :: 20 : 6 ::$) 10 ad 3; adeoque, pondus auri in corona ad pondus argenti, ut ($a \times \overline{e-b} : b \times \overline{a-e}$ :: $19 \times 10 : 10\frac{1}{3} \times 3$ ::) $190 : 31$; et pondus coronæ ad pondus argenti, ut $190 + 31 = 221$ ad 31. Innoteſcit igitur proportio ponderum coronæ et miſtorum inter ſe. Cæterum quo eruantur ipſa miſtorum pondera, et quæſtio determinata ſit, poſtulatur altera conditio, nempe ut innoteſcat coronæ pondus, de hoc vero ſilet *Vitruvius*, *loc. cit.* § 12.

Si miſtura præter aurum et argentum, aliod quodvis metallum colliquefactum contineret, etiamſi daretur coronæ pondus, ipſa miſtorum pondera neutiquam detexiſſet *Archimedes*, ut ex ſeq. *prob.* patebit.

P R O B. II.

101. *Datis miſturæ ex tribus metallis conflatæ pondere, et miſtorum gravitatibus ſpecificis, invenire proportionem miſtorum inter ſe.*

Sint x, y, z, miſtorum puta auri, argenti et æris moles; a, b, c horum gravitates ſpecificæ reſpective; miſturæ autem gravitas ſpecifica g, et pondus p: & inde duæ habentur æquationes $ax + by + cz = p$, et $ax + by + cz = g \times \overline{x + y + z}$; ex priore autem elicitur $x = \frac{p - by - cz}{a}$ & hoc valore x in alteram ſubſtituto, concinnando prodit $y = \frac{agz - cgz + gp - ap}{bg - ag}$. Quoniam igitur valoribus x et y immiſcetur z, hoc incognito, illi determinari nequeunt.

102. Sufficiant hæc in ſpecimen ſolutionum problematum hujuſmodi indeterminatorum. Pergamus jam methodum generalem ſolvendi problemata de numeris integris & affirmativis docere, quam invenit ſummus analyſta *De Moivre*. Vide *Dodſon's Math. Repoſ. Vol.* 2.

PROB. I.

103. *Quæruntur valores integri et affirmativi* x *et* y *in æquatione* $9x - 7y = 13$.

Tranſp. erit $7y = 9x - 13$

Adeoque - $y = \frac{9x - 13}{7} = x - 1 + \frac{2x - 6}{7}$ (quod patet dividendo $9x - 13$ per 7). Si igitur ex hoc valore y, per quæſtionem integro, dematur maximus integer quem ſub ſe continet ſcilicet quotiens $x - 1$, erit reſiduum $\frac{2x - 6}{7}$ integer. Finge hoc æquari a, et inde erit $2x - 6 = 7a$, et $x = \frac{7a + 6}{2} = 3a + 3 + \frac{a}{2}$. Sed hic valor x eſt quoque integer, ab hoc igitur dempto maximo integro $3a + 3$, erit reſiduum $\frac{a}{2}$ integer = puta b, et inde $a = 2b$. Comparato jam valore integro a abſque reſiduo, regrediendo ſubſtituatur $2b$ pro a in valorem præcedentem x, et prodibit $x = \frac{\overline{7 \times 2b} + 6}{2} = \frac{14b + 6}{2} = 7b + 3$,

et - $y = \frac{\overline{9 \times 7b + 3} - 13}{7} = \frac{\overline{9 \times 7b} + 14}{7} = 9b + 2$.

In hiſce autem valoribus x et y, deſignare b poteſt aut zero, aut integrum quemlibet majorem zero, ſic enim affirmativi exurgent horum valores, ſecus ſi minorem zero. Sit igitur

$b = 0$, erit $x = 0 + 3 = 3$, et $y = 0 + 2 = 2$
$b = 1$, - $x = 7 + 3 = 10$, - $y = 9 + 2 = 11$
$b = 2$, - $x = 14 + 3 = 17$, - $y = 18 + 2 = 20$,
&c. &c. &c.

Hinc in hujuſmodi æquationibus $fx - gy = h$, innumeri prodibunt quantitatum incognitarum valores, creſcente x per coefficientem y et creſcente y per coefficientem x, in infinitum.

PROB. II.

104. *Valente nummo quodam anglico* 21 *ſolidos, gallico autem* 17 *ſolidos, quot utriuſque* 100 *libras ſterlingas perſolvent; ſive in genere, quæruntur valores integri et affirmativi* x *et* y *in æquatione* $21x + 17y = 100 \times 20 = 2000$.

$$21x + 17y = 2000$$

Tranſp. et divid. erit $y = \frac{2000 - 21x}{17} = 117 - x + \frac{11 - 4x}{17}$

Sit $a = \frac{11 - 4x}{17}$, adeoq. $x = \frac{11 - 17a}{4} = 2 - 4a + \frac{3 - a}{4}$

Sit $b = \frac{3 - a}{4}$, adeoq. $a = 3 - 4b$.

Subſtituatur jam hic valor integer a reſiduo vacans, in valorem x, et evadet

$x = \frac{11 - 17 \times \overline{3 - 4b}}{4} = 17b - 10$, et $y = \frac{2000 - 21 \times \overline{17b - 10}}{17}$
$= 130 - 21b$.

Cæterum quo $17b - 10$ ($= x$) evadat affirmativus, ſtatuendus eſt b major nihilo, et quo $130 - 21b$ ($= y$) evadat affirmativus, limitem $\frac{130}{21} = 6\frac{4}{21}$ haud tranſcendere debet. Sortietur igitur b ſex valores integros, ſcil.

$$b = 1,\ 2,\ 3,\ 4,\ 5,\ 6,$$

$$\text{Adeoque} \begin{cases} x = 7,\ 24,\ 41,\ 58,\ 75,\ 92, \\ y = 109,\ 88,\ 67,\ 46,\ 25,\ 4. \end{cases}$$

Hinc in æquationibus formæ $fx + gy = h$, intra certos limites concludentur valores affirmativi x et y, creſcente x per coefficientem y et decreſcente y per coefficientem x.

PROB. III.

105. *Quæruntur valores integri et affirmativi* x *et* y *in æquatione* $5x + 8y = 1989$.

Tranſponendo et dividendo erit $x = \frac{1989 - 8y}{5}$, at hic valor x per quæſtionem debet eſſe integer, unde pergendo ſecundum methodum *Moivreanam* in præcedentibus adhibitam, erui poteſt ſolutio.

106. Citius autem methodo inſequenti integrum quem quærimus reſiduo vacantem eruere licet, quam idcirco in poſterum uſurpabimus. Sumantur multipli denominatoris 5 terminis numeratoris 1989 et $8y$ proximi, ſive ſupra ſive infra fuerint, ſcilicet 1990 et $10y$, eritque $\frac{1990 - 10y}{5}$ integer, adeoque differentia quoque $\frac{1 - 2y}{5}$ integer, = puta a, et inde evadet $y = \frac{1 - 5a}{2}$. Similiter, ab hujus numeratore dematur multiplus proximus denominatoris 2, ſcil. $4a$, eritque $\frac{1 - a}{2}$ integer = puta b, unde evadet $a = 1 - 2b$.

Prodibit igitur
$$\begin{cases} y = \dfrac{1 - 5 \times \overline{1 - 2b}}{2} = 5b - 2, \\ \text{et } x = \dfrac{1989 - 8 \times \overline{5b - 2}}{5} = 401 - 8b. \end{cases}$$

Adeoque ob $5b - 2\ (= y)$ affirmativum, neceſſe eſt ut b ſit major nihilo, at non tranſcendere debet limitem $\frac{401}{8} = 50\frac{1}{8}$, ſic enim negativus evaderet $401 - 8b\ (= x)$ Sortietur itaque b valores integros 50, totidemque x et y, ſcil.

$$\begin{array}{llllll} b = & 1, & 2, & 3, & \&c. & 50, \\ x = & 393, & 385, & 377, & \&c. & 1, \\ y = & 3, & 8, & 13, & \&c. & 248. \end{array}$$

107. Cæterum notandum eſt in æquationibus formæ $fx \pm gy = h$, quo valores x et y emergant integri, neceſſe eſt ut coefficientes f et g ſint ad ſe invicem primi, aliter enim dividendo per maximum horum diviſorem communem, terminus abſolutus h evadere poteſt fractionalis, ac idcirco ſolutio in numeris integris impoſſibilis. Sic habita æquatione $6x + 9y = 17$, dividendo per 3 erit $2x + 3y = 5\frac{2}{3}$.

Aliquando contractius ſolvi poſſunt hujuſmodi æquationes.

108. Sic propoſita æquatione $5x + 8y = 152$, ubi coefficientium alter ſcilicet 8 ſit diviſor termini abſoluti 152, erit

$$5x = (152 - 8y =)\ \overline{19 - y} \times 8$$

Adeoque $x : 19 - y :: 8 : 5$.

Erit igitur $x = 8,\ 16,\ 24,$ &c.

$19 - y = 5,\ 10,\ 15,$ &c.

Adeoque $y = 14,\ 9,\ 4.$

109. Propoſita æquatione $5x + 8y = 91 = 13 \times 7$, ubi ſumma coefficientium ſit termini abſoluti diviſor, manifeſtum eſt utramque quantitatem incognitam deſignari poſſe

per quotum 7, et inde ex lege incrementi et decrementi prodeunt utriuſque valores,

$$\begin{cases} x = 7, \; 15 \\ y = 7, \; 2. \end{cases}$$

Aut propoſita æquatione $5x - 8y = 81 = 27 \times 3$, ubi differentia coefficientium ſit termini abſoluti diviſor, manifeſtum eſt x et y deſignari poſſe per quotum 27, et inde facile habentur reliqui utriuſque valores.

110. Propoſita æquatione $5x + 8y = 198$, ubi differentia coefficientium ſit termini abſoluti diviſor, finge $x + y = s$, adeoque $5x + 5y = 5s$ et $8x + 8y = 8s$. Sed differentiæ inter haſce et datam dabunt

$$\begin{cases} 3y = 198 - 5s, & \text{adeoque erit } s < \frac{198}{5} = 39\frac{3}{5} \\ 3x = 8s - 198, & \text{adeoque erit } s > \frac{198}{8} = 24\frac{6}{8}. \end{cases}$$

Conſiſtere igitur debet s inter limites $39\frac{3}{5}$ et $24\frac{6}{8}$, et quoniam $y = \frac{198 - 5s}{3}$ et $x = \frac{8s - 198}{3}$, quo integri eruantur utriuſque valores, debet inſuper s eſſe ex multiplis denominatoris 3. Erunt igitur,

$s = 27, \; 30, \; 33, \; 36, \; 39,$
$x = 6, \; 14, \; 22, \; 30, \; 38,$
$y = 21, \; 16, \; 11, \; 6, \; 1.$

PROB. IV.

111. *Valentibus nummis aureis quibuſdam* 18¼ *ſolidos*, 22¾ *ſol. et* 29¼ *ſol. reſpective, quot cujuſque claſſis ſumendi ut omnium ſumma ſit* 1000, *valor autem* 1000 l. *Sive, deſignantibus* x, y, *et* z *nummos cujuſque claſſis quæſitos, invenire valores horum integros et affirmativos in æquationibus* $x + y + z = 1000$, *et* $73x + 91y + 117z = 80{,}000$.

Nam multiplicando æquationem $x + y + x = 1000$, ſucceſſive per 73 & 117,

$$\text{erunt} \begin{cases} 73x + 73y + 73z = 73000 \\ 117x + 117y + 117z = 117000 \end{cases}$$

Deinde ſubducendo priorem ab æquatione $73x + 91y + 117z = 80{,}000$, & hanc a poſteriore,

$$\text{erunt} \begin{cases} 18y + 44z = 7000 \\ 44x + 26y = 37000 \end{cases}$$

$$\text{ſive} \begin{cases} 9y + 22z = 3500, \text{adeoq.}\ y = \frac{3500 - 22z}{9}\ \text{et}\ z = \frac{3500 - 9y}{22} \\ 22x + 13y = 18500, \ \therefore\ y = \frac{18500 - 22x}{13}. \end{cases}$$

Sed $\frac{3500 - 22z}{9}$ $(= y)$ debet eſſe int. eſt autem $\frac{3501 - 18z}{9}$ int. adeoq. diff. $\frac{4z + 1}{9} = a$ int. adeoq. $z = \frac{9a - 1}{4}$. Similiter, erit differ. $\frac{a - 1}{4} = b$ int. adeoq. $a = 4b + 1$, adeoq. ſubſtit.

$$z = \frac{36b + 8}{4} = 9b + 2. \text{ Itemque}$$

$\frac{18500-22x}{13}$ $(=y)$ debet eſſe int. eſt autem $\frac{18499-26x}{13}$ int. adeoq. diff. $\frac{4x+1}{13}=c$ int. adeoq. $x=\frac{13c-1}{4}$. Et rurſus erit differ. $\frac{c-1}{4}=d$ integer, $\therefore\ c=4d+1$, et ſubſtit.

$$x=\frac{52d+12}{4}=13d+3.$$ Itemque

$\frac{3500-9y}{22}$ $(=z)$ debet eſſe int. eſt autem $\frac{3498}{22}$ int. adeoque

$\frac{9y-2}{22}=e$ int. adeoque $y=\frac{22e+2}{9}$, adeoque

$\frac{4e+2}{9}=f$ int. adeoque $e=\frac{9f-2}{4}$, adeoque

$\frac{f-2}{4}=g$ integer, adeoque $f=4g+2$, et

$$e=\frac{36g+16}{4}=9g+4,$$ adeoque

$$y=\frac{22\times\overline{9g+4}+2}{9}=22g+10.$$

Inveſtigatis jam valoribus z, x, et y in terminis b, d, et g reſpective; ob $9b+2=z$, manifeſtum eſt quod b deſignare poteſt aut zero aut integrum quemlibet affirmativum: quod perinde quoque verum erit de d et g. Hiſce autem inter ſe collatis, eruentur limites quidam valorum affirmativorum b, d, et g.

Nam, $22g+10\ (=y)=\frac{3500-22\times\overline{9b+2}}{9}=384-22b$.

Adeoq. $22g+22b=374$.

Sive - $g+b=17$. Et proinde b et g excedere nequeunt 17 reſpective.

Itemque $22g+10=\frac{18500-22\times\overline{13d+3}}{13}=1418-22d$.

Adeoque - $g+d=64$. Unde d nequit eſſe major quam 64 nec minor quam 47.

Erit igitur $b=0$, 1, &c. 17, adeoq. $z=2$, 11, &c. 155,
$g=17, 16$, &c. 0, - $y=384, 362$, &c. 10,
$d=47, 48$, &c. 64, - $x=614, 627$, &c. 835.

PROB. V.

112. *Invenire valores integros et affirmativos* x, y, *et* z *in æquatione* $17x+19y+21z=400$.

Erit $x=\frac{400-19y-21z}{17}$ int. ſed $\frac{408-17y-17z}{17}$ eſt int.

∴ differ. $\frac{8+2y+4z}{17}$ integer $=a$,

adeoq. $y=\frac{17a-4z-8}{2}$ int. ∴ $\frac{a}{2}$ int. $=b$, ∴ $a=2b$.

Unde ſubſtituendo,

erit $y=\frac{17\times 2b-4z-8}{2}=17b-2z-4$,

et $x=\frac{400-19\times\overline{17b-2z-4}-21z}{17}=28+z-19b$.

In hiſce valoribus y et x, definiendi ſunt z et b. Quo determinentur igitur limites z, obſervandum eſt, quod z nequit eſſe minor unitate, et ſi pro x et y ſubſtituatur in æquationem datam minimus utriuſque valor ſcilicet 1, evadet $z=\frac{400-17-19}{21}=17\frac{7}{21}$, et proinde nequit eſſe major quam 17.

Sit nunc $z=1$, evadet $y=17b-6$, et $x=29-19b$. At b nequit eſſe zero aut negativus, ſic enim prodiret negativus valor. y; neque tranſcendere poteſt $\frac{29}{19}=1\frac{10}{19}$, ſic enim foret x negativus. Unicus igitur valor b erit 1, et in hoc caſu $y = 17 - 6 = 11$, et $x = 29 - 19 = 10$. Similiter, ſit $z = 2, 3, 4, 5, 6$ ſucceſſive, erit ut ante 1, unicus valor qui dabit y et x affirmativos, et in his caſibus erit $y = 9, 7, 5, 3, 1$, et $x = 11, 12, 13, 14, 15$, reſpective.

Sit $z = 7, 8, 9, 10$ ſucceſſive, nullum ſortietur valorem b, qui non dabit y et x negativum, et inde rejiciendi ſunt hi valores z.

Sit z autem $= 11, 12, 13, 14$, poſito $b = 2$, evadet $y = 8, 6, 4, 2$, et $x = 1, 2, 3, 4$ reſpective. Reliqui valores z ſcilicet 15, 16, 17, perinde rejiciendi ſunt. Reſtant igitur decem valores integri et affirmativi x, y, et z, ſcil.

$z = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11,\ 12,\ 13,\ 14,$

$y = 11,\ 9,\ 7,\ 5,\ 3,\ 1,\ 8,\ 6,\ 4,\ 2,$

$x = 10,\ 11,\ 12,\ 13,\ 14,\ 15,\ 1,\ 2,\ 3,\ 4.$

PROB. VI.

113. *Invenire valores integros et affirmativos* x, y, *et* z *in æquatione* $5x + 7y + 11z = 224$.

Erit $x = \dfrac{224 - 7y - 11z}{5}$ int. ſed, $\dfrac{225 - 5y - 10z}{5}$ eſt int.

$$\therefore \frac{1 + 2y + z}{5} = a \text{ integer},$$

adeoq. $y = \dfrac{5a - z - 1}{2}$ $\therefore$ $\dfrac{a - z - 1}{2} = b$ integer,

et $a = 2b + z + 1$. Unde ſubſtit.

erit $$y = \frac{5 \times \overline{2b + z + 1} - z - 1}{2} = 5b + 2z + 2,$$

Et $$x = \frac{224 - 7 \times \overline{5b + 2z + 2} - 11z}{5} = 42 - 7b - 5z.$$

Cæterum z nequit eſſe minor unitate, nec major quam $\frac{224-5-7}{11} = 18\frac{4}{11}$, adeoque limites 1 et 18 haud tranſgredietur, ſit

$z = 1$, erit $y = 5b + 4$, et $x = 37 - 7b$, adeoq. $b = 0, 1, 2, 3, 4, 5$,
$z = 2$, $y = 5b + 6$, $x = 32 - 7b$, - $b = -1, 0, 1, 2, 3, 4$,
$z = 3$, $y = 5b + 8$, $x = 27 - 7b$, - $b = -1, 0, 1, 2, 3$,
$z = 4$, $y = 5b + 10$, $x = 22 - 7b$, - $b = -1, 0, 1, 2, 3$,
$z = 5$, $y = 5b + 12$, $x = 17 - 7b$, - $b = -2, -1, 0, 1, 2$,
$z = 6$, $y = 5b + 14$, $x = 12 - 7b$, - $b = -2, -1, 0, 1$,
$z = 7$, $y = 5b + 16$, $x = 7 - 7b$, - $b = -3, -2, -1, 0$,
$z = 8$, $y = 5b + 18$, $x = 2 - 7b$, - $b = -3, -2, -1, 0$,
$z = 9$, $y = 5b + 20$, $x = -3 - 7b$, - $b = -3, -2, -1$,
$z = 10$, $y = 5b + 22$, $x = -8 - 7b$, - $b = -4, -3, -2$,
$z = 11$, $y = 5b + 24$, $x = -13 - 7b$, - $b = -4, -3, -2$,
$z = 12$, $y = 5b + 26$, $x = -18 - 7b$, - $b = -5, -4, -3$,
$z = 13$, $y = 5b + 28$, $x = -23 - 7b$, - $b = -5, -4$,
$z = 14$, $y = 5b + 30$, $x = -28 - 7b$, - $b = -5$,
$z = 15$, $y = 5b + 32$, $x = -33 - 7b$, - $b = -6, -5$,
$z = 16$, $y = 5b + 34$, $x = -38 - 7b$, - $b = -6$,
$z = 17$, $y = 5b + 36$, $x = -43 - 7b$, - $b = -7$,
$z = 18$, $y = 5b + 38$, $x = -48 - 7b$, - $b = -7$.

Dantur itaque 59 valores b, quibus ſubſtitutis totidem prodeunt valores integri et affirmativi y et x, ſcil.

$z = 1$, $y = 4, 9, 14, 19, 24, 29$, $x = 37, 30, 23, 16, 9, 2$,
$z = 2$, $y = 1, 6, 11, 16, 21, 26$, $x = 39, 32, 25, 18, 11, 4$,
$z = 3$, $y = 3, 8, 13, 18, 23$, $x = 34, 27, 20, 13, 6$,
&c. &c. &c.
$z = 18$, $y = 3$, $x = 1$.

PROB. VII.

114. *Quot modis erogare licet* 1000 l. *adhibitis ſolummodo nummis* 5 *ſolidorum,* 21 *ſol. et* 27 *ſol. reſpective. Seu quæruntur omnes valores poſſibiles* x, y, et z *in æquatione* $5x + 21y + 27z = 20,000$.

$$\text{Erit } x = \frac{20,000 - 21y - 27z}{5} \therefore \frac{y + 2z}{5} = a \text{ int. et } y = 5a - 2z,$$

$$\text{et } x = \frac{20,000 - 21 \times \overline{5a - 2z} - 27z}{5} = 4000 - 21a + 3z.$$

Cæterum non poſtulat quæſtio x, y, et z in omni caſu ſimul eſſe adhibendos, poteſt igitur horum quivis evadere zero, unde erit minimus valor $z = 0$, et maximus valor, evaneſcentibus ſcilicet x et y, $\frac{20,000}{27} = 740$. His cognitis, innoteſcunt limites a, nam

$$5a - 2z \left\{ \begin{array}{l} \text{ſi } z = 0 \\ \text{ſi } z = 740 \end{array} \right\} \begin{array}{l} = 5a, \text{ unde prodit minor limes } a = 1 \\ = 5a - 1480, \text{ et major limes } a = \end{array}$$

$$= \frac{1480}{5} = 296.$$

			valores z.	
Sit $a = 1$,	erit $y = 5 - 2z$,	et $x = 3979 + 3z$,	3	ſcil. 0, 1, 2,
$a = 2$,	$y = 10 - 2z$,	$x = 3958 + 3z$,	6	0, 1, 2, 3, 4, 5,
$a = 3$,	$y = 15 - 2z$,	$x = 3937 + 3z$,	8	0, 1, &c.
$a = 4$,	$y = 20 - 2z$,	$x = 3916 + 3z$,	11	0, 1, &c.
$a = 5$,	$y = 25 - 2z$,	$x = 3895 + 3z$,	13	0, 1, &c.
&c.	&c.	&c.		

Manifeſtum eſt quod valores z reſtringuntur tantum per ſeriem valorum y donec numerus abſolutus in ſerie valorum x decreſcendo evaſerit negativus, quod fiet poſt terminum progreſſionis $\frac{3979}{21} + 1 = 190$. Sit

			valores z.
$a = 189$	$y = 945 - 2z$	$x = 31 + 3z$	473
$a = 190$	$y = 950 - 2z$	$x = 10 + 3z$	476
$a = 191$	$y = 955 - 2z$	$x = -11 + 3z$	$477 - 3 = 474$
$a = 192$	$y = 960 - 2z$	$x = -32 + 3z$	$480 - 10 = 470$
$a = 193$	$y = 965 - 2z$	$x = -53 + 3z$	$482 - 17 = 465$
$a = 194$	$y = 970 - 2z$	$x = -74 + 3z$	$485 - 24 = 461$
$a = 195$	$y = 975 - 2z$	$x = -95 + 3z$	$487 - 31 = 456$
&c.	&c.	&c.	&c.
$a = 295$	$y = 1475 - 2z$	$x = -2195 + 3z$	$737 - 731 = 6$
$a = 296$	$y = 1480 - 2z$	$x = -2216 + 3z$	$740 - 738 = 2$.

Ad terminum 190 *um* uſque liquet quod inter valores z, cenſeri poteſt zero, ſalvis valoribus poſſibilibus y et x, poſtea vero non item, ex eo quod valores x, ob terminum abſolutum jam negativum, emergent negativi ſeu impoſſibiles. Porro, ex valoribus ſubſequentibus z ope progreſſionis y elicitis tot auferri debent quot pariunt negativos valores x. Sic in termino 191 *mo*, tres valores z, ſcilicet 1, 2, 3 qui negativum exhibent $-11 + 3z = x$ demendi ſunt a 477 valoribus ope $955 - 2z = y$ comparatis, et inde reſtant 474 tantum valores.

Denique, notandum eſt quod termini 190 valorum z, ex binis progreſſionibus arithmeticis conflantur, ſcilicet 3, 8, 13, &c. et 6, 11, 16, &c. Ex binis etiam termini reliqui $296 - 190 = 106$, ſcilicet 474, 465, 456, &c. et 470, 461, 452, &c. Quatuor autem progreſſionum ſumma dabit omnes valores z et idcirco problematis ſolutiones,

$$3,\ 8,\ 13,\ \text{-}\ 473 = \frac{\overline{3+473}\times 95}{2} = 22610$$

$$6,\ 11,\ 16,\ \text{-}\ 476 = \frac{\overline{6+476}\times 95}{2} = 22895$$

$$474,\ 465,\ \text{-}\ 6 = \frac{\overline{474+6}\times 53}{2} = 12720$$

$$470,\ 461,\ \text{-}\ 2 = \frac{\overline{470+2}\times 53}{2} = \underline{12508}$$

$$70733$$

115. Huic quæſtioni 70734 ſolutiones tribuit *Simpſon Algeb. p.* 200, edit. 2dæ. Virum acutiſſimum ſane et cui multum debet Matheſis, præ feſtinatione effugit, zero minorem limitem q (ſeu a) deſignare non poſſe. Hoc enim poſito, valor generalis y, ſcilicet $5q - 2z$ evaderet negativus ſeu impoſſibilis. Pariter lapſus eſt ſtatuendo ſolutiones 60 problematis penultimi, 59 tantum admittentis; nam ubi $z = 14$, unicum ſortietur valorem y, ſcilicet 5, et x, ſcilicet 7. Sed

Opere in longo fas eſt obrepere ſomnum.

116. Si quærantur valores integri et affirmativi, x, y et z, quæſtio 70395 ſolutiones ſortietur, nam quatuor progreſſiens arithmeticæ evadent

2, 7, 12 - - - 472 = 22515
4, 9, 14 - - - 474 = 22705
474, 465, 456 - 6 = 12720
469, 460, 451 - 1 = 12455

70395.

117. Notandum eſt quod in quæſtionibus trium quantitatum incognitarum ſolvendis a methodo *Moivreana* deflexímus, utpote quæ ſolutiones juſto pauciores elicitura ſit, niſi quantitas z in generalibus valoribus y et x, prodeat iiſdem affecta ſignis; aut ſubſtituendo valorem binomium ad idem in utriſque revocetur ſignum: cæterum haud raro prodibit z diverſis affecta ſignis, ut in exemplis allatis; et aliquando nullo artificio, notante *Dodſon*, *p.* 22, ad eadem ſigna in utriſque revocari poteſt.

His veſtigiis inſiſtendo eruere poſſimus ſolutiones æquationum quatuor quantitates incognitas involventium, cæterum quum parum utiles ſint, adjungam potius problema cujus uſus in Chronologia egregius.

PROB. VIII.

118. *Invenire numerum quendam qui per diviſores duos pluresve datos ſucceſſive diviſus, reſidua ſortietur numeris quoque datis reſpective æqualia.*

Ex. gr. Quotus annus *Periodi Juliani* erit ſimul tertius *Cycli Solis*, ſeptendecimus *Cycli Lunæ*, et undecimus *Cycli Indictionis*; deſignantibus nimirum Cyclis hiſce annorum 28, 19, et 15 reſpective, diviſores datos, et numeris 3, 17, et 11, reſidua data.

Sit n numerus qui per 28 et 19 diviſus ſortietur reſidua 3 et 17 reſpective, eruntque $\frac{n-3}{28}$ et $\frac{n-17}{19}$ integri. Finge $x = \frac{n-3}{28}$ et $y = \frac{n-17}{19}$. Unde prodibit $n = 19y + 17 = 28x + 3$, et $y = \frac{28x-14}{19}$ integer, dempto autem integro $\frac{19x-19}{19}$ erit reſiduum $\frac{9x+5}{19} = a$ integer, adeoque $x = \frac{19a-5}{9}$ integer, et reſiduum $\frac{a+4}{9} = b$ integer, et $a = 9b - 4$. Unde ſubſtituendo, erit $x = 19b - 9$, $y = 28b - 14$, et $n = 532b - 249$. Hinc erit b affirmativus, & ſubſtituto minimo valore ejus, ſcilicet 1, evadet $n = 532 - 249 = 283$, qui utique conditiones diviſorum 28 et 19 implebit.

Porro, ſi huic numero 283 addatur aut diviſor primus 28, aut multiplus ejus quivis, adhuc explebit ſumma conditionem diviſoris primi. Similiter, ſi eidem 283 addatur aut diviſor ſecundus 19, aut multiplus ſecundi quilibet, explebit ſumma conditionem diviſoris ſecundi; et proinde ſi eidem addatur 532 factum ex utriſque diviſoribus in ſe ductis, aut quivis multiplus ejus, puta $532z$, deſignante nimirum z integrum quemcunque affirmativum, perinde explebit ſumma diviſoris utriuſque conditiones.

Finge nunc $532z + 283$ conditionibus diviſoris primi et ſecundi ſatisfacientem, eſſe numerum quæſitum, atque idcirco conditioni tertii quoque diviſoris 15 reſpondere; eritque $\frac{532z + 283 - 11}{15}$ integer, adeoque dempto integro

$\frac{525z + 270}{15}$ erit $\frac{7z + 2}{15}$ integer $= c$, et $z = \frac{15c - 2}{7}$; atque iterum $\frac{c - 2}{7} = d$ integer, et $c = 7d + 2$: unde ſubſtituendo erit $z = 15d + 4$. Hinc erit d vel zero, vel affirmativus; et minimus valor z, evaneſcente d, $= 4$. Evadet igitur numerus quæſitus $532z + 283 = 2128 + 283 = 2411$, qui nimirum annum periodi Juliani quæſitum indicabit.

Similiter, Datis diviſore quarto reſiduoque ſibi proprio, invenire licet numerum qui conditionibus quatuor diviſorum ſatisfaciet, ſi numero jam invento 2411 addatur multiplus quidam 7980 ex tribus diviſoribus datis conflati; atque ita deinceps.

Sin numerus hoc pacto inventus minimum multiplum communem omnium diviſorum datorum tranſcendat, non erit minimus qui cunctas explebit conditiones; dividendo autem hunc numerum per minimum multiplum communem, reſiduum peracta diviſione dabit minimum. Et ſi huic reſiduo addatur deinceps diviſor communis, inveniri poſſunt ad libitum numeri qui problematis conditiones exſolvent.

119. Traditis jam ſolutionibus problematum indeterminatorum, quæ ad æquationes ſimplices deducunt, nonnulla ex *Diophantæis*, quæ circa numeros quadratos verſantur, et ex quadraticarum reductione pendent, adjungere libet.

De numeris rationalibus tantum agit *Diophantus* ablegatis ſurdis.

PROB. I.

120. *Numerum quadratum datum in duos quadratos dividere.*

Sit quadratus dividendus 25, et q. quæſiti x^2 et y^2, adeoque $x^2 + y^2 = 25$. Finge jam latus alterutrius quadrati quæſiti eſſe æquale ſummæ vel differentiæ lateris quadrati dati et lateris alterius quæſiti quolibet numero præter unitatem affecti, puta $y = 5 \pm 2x$ vel $2x \pm 5$. Quadrando igitur hunc valorem y, et ſubſtituendo pro y^2, evadet

$$5x^2 \pm 20x + 25 = 25.$$

Et tranſponendo $5x^2 = \mp 20x$
Et dividendo - - $x = \mp 4$, adeoque $y\ (= 5 \pm 2x) = \mp 3$.

Et horum quadrata 16 et 9 conficiunt 25.

Similiter, ſit $y = 5 - \frac{1}{2}x$, per reductionem evadet $x = 4$ et $5 - \frac{1}{2}x = 3$. Sit $y = 5 - 3x$, erit $x = 3$ et $5 - 3x = -4$. Sit $y = 4x - 5$, eritque $x = \frac{40}{17}$ et $4x - 5 = \frac{75}{17}$ quorum quadrata $\frac{1600}{289}$ et $\frac{5625}{289}$ abſolvunt 25. Atque ita deinceps. Sed quo magis ab unitate recedunt coefficientes x, in valores fractos eo profuſiores excurrent latera quæſita.

121. Quo percipiatur ſingulare *Diophanti* artificium in hac ſolutione, obſervandum eſt, 1^{mo}. Latus quadrati dati in membrum unum lateris ficti adſumi, ad tollendum ex æquatione reſultante quadratum datum, quo pacto æquatio

quadratica, ſublato termino abſoluto deprimetur ad ſimplicem, et inde eruetur valor x per diviſionem ſolam, qui idcirco ſemper erit rationalis.

2. Debere coefficientem x membri alterius lateris ficti eſſe majorem minoremve unitate; nam ſi ponatur unitas, orietur valor x in æquatione reſultante æqualis lateri ipſius quadrati dividendi, et proinde ipſum latus fictum reperietur nihil. Sic poſito $y = 5 - x$, per reductionem prodibit $x = 5$, et $5 - x = 0$.

3. Poſſe latus fictum eſſe binomium vel reſiduum, ex priore enim emergent latera negativa, ex poſteriore eadem poſitiva, quorum utraque æquo jure quæſtionem ſolvent; utplurimum vero ponitur reſiduum, quam formam ipſe adhibet *Diophantus*.

PROB. II.

122. *Numerum datum ex duobus quadratis conflatum in alios duos quadratos dividere.*

Sit numerus dividendus $13 = 4 + 9$. Sumptis lateribus quadratorum datorum 2 et 3, ponantur latera quæſitorum quadratorum $x - 2$ et $2x - 3$. Horum quadrata ſimul $5x^2 - 16x + 13 = 13$, et per reductionem prodibit $x = \frac{16}{5}$; adeoq. $x - 2 = \frac{6}{5}$ et $2x - 3 = \frac{17}{5}$, quorum quadrata $\frac{36}{25}$ et $\frac{289}{25}$ ſimul $= \frac{325}{25} = 13$. At ſi ponantur latera $x + 2$ et $2x - 3$, erit $5x^2 - 8x + 13 = 13$, et per reductionem $x = \frac{8}{5}$ adeoque $x + 2 = \frac{18}{5}$ et $2x - 3 = \frac{1}{5}$

at horum quadrata quoque $\frac{324}{25}$ et $\frac{1}{25}$ conficiunt 13. Atque ita ſemper prodibunt bina quadratorum paria, quorum alterutrum quæſtioni ſatisfaciet, prout latera ficta ponantur vel utraque reſidua, vel alterum binomium exiſtente altero reſiduo.

123. Cæterum cavendum eſt 1mo. Ne idem ſtatuatur coefficiens x in utroque latere ficto; alioquin latera quadratorum quæſitorum non prodibunt diverſa a datis. Sic poſitis $x + 2$ et $x - 3$, evadent latera quæſita 3 et $\pm$ 2 quæ jam data ſunt.

2. Ne coefficientes x in utroque latere, ubi hæc latera ſunt reſidua, ſint inter ſe in ratione laterum datorum, aut altero exiſtente binomio, in ratione ſummæ datorum laterum ad eorundem differentiam; alioquin idem ſequetur abſurdum, quod ſcilicet latera quæſita emergent eadem ac data.

P R O B. III.

124. *Invenire numerum quadratum qui adjecto quolibet numero conficiat quadratum.*

Sit quadratus quæſitus x^2, conficiatque quadratum $x^2 + 3$. Finge latus poſterioris, $x - n$ (deſignante n numerum quemcunque) et quadrando, erit

$$x^2 - 2nx + n^2 = x^2 + 3,$$

Et tranſponendo - $2nx = n^2 - 3$

Adeoque - - - - - - $x = \frac{n^2 - 3}{2n}$.

Poſito igitur $n = 6$, erit $x = \frac{11}{4}$, adeoq. $x^2 = \frac{121}{16}$, et $x^2 + 3 = \frac{169}{16}$ quadratus nimirum cujus latus eſt $\frac{13}{4}$. Variante autem utcunque n, variari quæſtionis ſolutiones manifeſtum eſt. Similiter, *invenire licet quadratum a quo dempto numero dato reſiduum erit quadratus.*

125. In quæſtionibus prioribus negotium erat æquationem quadraticam ad ſimplicem deprimere, tollendo quadratum datum, in noviſſimâ vero, quadratum quæſitum. Æquationum quadraticarum reductionem nuſquam in ſex libris *Arithmeticorum* jam extantibus ex profeſſo tradit *Diophantus*, harum tamen regulam tali latuiſſe viro minime credendum eſt; & ipſe quidem pollicetur, Def. 11. ſe quadraticarum ſolutiones poſtea commonſtraturum. Satis profectò evincunt problematis inſequentis limitationes, cujus ſolutionem mira ſubtilitate expedit *Diophantus*, Lib. 5, Quæſt. 33, illum probe quadraticarum reductionem cognoviſſe.

Οκταδράχμους καὶ πενταδράχμους χοέας τὶς ἔμιξε
  Τοῖς προπολοῖσι πιεῖν χρῆστ' ἀποταξάμενος·
Καὶ τιμὴν ἀπέδωκεν ὕπερ πάντων τετράγωνον,
  Τὰς ἐπιταχθείσας δεξάμενον μονάδας,
Καὶ ποιοῦντα πάλιν ἕτερον σε φέρειν τετράγωνον,
  Κτησάμενον πλευρὰν σύνθεμα των χοέων.
Ὥστε διάστειλον τοὺς ὀκταδράχμους πόσοι ἦσαν
  Καὶ πάλι τοὺς ἑτέρους παῖ, λέγε πενταδράχμους.

Epigrammatis propoſitum tale eſt. Emit quidam duos cados vini, unius quidem choam drachmis 8, alterius vero choam drachmis 5, in uſum famulorum miſcendos; et pro omnibus chois perſolvit numerum quadratum, cui addendo numerum datum 60, fit alter quadratus, latus habens ſummam choarum. Quæritur numerus choarum utriuſque pretii.

Sit x ſumma choarum, et cum pretium ſummæ + 60 adæquat x^2, erit pretium ſummæ $x^2 - 60$. Finge hoc erogari pro chois 5 drachmarum ſolis, et prodibit numerus choarum quinque drachmis $\frac{x^2 - 60}{5}$ major quam x, (cum x contineat choas aliquot 8 drachmis) erit igitur $x^2 - 60$ major quam $5x$, ſive x^2 major quam $5x + 60$, et per reductionem, x major quam $\frac{5}{2} + \sqrt{\frac{25}{4} + 60}$, id eſt, major quam 10,7. Similiter, erit numerus choarum ſolis 8 drachmis $\frac{x^2 - 60}{8}$ minor quam x, adeoque x^2 minor quam $8x + 60$, et per reductionem, x minor quam 12,7. Oportet igitur valorem x inter limites 10,7, et 12,7 inveniri, et proinde *a fortiori*, inter limites integros arctiores 11 et 12, quos ponit *Diophantus*.

Sed altera quoque datur reſtrictio valoris x; ſcilicet, quod $x^2 - 60$ debet eſſe numerus quadratus. Finge igitur latus hujus $x - y$, et quadrando, erit $x^2 - 2xy + y^2 = x^2 - 60$, adeoque per prob. *præced.* $x = \frac{y^2 + 60}{2y}$. At neceſſe eſt $\frac{y^2 + 60}{2y}$ $(= x)$ eſſe majorem quam 11, ſive y^2 majorem quam $22y - 60$, adeoque per reductionem, y majorem quam $11 + \sqrt{61}$ ſive 18,7. et pari ratione, $\frac{y^2+60}{2y}$ eſſe minorem quam 12, ſive y^2 minorem quam $2y - 60$, adeoque per reductionem, y minorem quam 21,2. Oportet igitur numerum aſſumptum y inter limites 18,7 et 21,2. conſiſtere, adeoque inter limites integros *Diophantæos* 19 et 21.

Puta jam y medium obtinere locum inter 19 et 21, ſcilicet 20, et ſubſtituto hoc valore y in æquationem $x^2 - 2xy + y^2 = x^2 - 60$, prodibit $x^2 - 40x + 400 = x^2 - 60$, adeoque $x = 11\frac{1}{2}$, et proinde $x^2 - 60 = 132\frac{1}{4} - 60 = 72\frac{1}{4}$. Ergo ſumma choarum ſit $11\frac{1}{2}$, intra limites nempe præſcriptos, et totum pretium $72\frac{1}{4}$ drachmarum, ſcilicet numerus quadratus (utpote latus habens $8\frac{1}{2}$.)

Denique, quo inveniatur numerus choarum utriuſque pretii, ſit z numerus choarum 5 drachmis, et erit $11\frac{1}{2} - z$ numerus choarum 8 drachmis, pretium priorum erit $5z$ drachm. poſteriorum autem $92 - 8z$ drachm. adeoque pretium ſummæ $5z + 92 - 8z$ ſive $92 - 3z = 72\frac{1}{4}$, et proinde $3z = 19\frac{3}{4}$, et $z = 6\frac{7}{12}$, et inde $11\frac{1}{2} - z = 4\frac{11}{12}$. Numerus igitur choarum 5 drachmis evadit $7\frac{7}{12}$, choarum autem 8 drachmis $4\frac{11}{12}$.

Alii quoque dantur limites valoris y. Nam $11 \pm \sqrt{61}$ evadit 18,7 vel 3,3. et $12 \pm \sqrt{84}$ evadit 21,2 vel 2,8. Conſiſtere igitur poteſt y inter 3,3 et 2,8. Puta $y = 3$, et ſubſtituendo $x - 3$ ut antea, exurget $x = 11\frac{1}{2}$. Sin limites alterutros tranſgrediatur y, exulabit x a limitibus præſcriptis, et numerus choarum vel 8 drachmis vel 5 drachmis emerget negativus, quod impoſſibile eſt.

126. Pauca hæc exempla in ſpecimen ſagaciſſimi *Diophanti* ingenii ad quæſtiones vel ſpinoſiſſimas expediendas ſufficiant. Haud parum quidem intentionis animi lectoribus etiam mathematice doctis requiritur ad ſolutiones complures *Diophantæas* capiendas, præter materiæ ſubtilitatem, ex eo quod perpaucas notas algebraicas adhibet, adeo ut plures ſæpe quantitates incognitæ eadem deſignantur ſpecie. Sic in propoſitione noviſſima, quantitates quas notis x, y, et z, diſtinximus, omnes una ſpecie ς deſignat *Diophantus*.

127. Quo tempore inventa fuit Arithmetica ſpecioſa ſeu Algebra, in incerto eſt. Hoc ſaltem præſumendum eſt, inventorem non fuiſſe *Diophantum*, quippe qui medias in res, haud ſecus ac notas lectorem rapit, et nullis fere præmiſſis principiis, quæſtionibus ſolvendis ſtatim ſe accingit. Innuit tantum, idque obiter, quaſi demonſtratione non egens, " quod minus in minus dat plus, in plus autem, minus." Et præterea monet, Def. 10, " Expedit eum qui hoc negotii ſit aggreſſurus, in additione ſubductione et multiplicatione ſpecierum jamdudum exercitatum eſſe." Niſi forſan inſtituebantur elementa artis neceſſaria in alio quodam opere *Diophanti* jam deperdito, de *Arithmeticæ praxi*; de quo meminit *Theon*.

128. Priſcis Mathematicis *Platoni*, *Euclidi*, *Archimedi*, &c. hanc Artem innotuiſſe, recentiorum contenderunt multi. *Schootenius*, *Nonius* aliique, ad quorum ſententiam accedit quoque *Wallis*, cum certo ſcirent ex demonſtrationum et conſtructionum ſubtilitate et elegantia, analyſin *quandam* Veteribus probe cognitam eſſe, neque aliam noſſent præter eam quæ Algebræ ſpecioſæ ope inſtituitur, nullumque hujus veſtigium ante *Diophantum* inveniſſent; Veteres admirationis aucupandæ temere inſimularunt, utpote qui inventa ſua vulgari tantum ſyntheſeos forma tradebant,

hoc conſilio, ut artem ſuam analyticam a Poſteris celarent. Luculenter autem conſtat ex *Pappi Alexandrini* præfatione ad Lib. 7, *Collectionum Mathematicarum*, veteres Geometras in ſolutionibus Theorematum & Problematum propoſitorum inveſtigandis analyſin *geometricam* uſurpaſſe; eamque plenius expoſuiſſe *Euclidem*, nempe Elementorum ſcriptorem, *Apollonium Pergæum*; *Ariſtæum* ſeniorem, et *Eratoſthenem*; quorum opera analytica 33 libris contenta recenſet *Pappus*, qui & ipſe naturam analyſeos & ſyntheſeos Veterum optime explicat.

Analyſin Veterum ita collaudat *Halley* Geometer ſimul et Algebraiſta ſummus, in Præfatione ad editionem ſuam *Apollonii Pergæi*, *De Sectione Rationis*.

" Methodus hæc cum *Algebra Speciosa* facilitate conten-
" dit, evidentiâ vero & demonſtrationum elegantiâ eam
" longe ſuperare videtur: ut abunde conſtabit, ſi quis con-
" ferat hanc *Apollonii* doctrinam *De Sectione Rationis* cum
" ejuſdem Problematis analyſi Algebraicâ, quam exhibuit
" *Cl. Walliſius*, *Tom* 2, *Operum Mathemat. cap.* 54, *pag.* 220."

Conſulat quoque lector Appendicem ad *Simpſoni Conica*, *edit. 2da.* ubi trium propoſitionum ſolutiones Algebraicas cum iis quæ per analyſin geometricam fiunt conferre licet. Miro certe veteris analyſeos flagrabat amore *Halley*, qui immane quoddam, imo vix credibile eſt auſus, nempe opus interpretandi Verſionem Arabicam *Apollonii de Sectione Rationis* (*Apollonii* opere περὶ λόγȣ ἀποτομῆς deperdito) linguæ Arabicæ ipſe prorſus ignarus! & quod vel Arabice doctiſſimum abſterruerit, codex ille peſſimâ manu exaratus, punctiſque diacriticis deſtitutus, & diagrammatum lineas literis male ſignatas & diſtinctas habens, mendis lacuniſque ſcatebat. Verùm beneficio ſchedarum aliquot, quas, paucis ante annis, latine verterat *D. Bernardus*, Aſtronomiæ Profeſſor *Savilianus*, quæ, clavis ad inſtar, aditum illi aperuerunt ad *Apollonii* mentem inveſtigandam; excerptis vocibus illis de quibus ex verſione *Bernardi* liquido conſtabat, dein ad argumentum reſpiciendo, & notas obſcuriores iterum iterumque ſecum evolvendo, quid voluerunt paulatim deprehendit, & hac quaſi *deciphrandi* methodo totum librum perlegit, eundemque denuo pedetentim percurrendo, opus integrum abſque alterius cujuſpiam auxilio ad finem perduxit.

129. Obſervandum reſtat, quod reſolutio æquationum indeterminatarum in applicatione Algebræ ad Geometriam apprime utilis eſt, cum præcipuarum curvarum æquationes ſunt indeterminatæ. Methodus tamen generalis tales ſolvendi hactenus deſideratur ultra problemata primi gradus. Licet enim occurrunt ſolutiones problematum ſecundi, tertii et etiam quarti gradus apud *Diophantum* ejuſque Commentatores, ſunt tantum particulares. Qui plura autem de hiſce deſiderat, conſulat diſſertationes *Euleri*, *Acad. Petrop. Tom* 6, *veter.* et *Tom* 9, *nov.* Et *La Grange Acad. Berlin. Tom* 23.

ANALYSIS ÆQUATIONUM.

PARS TERTIA.

130. Hactenus expoſuimus modos ordinandi & concinnandi æquationes, eaſdemque reducendi ſi modo ſecundum non tranſcendant gradum. Antequam vero reſolutionem æquationum dimenſionum ſuperiorum aggrediamur, neceſſe erit *præparationem æquationum* inſtituere, præmittendo aliqua de naturâ radicum & præcipuis tranſmutationibus æquationum.

DE NATURA RADICUM ÆQUATIONIS.

131. *Radix eſt numerus vel ſpecies qui ſi in æquatione pro ſpecie radicem ſignificante ſubſtituatur, efficiet omnes terminos evaneſcere.*

Sic æquationis $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, unitas eſt radix, quoniam ſcripta pro x producit $1 - 1 - 19 + 49 - 30$, hoc eſt nihil. Sed

132. *Æquationis ejuſdem plures eſſe poſſunt radices.*

Nam ſi in hac eadem æquatione $x^4 - x^3$, &c. $= 0$, pro x ſcribas numerum 2, & pro poteſtatibus x ſimiles poteſtates numeri 2, producetur $16 - 8 - 76 + 98 - 30$, hoc eſt nihil. Atque ita ſi pro x ſcribas numerum 3 vel numerum negativum -5, utroque caſu producetur nihil, terminis affirmativis & negativis in his quoque caſibus ſe mutuo deſtruentibus. Proinde cum numerorum 1, 2, 3 vel — 5, quilibet ſcriptus in æquatione pro x impleat conditionem ipſius x, efficiendo ut termini omnes æquationis conjunctim æquentur nihilo, erit quilibet eorum radix æquationis.

Atque ita æquationis $x^3 = 1$, ſeu $x^3 - 1 = 0$, tres radices erunt 1, $\frac{-1+\sqrt{-3}}{2}$, vel $\frac{-1-\sqrt{-3}}{2}$, nam ſi ſcribas 1 pro x, producetur $1 - 1$, hoc eſt nihil; ſi ſcribas $\frac{-1+\sqrt{-3}}{2}$ pro x, producetur $\frac{8}{8} - 1$, hoc eſt nihil; et ſimiliter, ſi ſcribatur $\frac{-1-\sqrt{-3}}{2}$ pro x. In ſpeciebus quoque idem fiet; ſic æquationis $x^4 - px^3 (-p^2 + q)x^2 + p^3x - p^2q = 0$, ſunt p, $-p$, $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$, vel $+\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$ radices; nam ſcribendo p pro x producetur $p^4 - p^4 - p^4 + p^2q + p^4 - p^2q$, hoc eſt nihil; & ſic deinceps.

Hinc liquet quantitates reales & impoſſibiles æque fungi poſſe munere radicum, efficiendo omnes æquationis terminos evaneſcere.

133. *Eandem vero æquationem plures habere poſſe radices æquum eſt, cum plures eſſe poſſunt ejuſdem problematis ſolutiones.* § 34.

134. Raro tamen *omnes* æquationis radices, etiamſi *reales & affirmativæ* ſint, problema a quo deducta eſt iſta æquatio ſolvent. § 35 & 44. Sed hoc fit ob limitationem aliquam in problemate quod ad æquationem nil ſpectat. Nam ſi æquatio non complectitur omnes conditiones problematis,

non neceſſe eſt ut omnium conditionum teneatur limitibus, quicquid amplius eſt in problemate quam in æquatione poteſt illud limitibus arctare, hanc non item. § 32. Veruntamen neceſſe eſt æquationem qua reſpondetur tot habere radices quot ſunt quæſitæ quantitatis caſus diverſi ab iiſdem datis pendentes, & eadem argumentandi ratione determinandi. Nam quiſque caſus diverſus ſolutionem diverſam poſtulat. Et quidem caſus problematum indigitare a datis diverſos, eſt proprium munus radicum *negativarum*. § 43 & 93. Hinc in quæſtionibus geometricis et phyſicis radices affirmativæ & negativæ ad plagas oppoſitas tendere ſolent. § 40, & 42.

135. Hæc æquationum copia ſimul quæſita & non quæſita complectentium inveniendis diverſis caſibus ejuſdem problematis generalis inſervire poteſt. Nihilominus fatendum eſt, incommodo ſæpe laborare hanc copiam, ubi ſolutionem quæſitam aliis nexam atque implicitam non ſine calculo valde operoſo ſecernere atque evolvere licet; ut bene quidem monet *D'Alembert* *. " Les quantités né-
" gatives — montrent tout à la fois & la richeſſe de cette
" ſcience, qui fait trouver dans la ſolution du problême,
" juſqu' aux choſes qu'on ne demandoit pas, & en même
" tems, ſi on oſe le dire, l'imperfection du calcul, qui en
" donnant ce qu'on ne cherche pas, & qu'on ne lui de-
" mande point, ne donne pas toujours ce qu'on lui de-
" mande avec toute la perfection qu'on pourroit exiger.
" C'eſt qui n'arrive que trop dans les queſtions algébri-
" ques ; la ſolution d'un problême, qui n'en a quelquefois
" réellement qu'une ſeule poſſible (dans le ſens où il a été
" propoſé) eſt ſouvent incorporée & comme amalgamée

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* *Melanges de Literature*, vol. 5, p. 230.

" avec pluſieurs autres ſolutions de problêmes analogues,
" mais differens ; ſolutions qui enveloppant & maſquant,
" pour ainſi dire, la premiere, la rendent plus difficile à
" découvrir. Ceux qui ont quelque connoiſſance de ce
" qu'on appelle en Algebre la théorie des équations, ſavent
" par expérience la vérité de ce que nous venons de dire."

136. *Æquationum vero radices ſæpe impoſſibiles eſſe æquum eſt, ne caſus problematum qui ſæpe impoſſibiles ſunt exhibeant poſſibiles*, § 34 & 46.

Et hic quidem obſervandum eſt, quod in *Algebrâ* radix impoſſibilis eſt tantum nota ſeu ſymbolum operationis quæ in certo quodam caſu fieri nequit, ejuſque idcirco munus eſt *geneſin* quantitatis quæſitæ, & *limites* intra quos conſiſtere poteſt, indicare: *Geometria* vero radices impoſſibiles proprie non agnoſcit, ex eo quod ſolutio quæſtionis geometrice tantum exponi poteſt ubi ſit realis. Ut ſi rectæ & circuli interſectio determinanda eſſet, deſignantibus p perpendiculum a centro in rectam demiſſum, r radium ductum ad interſectionem, & x diſtantiam interſectionis a p, habetur æquatio interſectionem definiens $x^2 = r^2 - p^2$, per *Eucl. Lib.* 1, *Prop.* 47. Erit autem interſectio poſſibilis ſi p ſit minus quam r, ſin majus, fiet impoſſibilis ; & æquationis duæ radices quæ interſectiones duas determinant, nempe $x = \pm \sqrt{r^2 - p^2}$, debent eſſe perinde poſſibiles vel impoſſibiles ut rem ipſam vere experimant. Cæterum haſce radices geometrice tantum exponere fas eſt ubi interſectio eſt poſſibilis.

137. Aliquando fieri poteſt, velut in cubicarum *caſu irreducibili*, quod radices *reales* ſub *ſpecie impoſſibili* adumbrentur ; cæterum hoc fit ex eo quod conditiones in ſolutionem æquationis adſcitæ, ſibi in iſto caſu refragantur.

DE NUMERO RADICUM ÆQUATIONIS.

138. *Poteſt æquatio tot habere radices quot ſunt dimenſiones ejus, & non plures; ſi modo termini ejus ex ſimplicibus poteſtatibus integris conſtantur.*

Æquationem ſimplicem unam ſortiri radicem, quadraticam vero binas, ex Parte I. manifeſtum erat; unde in æquationibus dimenſionum ſuperiorum, ſimilem eſſe rationem radicum, per analogiam conjicere liceret. Cæterum numerus et natura radicum ex *generatione æquationis* optime intelligetur.

Ut ſi ſcire vellemus quomodo generetur æquatio cujus radices ſint 1, 2, 3, vel -5, obſervando retrogradam compoſitionis viam, ſupponamus x ambigue ſignificare numeros illos, ſeu eſſe $x = 1$, $x = 2$, $x = 3$, vel $x = -5$, adeoque per tranſpoſitionem $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, vel $x + 5 = 0$. Et multiplicando hæc in ſe, prodibit multiplicatione $x - 1$ in $x - 2$ hæc æquatio $x^2 - 3x + 2 = 0$, quæ duarum eſt dimenſionum ac duas habet radices 1 & 2; nam ſcriptâ 1 pro x evadit factor prior $1 - 1 = 0$, vel ſcripto 2 pro x evadit factor poſterior $2 - 2 = 0$, unde in alterutro caſu factum ſub utriſque nihil erit, ſcil. $1 - 3 + 2 = 0$, vel $4 - 6 + 2 = 0$. Porro, ductis in ſe tribus factoribus $x - 1$, $x - 2$, & $x - 3$, generatur $x^3 - 6x^2 + 11x - 6 = 0$, æquatio trium dimenſionum totidemque radicum 1, 2 vel 3; nam ſi x ambigue ſignificat 1, 2 vel 3, evaneſcet primus, ſecundus aut tertius factor reſpective, at quolibet factore evaneſcente evaneſcet factum ſub tribus. Denique ductis in ſe quatuor factoribus $x - 1$, $x - 2$, $x - 3$. & $x + 5$ fit $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, ut ſupra, æquatio quatuor dimenſionum totidemque radicum 1, 2, 3, vel -5. Nam ſi x ambigue, &c. Cum igitur hæc æquatio ex tot factoribus ſimplicibus generatur quot ſunt dimenſiones ejus, & factor quiſque ſimplex unam ſuppeditat radicem, ſequitur eam tot habere radices quot ſunt dimenſiones ejus. Non autem plures, cum præter hoſce numeros nullus eſt cujus ſubſtitutione factor aliquis evadit nihil; ubi vero factorum nullus nihil eſt, quod ſub omnibus continetur nihil eſſe non poteſt. Et ſimile eſt ratiocinium de omnibus æquationibus. Nam tali multiplicatione imaginari poſſumus omnes generari, quamvis factores ab invicem ſecernere ſolet eſſe difficillimum, & ipſum ſit quod æquationem reſolvere & radices extrahere: habitis enim radicibus habentur factores.

Sed revocare gradus ——
Hic labor, *hoc opus.*

139. Poſſunt quoque æquationes plurium dimenſionum, ex aliis inferiorum dimenſionum, illarum dimenſiones ſimul æquiparantium generari. Nam æquationem cubicam ex quadraticâ & ſimplici in ſe ductis; biquadraticam ex cubica & ſimplici vel ex duabus quadraticis in ſe ductis generari concipiamus; atque ita deinceps. Sic æquationis $x^4 - x^3 - 19x^2$ &c. factores ſunt $x^3 - 6x^2$ &c. et $x + 5 = 0$, vel $x^2 - 3x + 2 = 0$, & $x^2 + 2x - 15 = 0$. Itemque æquationis $x^3 - 1 = 0$, factores ſunt $x^2 + x + 1 = 0$, & $x - 1 = 0$. Factorum vero quadraticorum, &c. radices erunt radices æquationum.

140. Hanc generationem æquationum, in quadraticis ſaltem, ex principiis *Geometricis* inſuper confirmare & illuſtrare jam aggrediemur.

Nam ad quadraticas puras quod attinet, ſi habentur duæ quantitates x & q, quarum major ſit x, erit $x^2 - q^2 = (x + q)(x - q)$, per *Eucl. Lib.* 2. *Prop.* 5. At ſi x fiat æqualis q, evaneſcet tum $x^2 - q^2$ tum $x - q$. Unde legitime concludamus æquationem algebraicam $x^2 - q^2 = 0$, generari poſſe ex multiplicatione factorum ſimplicium $x + q$ & $x - q$, quorum alter poteſt eſſe nihil. Contra, $x^2 + q^2 = \frac{(x + q)^2 + (x - q)^2}{2}$, per *Eucl. Lib.* 2. *Prop.* 9. Minus recte igitur concludamus conſtitui poſſe æquationem algebraicam $x^2 + q^2 = 0$, neutiquam evaneſcente $\frac{(x + q)^2 + (x - q)^2}{2}$; aut generari poſſe multiplicatione duorum factorum ſimplicium, cum ex ſemiſumma quadratorum ſummæ & differentiæ x & q revera conflatur. Ergo prodeunt hujus factores $x \pm q\sqrt{-1}$ impoſſibiles, ut æquum eſt.

Eadem quoque obtinent in quadraticis affectis. Nam ſi data quædam quantitas p dividatur duas in partes inæquales quarum una ſit x, alteram deſignabit $p - x$, et deſignante q rectangulum partium, erit $q = px - x^2$, per *Eucl. Lib.* 2. *Prop.* 3, et horum differentia nempe $x^2 - px + q$ æquatur nihilo. Unde legitime concludamus æquationem algebraicam $x^2 - px + q = 0$, generari poſſe multiplicatione factorum $x - \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$ & $x - \frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$, quorum alteruter erit nihil, ob $x - \frac{1}{2}p = \pm \sqrt{\frac{1}{4}p^2 - q}$; & quorum uterque erit poſſibilis, cum q in hoc caſu eſt neceſſario minus quam $\frac{1}{4}p^2$, per *Eucl. Lib.* 2. *Prop.* 5. Contra, deſignantibus x & q latera trianguli angulum acutum comprehendentia, p ſegmentum majoris lateris x interceptum inter perpendiculum ab angulo oppoſito & angulum acutum, exponet $x^2 - 2px + q^2$ quadratum lateris reliqui angulo acuto oppoſiti, per *Eucl. Lib.* 2. *Prop.* 13.

At hoc latus, ſalvo triangulo nunquam evaneſcere poteſt. Ergo æquationis $x^2 - 2px + q^2 = 0$, factores $x - p \pm \sqrt{p^2 - q^2}$ prodeunt impoſſibiles, cum q^2 in hoc caſu, eſt neceſſario majus quam p^2, per *Eucl. Lib.* 1. *Prop.* 47. Et ex hypotheſi impoſſibili concluſionem impoſſibilem colligi mirum eſſe non debet.

141. Uſum æquationum originalium $x - a = 0$, $x^2 - 20x - 9 = 0$, in compoſitionem æquationum ſuperiorum dimenſionum, qualem explicuimus, primus patefecit ingens ille algebræ hodiernæ repertor ac inſtaurator *Vieta* *; qui methodum generalem invenit pro educendis radicibus ex æquatione qualibet, *ubique*, ut ait, *obſervando retrogradam compoſitionis viam.* Hujuſque veſtigiis inſiſtentes *Harriott*, *Oughtred*, aliique tam noſtrates quam extranei quæcunque de hac re ſcriptis mandarunt, a *Vieta* deſumpta debent agnoſcere; judice *Halley*, *Phil. Tranſ. An.* 1687. Nº 190. *Art.* 2. *Montucla*, &c.

Ex generatione æquationum, regulam inſequentem in analyſi multis nominibus perutilem, atque uberrimam elicuit *Harriott*.

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* *Vieta* vir fuit ingenioſa & profunda meditatione, cujus vi nihil illi inacceſſum in abſtruſioribus ſcientiis, nihil quod acumine mentis poſſet confici, difficile confectu fuit. Hujus tale dedit indicium. Bello inter Gallos Hiſpanoſque ingruente, literæ pleræque Hiſpanorum quibus præfectos ad Belgiam de conſiliorum ſuorum rationibus monebant, interceptæ, ob ingentem notarum multitudinem, (utebantur enim amplius 500, exoleto & incognito charactere exaratis) eos qui vulgo in his rebus induſtriam ſuam exercent prorſus eluſerunt. Itaque ad Vietam regis juſſu miſſæ ſunt, qui familiari ſibi in ſtudiis gravioribus meditatione illud totum expiſcatus eſt, & pleraſque deinceps alias maximi momenti, reperto ſemel arcano, nullo negotio interpretatus eſt. Quod res Hiſpanas totum biennium valde conturbavit. Qui, re tandem per literas viciſſim interceptas detectâ, *Magicis Artibus*, nam aliter fieri non potuiſſe, a Rege id factum, paſſim publicabant.

DE COEFFICIENTIBUS & SIGNIS TERMINORUM ÆQUATIONIS.

142. *Ubi coefficiens* primi *ſeu altiſſimi termini ſit unitas, coefficiens* ſecundi *termini æquationis, ſi ſignum ejus mutetur æqualis ſit aggregato omnium radicum ſub ſignis propriis ; ea* tertii, *æqualis aggregato rectangulorum ſub ſingulis binis radicibus ; ea* quarti *ſi ſignum ejus mutetur, æquales aggregato contentorum ſub ſingulis terni radicibus ; ea* quinti, *æqualis aggregato contentorum ſub ſingulis quaternis, &c. et denique ea* ultimi *æqualis contento unico ſub omnibus radicibus.*

Aſſumamus $x = a$, $x = b$, $x = -c$, vel $x = d$, ſeu tranſponendo, $x - a = 0$, $x - b = 0$, $x + c = 0$, vel $x - d = 0$, et ex horum continua multiplicatione generemus æquationes, ut ſupra.

$$\begin{array}{l} x - a \\ \times\, \underline{x - b} \end{array}$$

$$= \left\{ x^2 \begin{matrix} -a \\ -b \end{matrix} x + ab = 0. \right.$$ Æquatio quadrat.

$$\times\, \overline{x + c}$$

$$= \left\{ x^3 \begin{matrix} -a \\ -b \\ +c \end{matrix} x^2 \begin{matrix} +ab \\ -ac \\ -bc \end{matrix} x + abc = 0. \right.$$ Æquatio cubica.

$$\times\, \overline{x - d}$$

$$= \left\{ x^4 \begin{matrix} -a \\ -b \\ +c \\ -d \end{matrix} x^3 \begin{matrix} +ab \\ -ac \\ +ad \\ -bc \\ +bd \\ -cd \end{matrix} x^2 \begin{matrix} +abc \\ -abd \\ +acd \\ +bcd \end{matrix} x - abcd = 0. \right.$$ Æq. biquadrat.

&c.

143. Ex harum æquationum inſpectione conſtat, quod coefficiens ſecundi termini ſub ſignis mutatis, nimirum $+ a + b$ in quadratica, $+ a + b - c$ in cubica, $+ a + b - c + d$ in biquadratica, eſt ſumma omnium radicum; coefficiens tertii $ab - ac + ad$, &c. ſumma rectangulorum ſub ſingulis binis; ea quarti ſub ſignis mutatis, $- abc + abd$, &c. ſumma contentorum ſub ſingulis ternis; atque ea ultimi, nimirum $+ ab$ in quadratica, $- abc$ in cubica, $- abcd$ in biquadratica, contentum unicum ſub omnibus.

Et ſic univerſaliter: æquationis cujuſcunque $x^{n} . Px^{n-1} . Qx^{n-2} . Rx^{n-3} . Sx^{n-4}$ &c. $= 0$, coefficiens P, ſi ſignum ejus mutetur, æqualis ſit ſummæ radicum a, b, c, &c. ſub ſignis propriis; Q æqualis ſummæ rectangulorum ſub ſingulis binis ab, ac, &c. bc, bd, &c. cd, &c. R, ſi ſignum ejus mutetur, æqualis ſummæ contentorum ſub ſingulis ternis abc, abd, &c. bcd, bce, &c. cde, &c. S æqualis ſummæ contentorum ſub ſingulis quaternis $abcd$, &c. $bcde$, &c. $cdef$, &c. Et ſic in infinitum.

144. Hic obſervandum eſt, quod mutentur ſigna terminorum alternorum a ſecundo incipientium, quorum coefficientes ſunt aggregata contentorum ſub *impari* numero radicum ſignis mutatis; reliquorum vero terminorum non item. Nam in priore caſu, mutatis ſignis radicum numero *imparium*, mutantur ſigna ſingulorum contentorum, adeoque et aggregati contentorum in unoquoque termino; at in poſteriore caſu, ſive ſervatis ſive mutatis ſignis radicum numero *parium*, manent ſingulorum contentorum adeoque et aggregati ſigna. Sic $a \times b$ vel $- a \times - b = + ab$. At $a \times b \times - c = - abc$, & $- a \times - b \times c = + abc$. Et ſic deinceps.

Cor. I. Hinc cujuſlibet termini æquationis, puta termini m^1, ſive inter terminos pares ſive impares inveniatur, coefficiens æqualis erit aggregato contentorum ſub ſingulis $\overline{m-1}$ radicibus, cum ſignis mutatis. Eritque numerus terminorum æquationis omnes ſuos terminos habentis, uno plus quam numerus radicum ſeu dimenſionum.

Cor. II. Si omnes radices æquationis ſint affirmativæ, termini alterni a ſecundo incipientes erunt negativi, cæteri affirmativi; ſin omnes radices ſint negativæ, termini omnes erunt affirmativi.

Cor. III. Si radices ſint partim affirmativæ partim negativæ, præpollentibus affirmativis erit ſecundus terminus negativus, & vice verſa; æquipollentibus vero affirmativis & negativis, evaneſcet. Itemque, præpollentibus rectangulis affirmativis &c. erit tertius terminus affirmativus, & vice verſa; æquipollentibus vero evaneſcet. Et ſic deinceps.

Cor. IV. Si ultimus terminus æquationes deſit, ex radicibus ejus una erit nihil, aliter enim non evaneſceret contentum ſub omnibus. Si penultimus quoque deſit, ex radicibus duæ evaneſcent, aliter enim non evaneſcerent ſingula contenta ex quibus ſimul ſumptis conficitur penultimi coefficiens. Si tres poſtremi termini deſint, tres ex radicibus evaneſcent. Et ſic deinceps.

Cor. V. Æquationis cujuſcunque terminos nec fractos nec ſurdos habentis nulla radix fracta eſſe poteſt. Nam æquationis $x^n - Px^{n-1} + Qx^{n-2} - \&c. + U = 0$, radix ſit fractio $\frac{a}{b}$, ad minimos terminos ejus reducta; &

ſubſtituendo $\frac{a}{b}$ pro x, prodibit $\frac{a^n}{b^n} - \frac{Pa^{n-1}}{b^{n-1}} + \frac{Qa^{n-2}}{b^{n-2}}$ — &c. + $U = 0$; deinde multiplicando omnes terminos per b^{n-1} & tranſponendo, erit $\frac{a^n}{b} = Pa^{n-1} - Qa^{n-2}b$ + &c. — $Ub^{n-1} = 0$, cujus membrum poſterius eſt integer, adeoque & prius $\frac{a^n}{b}$. Unde erit b aut unitas, aut ex diviſoribus a^n. At in priore caſu non foret $\frac{a}{b}$ fractio, nec ad minimos terminos reducta in poſteriore caſu; contra hypotheſin. Poſſunt tamen radices eſſe ſurdæ. Sic æquationis $x^2 - x - 1 = 0$, radices ſunt $\frac{1 + \sqrt{5}}{2}$ & $\frac{1 - \sqrt{5}}{2}$.

Cor. VI. Æquationis cujuſcunque terminos nec fractos nec ſurdos habentis omnes radices non ſurdæ & radicum binarum rectangula, ternarumque aut plurium contenta erunt aliqui ex diviſoribus *integris* ultimi termini; atque adeo ubi conſtiterit nullum ultimi termini diviſorem eſſe aut radicem æquationis aut duarum radicum rectangulum pluriumve contentum, ſimul conſtabit nullam eſſe radicem radicumve rectangulum aut contentum niſi quod ſit ſurdum.

Cor. VII. Æquationis cujuſcunque terminos nullos impoſſibiles habentis radices impoſſibiles, ſiquæ ſint, erunt 1. Numero pares. Nam ultimus terminus utpote contentum ſub omnibus radicibus impoſſibilitate non vacaret, ſi compoſitionem ejus ingrederetur numerus impar factorum impoſſibilium. 2. Paris cujuſque partes impoſſibiles eædem erunt & contrariis ſignis affectæ. Aliter enim hæ partes per additionem non evaneſcerent ex termino ſecundo æquationis. 3. Paris cujuſque partes reales, ſi quæ ſint, erunt eædem & iiſdem ſignis affectæ. Nam rectangula & contenta impoſſibilia ſub ſingulis binis terniſve, &c. radicibus, ex reliquis terminis haud exularent niſi hæc contenta forent eadem & contrariis ſignis affecta; ſed factores horum impoſſibiles per *part. noviſſ.* ſunt iidem & contrariis ſignis affecti; ergo factores reales neceſſario erunt iidem & iiſdem ſignis affecti.

Neceſſe eſt igitur ut hujuſmodi æquationum radices impoſſibiles, ſi quæ ſint, induant vel formam $\pm a + b\sqrt{-1}$ ubi ex parti reali & impoſſibili conſtantur, vel $\pm b\sqrt{-1}$ ubi nullam ſortiuntur partem realem. Ex. gr.

Aſſumamus $x = a + b\sqrt{-1}$, $x = a - b\sqrt{-1}$, $x = +b\sqrt{-1}$, vel $x = -b\sqrt{-1}$. Et ex factorum $x - a \mp b\sqrt{-1}$ vel $x \mp b\sqrt{-1} = 0$, continua multiplicatione generemus æquationes; eritque

$$\begin{array}{l} \; x - a - b\sqrt{-1} \\ \times\; x - a + b\sqrt{-1} \\ \hline \end{array}$$

$$x^2 - 2ax \begin{array}{l} + a^2 \\ + b^2 \end{array} = 0. \quad \text{Æquatio quadratica.}$$

$$\begin{array}{l} \times\; x - b\sqrt{-1} \\ \hline \end{array}$$

$$x^3 \begin{array}{l} -2a \\ -b\sqrt{-1} \end{array} x^2 \begin{array}{l} + a^2 \\ + b^2 \\ + 2ab\sqrt{-1} \end{array} x \begin{array}{l} - a^2b\sqrt{-1} \\ - b^3\sqrt{-1} \end{array} = 0. \quad \text{Æq. cub.}$$

$$\begin{array}{l} \times\; x + b\sqrt{-1} \\ \hline \end{array}$$

$$x^4 - 2ax^3 \begin{array}{l} + a^2 \\ + 2b^2 \end{array} x^2 - 2ab^2x \begin{array}{l} + a^2b^2 \\ + b^4 \end{array} = 0. \quad \text{Æq. biquadratica.}$$

Ubi obſervandum, quod in æquationibus parium dimenſionum pelluntur quantitates impoſſibiles ex ſecundo termino per additionem ſolam, ex ultimo termino per multiplicationem ſolam, & ex terminis intermediis per additionem & multiplicationem conjunctim; contra, in terminis æquationum imparium dimenſionum ſemper hærent quantitates impoſſibiles.

Cor. VIII. Æquationis cujuſcunque dimenſionis imparis omnes ſuos terminos reales habentis erit una ſaltem radix realis, cum numerus radicum impoſſibilium ſemper ſit par, *per cor. præced.*

Cor. IX. Deſignante P, ſi ſignum ejus mutetur, ſummam radicum *a*, *b*, *c*, &c, ſub ſignis propriis, mutando inſuper ſigna alternorum coefficientium a ſecundo incipientium R, T, &c. facile invenire poſſumus ſummam quadratorum, cuborum, biquadratorum, &c. ex ſingulis radicibus, ope formularum § 79. Ut in æquatione $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, ubi coefficiens ſecundi termini eſt -1, tertii -19, quarti $+49$, quinti -30; ponendum erit $+1 = P$, $-19 = Q$, $-49 = R$, $-30 = S$. Et inde orientur

$A = P = 1$, *ſumma radicum.*

$B = PA - 2Q = 1 + 38 = 39$, *ſumma quadratorum.*

$C = PB - QA + 3R = 39 + 19 - 147 = -89$, *ſumma cuborum.*

$D = PC - QB + RA - 4S = -89 + 741 - 49 + 120 = 723$, *ſum. biq.*

Nimirum æquationis illius radices ſunt 1, 2, 3 & -5, et harum ſumma $1 + 2 + 3 - 5 = 1$, ſumma quadratorum $1 + 4 + 9 + 25$ eſt 39, ſumma cuborum $1 + 8 + 27 - 125$ eſt -89, & ſumma biquadratorum $1 + 16 + 81 + 625$ eſt 723.

Paulo aliter, et commodius quidem, calculum inſtituit *Newtonus*; mutat enim ſigna *omnium* coefficientium, quo evitetur confuſio (ut videtur) ſi qua forte inter operandum ſuboriatur, mutandis tantum ſignis coefficientium alternarum P, R, T, &c. Dein, quo compenſetur mutatio ſignorum reliquarum Q, S, V, &c. mutat quoque ſigna terminorum formulis in ſingulis negativorum quos nempe ſubeunt hæ coefficientes Q, S, V, &c. quo pacto *omnes* formularum termini prodeunt affirmativi; ſcilicet, $B = PA + 2Q$, $C = PB + QA + 3R$, &c.

Cor. X. Facile determinari poteſt numerus contentorum ſub ſingulis binis, ternis, pluribuſve radicibus, in unoquoque coefficiente P, Q, R, &c. ope Theorematis Binomii. Nam ſi omnes radices *a*, *b*, *c*, &c. ſint inter ſe *æquales*, ſingulas deſignabit *a*, eritque $(x - a)^n = x^n - nax^{n-1} + \frac{n}{1} \cdot \frac{n-1}{2} a^2x^{n-2} - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^3x^{n-3} + \&c. = 0$, adeoque P coefficiens ſecundi termini $= na$, $Q = n \cdot \frac{n-1}{2} a^2$, $R = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^3$ &c. Exponet igitur n ſummam radicum a; $n \cdot \frac{n-1}{2}$ ſummam rectangulorum ſub ſingulis binis aa; $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$ ſummam contentorum ſub ſingulis ternis aaa, &c. Verum hoc quoque ita ſe habebit, ſi radices *a*, *b*, *c*, &c. ſint *inæquales*; cum hæc conditio nec mutabit numerum radicum ſingula contenta componentium, nec ipſorum contentorum numerum in unoquoque termino æquationis. Ex. gr. Sit numerus radicum $n = 4$; erit numerus contentorum ſub ſingulis binis $\frac{4 \cdot 3}{1 \cdot 2} = 6$; numerus contentorum ſub ſingulis ternis $\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} = 4$; ſub ſingulis quaternis $\frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} = 1$; ut in æquatione biquadratica ſupra allatâ. Eſtque hæc regula in inveſtigandis plurium quantitatum *combinationibus* diverſis admodum utilis.

Cor. XI. Æquationis cujuſcunque $x^n \cdot Px^{n-1} \cdot Qx^{n-2}$ &c. $= 0$, omnes radices reales habentis, quadratum coefficientis termini cujuſque medii majus erit rectangulo coefficientium terminorum hinc inde conſiſtentium, ſi modo hoc rectangulum affirmativum ſit. Nam

I. Si radices ſint eædem & iiſdem ſignis affectæ; deſignabit na, $n \cdot \frac{n-1}{2} a^2$, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^3$, &c. ſeriem coefficientium P, Q, R, &c. reſpective; per *cor. præced.* atque ideo, rationem trium quarumvis coefficientium ſucceſſivarum, ſi dividantur ſigillatim per primam ex his, exponet 1, $\frac{n-m}{m+1} a$, $\frac{n-m}{m+1} \cdot \frac{n-\overline{m+1}}{m+2} a^2$. (ubi m deſignat integrum quemvis minorem n.) Sed $\overline{\frac{n-m}{m+1} a}\Big|^2$ ſeu $\frac{n-m}{m+1} \cdot \frac{n-m}{m+1} a^2$ majus eſt quam $1 \times \frac{n-m}{m+1} \cdot \frac{n-\overline{m+1}}{m+2} a^2$, ob $\frac{n-m}{m+1}$ majus quam $\frac{n-\overline{m+1}}{m+2}$. Ergo, quadratum &c. majus eſt rectangulo, &c.

II. Si radices ſint diverſæ at iiſdem ſignis affectæ, ſumptis tribus quibuſvis coefficientibus ſucceſſivis, puta R, S, T, erunt

Formæ terminorum in S^2.

1. $a^2b^2c^2d^2 + a^2b^2c^2e^2 + \&c. + b^2c^2d^2e^2 + \&c.$ [C4]
2. $a^2b^2c^2de + a^2b^2c^2df + \&c. + b^2c^2d^2ef + \&c.$ [αC5]
3. $a^2b^2cdef + a^2b^2cdeg + \&c. + b^2c^2defg + \&c.$ [βC6]
4. $a^2bcdefg + a^2bcdefh + \&c. + b^2cdefgh + \&c.$ [γC7]
5. $abcdefgh + abcdefgk + \&c. + bcdefghk + \&c.$ [δC8]

De coefficientibus formarum in S^2.

1. $a^2b^2c^2d^2$ ſeu $abcd \times abcd$. Cum ex quatuor radicibus quatuor fiunt quadrata, erit numerus combinationum $\frac{4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4} = 1$; adeoque quiſque terminus hujus formæ ſemel tantum occurrit: & totum numerum terminorum ejus exponet C4 ſeu numerus combinationum ſub ſingulis radicibus quaternis in æquatione (n) radicum, $= n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}$. per *cor. præced.*

2. $a^2b^2c^2de$ ſeu $abcd \times abce$. Cum ex quinque radicibus ternæ quævis fiunt quadrata, erit numerus combinationum $\frac{5\cdot4\cdot3}{1\cdot2\cdot3}$; quarum quæque bis occurrit, cum binæ reliquæ radices modis $\frac{2}{1}$ in factoribus diſponi poſſunt, ſcilicet

$abcd \times abce$ vel $abce \times abcd$: erit igitur coefficiens termini cujuſque $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \times \frac{2}{1} =$ puta α, & totus numerus terminorum hujus formæ, $\alpha C5$.

3. a^2b^2cdef ſeu $abcd \times abef$. Cum ex ſex radicibus binæ quæque fiunt quadrata, prodit numerus combinationum $\frac{6 \cdot 5}{1 \cdot 2}$; & quatuor reliquæ modis $\frac{4 \cdot 3}{1 \cdot 2}$ in factoribus diſponi poſſunt: erit igitur coefficiens termini cujuſque hujus formæ $\frac{6 \cdot 5}{1 \cdot 2} \times \frac{4 \cdot 3}{1 \cdot 2} = \beta$, & totus numerus terminorum, $\beta C6$. Similiter,

4. $a^2bcdefg$ ſeu $abcd \times aefg$. Erit coefficiens termini cujuſque hujus formæ $\frac{7}{1} \times \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \gamma$, & totus numerus terminorum $\gamma C7$.

5. $abcdefgh$ ſeu $abcd \times efgh$. Erit coefficiens termini cujuſque hujus formæ $\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = \delta$, & totus numerus terminorum $\delta C8$.

Formæ terminorum in RT.

1. $a^2b^2c^2de + a^2b^2c^2df +$ &c. $+ b^2c^2d^2ef +$ &c. $[\frac{1}{2}\alpha C5]$
2. $a^2b^2cdef + a^2b^2cdeg +$ &c. $+ b^2c^2defg +$ &c. $[\frac{2}{3}\beta C6]$
3. $a^2bcdefg + a^2bcdefh +$ &c. $+ b^2cdefgh +$ &c. $[\frac{3}{4}\gamma C7]$
4. $abcdefgh + abcdefgk +$ &c. $+ bcdefghk +$ &c. $[\frac{4}{5}\delta C8]$

Ubi obſervandum, quod omnes formas in S^2 præter primam ſortitur RT.

De coefficientibus formarum in RT.

1. $a^2b^2c^2de$ ſeu $abc \times abcde$. erit coefficiens $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = \frac{1}{2}\alpha$, adeoque totus numerus terminorum, $\frac{1}{2}\alpha C5$.
2. a^2b^2cdef ſeu $abc \times abdef$. coeff. erit $\frac{6 \cdot 5}{1 \cdot 2} \times \frac{4}{1} = \frac{2}{3}\beta$, adeoq. &c.
3. $a^2bcdefg$ ſeu $abc \times adefg$. coeff. erit $\frac{7}{1} \times \frac{6 \cdot 5}{1 \cdot 2} = \frac{3}{4}\gamma$, adeoq. &c.
4. $abcdefgh$ ſeu $abc \times defgh$. coeff. erit $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = \frac{4}{5}\delta$, adeoq. &c.

Hinc erit

$$\begin{array}{rl} S^2 = & C4 + \alpha C5 + \beta C6 + \gamma C7 + \delta C8 \\ RT = & \quad\;\; + \tfrac{1}{2}\alpha C5 + \tfrac{2}{3}\beta C6 + \tfrac{3}{4}\gamma C7 + \tfrac{4}{5}\delta C8 \\ \hline S^2 - RT = & C4 + \tfrac{1}{2}\alpha C5 + \tfrac{1}{3}\beta C6 + \tfrac{1}{4}\gamma C7 + \tfrac{1}{5}\delta C8. \end{array}$$ affirm.

Et ſic univerſaliter, deſignante Y ſummam productorum ſub ſingulis (m) radicibus in æquatione (n) dimenſionum, & deſignantibus α, β, γ, δ, &c. correſpondentes coefficientes numerales quantitatum $C\overline{m+1}$, $C\overline{m+2}$, $C\overline{m+3}$, &c. erit, $Y^2 - XZ = Cm + \frac{1}{2}\alpha C\overline{m+1} + \frac{1}{3}\beta C\overline{m+2} + \frac{1}{4}\gamma C\overline{m+3}$ &c. uſque ad $m+1$ terminos. Ergo, &c.

III. Si radices ſint diverſæ & diverſis ſignis affectæ; ob ingentem æquationum varietatem, hunc caſum univerſaliter demonſtrare difficillimum. Veritas tamen ejus ita per inductionem conſtabit.

In æquatione quadratica $x^2 \pm px + q = 0$, ubi radices ſunt reales, erit $\frac{1}{4}p^2$ æquale aut majus quam q, ſive p^2 æquale aut majus quam $4q$, per § 29. Adeoque *a fortiori*, p^2 majus quam q.

Et multiplicando quadraticam $x^2 + px + q = 0$ per $x - c = 0$, generabitur æquatio cubica $x^3 + (p - c)$ $x^2 + (q - pc)\,x - qc = 0$, & deſignante m numerum æqualem aut majorem quaternario, erit

$$\begin{aligned} P^2 &= p^2 - 2pc + c^2 = mq - 2pc + c^2. \\ Q &= \qquad\qquad\qquad\quad q - pc \\ \hline \text{Adeoque } P^2 - Q &= \qquad\quad \overline{m-1}\,q - pc + c^2. \text{ affir.} \end{aligned}$$

Ubi $q - pc$ eſt affirmativum, per hypoth. et $\overline{m-1}\,q$ eſt majus quam q, adeoque $\overline{m-1}\,q - pc$ eſt affirmativum, & *a fortiori*, $\overline{m-1}\,q - pc + c^2$ affirmativum.

Erit quoque

$$\begin{aligned} Q^2 &= p^2c^2 - 2pqc + q^2 = mqc^2 - 2pqc + q^2 \\ PR &= \qquad\qquad\qquad\quad qc^2 - pqc \\ \hline Q^2 - PR &= \qquad\quad \overline{m-1}\,qc^2 - pqc + q^2 \text{ affirm.} \end{aligned}$$

Et ſimiliter probare licet in æquatione cubica $(x^2 - px + q)$ $(x + c) = 0$, &c.

Multiplicando autem quadraticam $x^2 + px + q = 0$ per quadraticam $x^2 - gx + b = 0$, generabitur biquadratica $x^4 + (p - g)\,x^3 + (q - pg + b)\,x^2 + (pb - qg)\,x + qb = 0$, adeoque ob $g^2 =$ vel $> 4b$, $=$ puta nb, erit

$$P^2 = p^2 - 2pg + g^2 = mq - 2pg + nb$$
$$Q = \quad q - pg + b$$

Adeoq. $P^2 - Q = \quad \overline{m-1}\,q - pg + \overline{n-1}\,b$ affir.

Itemque,

$$Q^2 = p^2g^2 - 2pg(q+b) + (q+b)^2 = (mn+2)qb - 2pg(q+b) + q^2 + b^2$$
$$PR = p^2b - pg(q+b) + qg^2 = (m+n)qb - pg(q+b)$$

Adeoq. $Q^2 - PR = (\overline{mn+2} - \overline{m+n})\,qb - pg\,(q+b) + q^2 + b^2.$

affirmativum nempe, ob $\overline{mn+2} - \overline{m+n}$ majus quam $\overline{m+n}$, cum m & n ſint quaternario majores aut ſaltem æquales, reſpective. Itemque,

$$R^2 = p^2b^2 - 2pqbg + q^2g^2 = mqb^2 - 2pqgb + nq^2b$$
$$QS = \quad qb^2 - pqgb + q^2b$$

$R^2 - QS = \quad \overline{m-1}\,qb^2 - pqgb + \overline{n-1}\,q^2b$
affirmativum.

Et multiplicando biquadraticam $x^4 + (p-g)\,x^3 +$ &c. $= 0$ per $x - e = 0$, in æquatione reſultante $x^5 + (p - g - e)\,x^4 + (q - pg + b - pc + cg)\,x^3 +$ &c. $= 0$, erit

$$P^2 = p^2 - 2pg - 2pe + 2ge + g^2 + e^2 = mq + 2 \times \overline{-pg - pe + ge} + nb + e^2$$
$$Q = q - pg - pe + ge + b \quad = q + 1 \times \overline{-pg - pe + ge} + b$$

$P^2 - Q = \quad \overline{m-1}\,q + 1 \times \overline{-pg - pe + ge} + \overline{n-1}\,b + e^2$
affirmativum. Atque ſic deinceps.

Cor. XII. Hinc in æquatione quacunque ſi quadratum coefficientis termini cujuſvis medii ſit æquale aut minus rectangulo coefficientium terminorum hinc inde conſiſtentium, ubi hoc rectangulum ſit affirmativum; indicio eſt, æquationem radices impoſſibiles ſortiri. Veruntamen non valet *converſa*, ubi enim radices ſunt impoſſibiles, poteſt quadratum &c. majus eſſe hoc rectangulo. Sic in æquatione quadraticâ $x^2 \pm px + q = 0$, radices erunt impoſſibiles ubi $\frac{1}{4}p^2$ ſit minus quam q, at ſalvâ hac conditione, poteſt eſſe p^2 majus, æquale, vel minus quam q.

Hinc quoque defectus termini cujuſvis medii, ſi modo termini hinc inde conſiſtentes iiſdem ſignis afficiantur, eſt nota radicum impoſſibilium. Nam quadratum, &c. in hoc caſu evaneſcit, atque ideo minus prodit rectangulo, &c.

DE NUMERO RADICUM AFFIRMATIVARUM & NEGATIVARUM ÆQUATIONIS.

145. *Æquatio* cujuſcunque *dimenſionis cujus ultimus terminus eſt negativus, unam ſaltem habet radicem realem* & affirmativam.

Nam in æquatione $x^n \,.\, x^{n-1}$ &c. — $U = 0$, ſi x ſit nihil, evaneſcentibus omnibus terminis præter ultimum, erit valor reſultans negativus; contra; creſcente x a nihilo ad magnitudinem infinitam uſque, terminis affirmativis tandem exſuperantibus negativos, valor reſultans erit affirmativus: ergo iſte valor affirmativus x qui inter creſcendum efficiet omnes terminos ſe deſtruere, erit radix.

146. *Æquatio quæcunque dimenſionis* imparis *cujus ultimus terminus eſt negativus, unam ſaltem habet radicem realem* & affirmativam, *per* præced.

—*Quod ſi ultimus terminus ſit affirmativus, unam ſaltem habet radicem realem &* negativam.

Subſtituendo enim $-x$ pro $+x$, primus terminus evadet negativus, & terminis omnibus novæ æquationis ad alteram partem tranſlatis, quo terminus altiſſimus evadat affirmativus, fiet ultimus terminus negativus, nova æquatio igitur ſortietur unam radicem realem & affirmativam, & proinde data æquatio unam realem & negativam.

147. *Æquatio quæcunque dimenſionis* paris *cujus ultimus terminus eſt negativus, neceſſario habet duas radices reales, unam* affirmativam, *alteram* negativam.

Habet enim unam radicem affirmativam, per § 145. Et ſi ſubſtituatur $-x$ pro $+x$, cum terminus altiſſimus (utpote dimenſionis paris) ſit affirmativus, nova æquatio habebit unam radicem realem & affirmativam, adeoque data æquatio alteram quoque radicem realem & negativam.

148. *Æquationis cujuſcunque unam tantum mutationem ſignorum* de $+$ in $-$ *habentis, unica erit radix realis* affirmativa.

Nam quoniam ultimus terminus in hoc caſu, neceſſario erit negativus, æquatio unam habebit radicem realem affirmativam, per § 145.

Quod ſi X deſignet ſummam terminorum affirmativorum, & Y ſummam negativorum, fiet æquatio $X - Y = 0$. Et quoniam per hypotheſin, eſt una tantum mutatio ſignorum, manifeſtum eſt, quod *minima* poteſtas x in polynomio X, puta x^m, neceſſario tranſcendet *omnem* poteſtatem x in polynomio Y. Adeoque dividendo per x^m, quantitas $\frac{X}{x^m}$ continebit ſolummodo poteſtates affirmativas x, & $\frac{Y}{x^m}$ poteſtates negativas x. Unde ſequitur, quod creſcente vel decreſcente x, perinde augebitur vel minuetur valor $\frac{X}{x^m}$; niſi X terminum unicum, nempe altiſſimum contineat, quo in caſu, non variabitur $\frac{X}{x^m}$. Contra, creſcente x, neceſſario minuetur valor $\frac{Y}{x^m}$, decreſcente vero x, augebitur. Sit jam a radix realis affirmativa æquationis, eritque $x = a$, $X = Y$, itemque $\frac{X}{x^m} = \frac{Y}{x^m}$. Adeoque ſubſtituendo pro x numeros majores quam a, erit ſemper $\frac{X}{x^m} > \frac{Y}{x^m}$ adeoque $\frac{X}{x^m} - \frac{Y}{x^m}$ ſeu $X - Y$ affirmativum; & ſubſtituendo pro x numeros minores quam a, erit ſemper $\frac{X}{x^m} < \frac{Y}{x^m}$ adeoque $X - Y$ negativum. Ergo impoſſibile eſt æquationem ſortiri radicem majorem minoremve quam a *.

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* Vide *Diſſert.* Sur la reſolution des æquations numeriques, par *M. De la Grange*, *Berlin. Acad.* an. 1769, p. 314, &c.

149. *Æquationis nullas radices impoſſibiles habentis tot ſunt radices* affirmativæ *quot ſignorum in continua ſerie* mutationes *de* + *in* —, & *de* — *in* +; *cæteræ* negativæ *ſunt.*

Hanc regulam in æquationum tranſmutationibus & limitibus inveſtigandis perutilem, tentando & inductione autorem inveniſſe, ſive *Harriott* fuit, ſive ut pleriſque placet, *Carteſius* *, veriſimile eſt. Demonſtrationem ejus, ex generatione æquationum, ita colligere licet.

In æquatione ſimplici $x \pm a = 0$ eſt una variatio ſignorum *de* + *in* — ubi radix eſt affirmativa, & una ſucceſſio *de* + *in* + ubi radix eſt negativa.

Et ſi multiplicetur $x + a = 0$, per $x - b = 0$, in æquatione quadr. reſultante $x^2 \begin{matrix} + a \\ - b \end{matrix} x - ab = 0$, cujus una radix eſt affirmativa & altera negativa, prodit ſeries ſignorum +, ±, —, prout *a* ſit majus, æquale aut minus quam *b*, ſcilicet, vel +, +, —, vel +, —, —; adeoque in utroque caſu, una variatio ſignorum *de* + *in* —, & una ſucceſſio, vel *de* + *in* +, vel *de* — *in* —.

His veſtigiis inſiſtendo demonſtrare poſſimus regulam in æquationibus cubicis, ut facit *M^c^Laurin*, *Algeb. Part.* 2, § 20. Verùm methodum eandem proſequi in æquationibus altiorum dimenſionum, propter ambiguitatem ſignorum terminorum ſucceſſivorum, impoſſibile foret. Demonſtratio autem fiet univerſalis ope Theorematis inſequentis, quod ex *Meditationibus Algebraicis* D. *Waring*, p. 74. expreſſimus.

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* Regulam ſic effert *Carteſius*, *Geom. Lib.* 3, *p.* 70. "Ex quibus etiam cog- "noſcitur quot veræ & quot falſæ radices in unaquaque æquatione haberi poſ- "ſunt. Nimirum, tot in ea veras haberi poſſe, quot variationes reperiuntur ſig- "norum + & —; & tot falſas quot vicibus ibidem deprehendentur duo ſigna "+ vel duo ſigna —, quæ ſe invicem ſequuntur." Contenderunt nonnulli *Carteſio* latuiſſe limitationem regulæ ad æquationes quæ omnes radices habent reales; at minus recte, ut ex ipſius verbis colligere licet, *ex quibus*, nimirum ex generatione æquationum radices reales habentium, de quibus paulo ante egerat; itemque ex eo quod non ſimpliciter adhibet *haberi*, at cum reſtrictione, *haberi-poſſe*, nimirum in caſu prædicto.

150. *Ubi æquationis radices nullæ impoſſibiles ſunt, ſi multiplicetur æquatio per* $x - p = 0$, *ut una adhuc radix affirmativa addatur prioribus, augebitur unitate numerus mutationum ſignorum in nova æquatione; quod ſi multiplicatio ſiat per* $x + p = 0$, *ut una adbuc radix negativa addatur prioribus, idem erit numerus mutationum ſignorum in nova ac in data æquatione.*

Sit $x^n - Px^{n-1} - Qx^{n-2} + Rx^{n-3} + Sx^{n-4} + Tx^{n-5} + \&c. = 0$

$$\begin{array}{l} \times x - p \\ \hline x^{n+1} \begin{matrix}-P\\-p\end{matrix} x^n \begin{matrix}-Q\\+pP\end{matrix} x^{n-1} \begin{matrix}+R\\+pQ\end{matrix} x^{n-2} \begin{matrix}+S\\-pR\end{matrix} x^{n-3} \begin{matrix}+T\\-pS\end{matrix} x^{n-4} \pm \&c. = 0. \end{array}$$

Hic obſervandum eſt, quod binæ partes coefficientis termini cujuſque novæ æquationis iiſdem afficientur ſignis, ubi termini imminentis & præcedentis in data æquatione ſigna ſunt diverſa; contrariis verò afficientur ſignis, ubi termini imminentis & præcedentis ſigna ſunt eadem. Sic coefficiens ſecundi termini novæ æquationis eſt $-P-p$, exiſtente ſigno ſecundi termini datæ æquationis negativo, primi autem affirmativo; at coefficiens tertii termini eſt $-Q+pP$, negativum nempe, inſtar termini imminentis, ſi Q ſit majus quam pP, affirmativum ſi ſecus.

Unde, ſi trium pluriumve terminorum ſucceſſivorum datæ æquationis ſigna ſint eadem, erunt duorum pluriumve terminorum ſucceſſivorum novæ æquationis ſigna ambigua.

Quod ſi talium cujuſlibet ſignum diverſum prodeat a ſigno termini imminentis datæ æquationis, termini proxime inſequentis quoque ſignum diverſum erit a ſigno termini huic imminentis. *Ex. gr.* Sit terminus quintus novæ æquationis $(S - pR)\,x^{n-3}$ negativus,

Erit - - - - - - S minus quam pR
Sed per *Cor.* XI, eſt RT minus quam S^2
Adeoque - - - RST minus quam pRS^2
Sive - - - - T minus quam pS.

Et proinde terminus ſextus $(T - pS)\,x^{n-4}$ negativus erit. Et ſic deinceps. Erit igitur ſeries ſignorum

datæ æquat. $+, -, -, +, +, +, +,$ &c.
novæ æquat. $+, -, \mp, +, -, -, -,$ &c.

Nimirum a termino primo ad quartum, idem eſt numerus variationum in utriſque, idque ſive tertius terminus novæ æquationis ſit affirmativus ſive negativus; a quarto verò ad quintum in nova æquatione, accedit una variatio quæ non eſt in data; in reliquis terminis idem prodit numerus ſucceſſionum in utriſque. Ergo, &c.

Simili ratiocinio conſtat, quod ſi multiplicetur æquatio $x^n - Px^{n-1} -$ &c. $= 0$ per $x + p = 0$, idem reſultabit numerus mutationum ſignorum in nova ac in data æquatione.

Sic multiplicando quadraticam $x^2 + px + q = 0$ per $x - c = 0$, in æquatione cubica reſultante $x^3 + (p - c)$

$x^2 + (q - pc)x - qc = 0$, cujus una radix eſt affirmativa, reliquæ negativæ; ſit p majus quam c, poteſt eſſe q majus vel minus quam pc, adeoque ſeries ſignorum +, +, ±, —. Quod ſi p ſit minus quam c, erit quoque q minus quam pc, & ſeries ſignorum +, —, —, —. Nimirum in utroque caſu una variatio ſignorum & duæ ſucceſſiones, ut æquum eſt. Atque ita in æquationibus altiorum dimenſionum regulam demonſtrare licet.

151. Ubi vero radices aliquæ impoſſibiles ſunt, non valet regula *Carteſiana*, niſi quatenus impoſſibiles illæ quæ nec negativæ ſunt nec affirmativæ pro *ambiguis* habeantur. Ut in æquatione $x^3 + qx - r = 0$, interponendo $\pm 0x^2$ quo compenſetur defectus ſecundi termini, erit ſeries ſignorum vel +, +, +, —, vel +, —, +, —; nimirum in priore caſu, ſi ſigna ſpectes, una radix affirmativa & duæ negativæ, in poſteriore, tres affirmativæ. Sunt ergo duæ *impoſſibiles*, quæ pro ambiguitate ſua, priore caſu negativæ, poſteriore affirmativæ eſſe videntur.

Aliquando augendo numerum radicum æquationis detegentur radices impoſſibiles, ſi quæ in ea deliteſcunt. Sic in æquatione $x^3 + px^2 + 3p^2x - q = 0$, ſigna indicant radicem unam eſſe affirmativam, & duas negativas; multiplicando autem per $x - 2p = 0$, prodit $x^4 - px^3 + p^2x^2$ $(- 6p^3 - q)x + 2pq = 0$, quæ habere deberet duas affirmativas ac duas negativas radices, habet tamen ſi mutationum ſignorum ſpectes, affirmativas quatuor; ſunt ergo duæ impoſſibiles. Cæterum in his & talibus exemplis radices impoſſibiles exiſtere, ex Cor. XII, ſtatim innoteſcit.

152. Regulam ſatis facilem tradit *Newtonus* pro inveniendo numero radicum impoſſibilium, at minus accuratam,

quam idcirco prætereundam cenſui. De hac tale profert judicium *Waring*, in hiſce verſatiſſimus, *Med. Algeb.* p. 47. "Hæc methodus in quadraticis æquationibus verum præ-"bet numerum impoſſibilium radicum; in cubicis autem "probabilitas inveniendi impoſſibiles radices non videtur "majorem habere rationem ad probabilitatem fallendi "quam 2 ad 1. In æquationibus autem multo ſuperiorum "dimenſionum verum impoſſibilium radicum numerum "perraro deteget."

Sic in æquatione $x^3 - 2x + 4 = 0$, ſignorum ſubſcriptorum ſeries ſecundum regulam *Newtonianam*, indicaret nullas dari radices impoſſibiles, licet factorem quadraticum habet $x^2 - 2x + 2 = 0$ cujus radices per eandem reg. ſunt impoſſibiles.

DE TRANSMUTATIONIBUS ÆQUATIONUM.

153. I. *Æquationis cujuſvis radices omnes affirmativas in negativas, & negativas in affirmativas mutare.*

Fiet hoc mutando tantum ſigna terminorum alternorum a ſecundo incipientium. Nam ſi ſubſtituatur $-x$ pro $+x$ in æquatione quacunque dimenſionis *paris*, mutabuntur ſigna terminorum alternorum a ſecundo incipientium, quos nempe occupant impares poteſtates x; ſignis reliquorum manentibus. At ſi ſubſtituatur $-x$ pro x in æquatione dimenſionis *imparis*, mutabuntur tantum ſigna terminorum alternorum a primo incipientium, adeoque tranſponendo omnes terminos cum ſignis mutatis, quo terminus altiſſimus novæ æquationis evadat affirmativus, mutabuntur, ut in præcedenti caſu, ſigna terminorum alternorum tantum a ſecundo incipientium.

Sic æquatio $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, cujus ſi ſigna ſpectes, tres ſunt radices affirmativæ, mutando ſigna ſecundi & quarti termini, fit $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, ejuſque radicum una tantum affirmativa, reliquæ negativæ, ut æquum eſt; adeoque in priore caſu, prodeunt radices $+1$, $+2$, $+3$, & -5, in poſteriore, -1, -2, -3, $+5$, mutatis nimirum radicibus affirmativis in negativas, & vice verſâ.

Sic quoque æquatio $x^5 - 4x^4 + 4x^3 - 2x^2 - 5x - 4 = 0$, tres radices affirmativas habens, mutando ſigna ſecundi quarti & ſexti termini, fit $x^5 + 4x^4 + 4x^3 + 2x^2 - 5x + 4 = 0$, cujus tres prodeunt radices negativæ, reliquæ affirmativæ. Ex quinque autem radicibus duæ ſunt impoſſibiles, quæ in priore æquatione inter affirmativas latebant, in poſteriore latent inter negativas; ita ut his deductis reſtet unica tantum radix vere negativa.

154. Poſſumus quoque ſupponere radicem æquationis ex cognita & incognita aliqua quantitate *utcunque componi*, & perinde pro eâ ſubſtituere quod æquipollens eſſe fingitur. Et hoc pacto tranſmutationes aliæ eliciendæ ſunt quæ diverſis uſibus inſerviunt.

155. II. *Radices æquationis data quadam quantitate augere, vel diminuere, vel de data quantitate ſubducere.*

Supponamus radicem æquationis æqualem eſſe ſummæ vel differentiæ cognitæ & incognitæ alicujus quantitatis, deinde, ſubſtituendo pro radice in æquatione datâ id quod æquipollens eſſe fingitur, emerget æquatio nova cujus radices evadent majores vel minores radicibus datæ hac cognitâ quantitate. Sic in æquatione $x^3 - px^2 + qx - r = 0$, ſi radices cognita quantitate c augeri vellem, ponendo $x + c = y$, & perinde ſubſtituendo $y - c$ pro x, &c. emerget æquatio,

$$\left.\begin{array}{r} x^3 - \\ -px^2 \\ +qx \\ -r \end{array}\right\} \left.\begin{array}{llll} y^3 & -3cy^2 & +3c^2y & -c^3 \\ & -py^2 & +2pcy & -pc^2 \\ & & +qy & -qc \\ & & & -r \end{array}\right\} = 0.$$

Et ſimiliter, ſi radices cognita quantitate c diminui vellem, ſubſtituendo $y + c$ pro x, aut ſi radices de cognita quantitate c ſubducere vellem, ſubſtituendo $c - y$ pro x, fiet quæſitum.

Atque in genere, radices æquationis (n) dimenſionum augere, &c. poſſumus cognita quadam quantitate c, ſubſtituendo pro poteſtatibus x ſingulis in terminis æquationis datæ poteſtates correſpondentes binomii $y + c$, ductas in ſingulorum terminorum coefficientes p, q, r, s, &c. reſpective.

156. Cæterum æquationis transformatæ termini abſque involutione facile conflari poſſunt.

Nam ſignis terminorum probè obſervatis, ſubſtituendo tantum c pro x in æquatione data habebitur terminus ultimus. Et multiplicando termini ultimi partem unamquamque per indicem c in iſta parte & dividendo factum per c, habebitur coefficiens penultimi. Dein rurſus multiplicando penultimi partem unamquamque per indicem c in iſta parte, jam unitate minorem quam prius, & factum dividendo per $2c$, habebitur coefficiens antepenultimi. Atque ita, ſemper multiplicando termini noviſſimi partem unamquamque per numeros unitate minores quam prius, & factum dividendo per c ductum in numeros unitate majores quam prius, habebitur termini inſequentis coefficiens.

157. Atque ita *Newtonus* methodum generalem inveſtigandi *limites* inter quos conſiſtunt radices æquationum docens, brevitatis ſtudioſus, ex ipſa æquatione data, tan-

quam ex termino ultimo, coefficientes terminorum reliquorum æquationis transformatæ immediate deducit. *Multiplicando utique æquationis terminum unumquemque per numerum dimenſionum ejus, & dividendo factum per radicem æquationis, &c.* Niſi quod facta ſucceſſive prodeuntia dividit per radicem æquationis ſolam, quæ dividi debuerunt per duplam radicem, triplam radicem, &c. reſpective. Et proinde, minuendi gratiâ, terminos reſultantes per maximos diviſores communes, ſiquos ſortiuntur, inſuper dividit.

158. 1. *Hoc pacto efficere poſſumus ut radicum aliquæ vel omnes quæ prius erant negativæ evadant affirmativæ*, augendo utique radices donec augmentum c aliquos vel omnes radices negativas tranſcendat; *vel etiam ut ex affirmativis aliquæ vel omnes evadant negativæ*, minuendo utique radices, donec decrementum, &c. Nam ſi omnis valor y fiat affirmativus, neceſſe eſt ut $x + c = y$, ſit affirmativus, adeoque ut c majus ſit omni radice negativa x. Atque ita in cæteris.

Sic in æquatione $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, ſi radices unitate augeri vellem, ſcribendo $y - 1$ pro x, æquationis prodeuntis $y^4 - 5y^3 - 10y^2 + 80y - 96 = 0$, radices erunt 2, 3, 4, -4, quæ prius erant 1, 2, 3, -5, unitate jam factæ majores. Pro x vero ſcribendo $y - 6$, prodiiſſet æquatio $y^4 - 25y^3 + 215y^2 - 695y + 504 = 0$, cujus radices fuiſſent 7, 8, 9, 1, omnes nimirum affirmativæ. Quod ſi pro x ſcripſiſſem $y + 1\frac{1}{2}$ prodiiſſet æquatio $y^4 + 5y^3 - 10y^2 - \frac{5}{4}y + \frac{39}{16} = 0$, cujus radices ita diminutæ evaſiſſent $-\frac{1}{2}$, $+\frac{1}{2}$, $+1\frac{1}{2}$, $-6\frac{1}{2}$, duæ nimirum affirmativæ ac duæ negativæ. Et pro eodem ſcribendo $y + 4$, radices jam numero quaternario diminutæ in æquatione reſultante $y^4 + 15y^3 + 65y^2 + 105y + 54 = 0$, evaſiſſent -3, -2, -1, -9, negativæ omnes.

159. 2. *Et hoc modo augendo vel diminuendo radices ſiquæ impoſſibiles ſunt, hæ aliquando detegentur.*

Sic in æquatione $x^3 - 3a^2x - 3a^3 = 0$, erit ſeries ſignorum $+$, $\pm$, $-$, $-$, ſcilicet in utroque caſu, ſi ſigna ſpectes, una radix affirmativa, reliquæ negativæ. At ſi augeas radices quantitate a, ſcribendo $y - a$ pro x, in æquatione reſultante $y^3 - 3ay^2 - a^3 = 0$, erit ſeries ſignorum $+$, $-$, $\pm$, $-$, nimirum $+$, $-$, $+$, $-$, vel $+$, $-$, $-$, $-$, adeoque in priore caſu tres radices affirmativæ, in poſteriore una tantum. Sunt ergo duæ impoſſibiles, quæ pro ambiguitate ſua modo inter affirmativas, modo inter negativas latent.

160. 3. *Eadem operatione poſſumus etiam ſecundos terminos æquationum tollere.*

Nam ſi ſecundus terminus æquationis ſit negativus, conſtat per COR. II. quod radices affirmativæ præpollent negativis, radices igitur minuendæ ſunt donec affirmativæ æquipollent negativis; aut ſi terminus ſecundus ſit affirmativus, radices augendæ ſunt donec negativæ æquipollent affirmativis; hoc vero fiet in utroque caſu, ſi coefficientem ſecundi termini æquationis propoſitæ cum proprio ſigno per numerum dimenſionum æquationis diviſam ſubducamus de quantitate quæ pro novæ æquationis radice ſignificanda aſſumitur, & reſiduum ſubſtituamus pro radice æquationis propoſitæ. Nam,

Proponatur æquatio $x^n - px^{n-1} + qx^{n-2}$ &c. $= 0$, cujus ſecundus terminus eſt negativus; fiat $x - c = y$, &

ſubſtituendo $y + c$ pro x, proveniet æquatio

$$y^n \left.\begin{matrix}+nc\\ -p\end{matrix}\right\} y^{n-1} \left.\begin{matrix}+n\cdot\overline{\frac{n-1}{2}}\,c^2\\ -p\cdot\overline{n-1}\,c\\ +q\end{matrix}\right\} y^{n-2}\ \&c. = 0.$$

Fingatur jam coefficientem ſecundi termini hujus nihil eſſe, ſeu $nc - p = 0$, adeoque $c = \frac{p}{n}$; & ſubſtituto hoc valore c, fiet $nc - p = \frac{np}{n} - p = p - p = 0$, adeoque evaneſcet ſecundus terminus novæ æquationis. Et ſimiliter, ſi ſecundus terminus datæ æquationis ſit affirmativus, vice $y - c$ ſubſtituendo $y - \frac{p}{n}$ pro x, evaneſcet $-\frac{np}{n} + p$ coefficiens ſecundi termini æquationis novæ. In omni autem caſu valet hæc regula, ſive radices ſint reales ſive impoſſibiles. Ut ſi proponatur æquatio $x^3 - 4x^2 + 4x - 6 = 0$, cujus una radix eſt ſurda, reliquæ impoſſibiles, ſubſtituto $y + \frac{4}{3}$ pro x, provenit

$$\begin{array}{rrrr}
y^3 & +\,4y^2 & +\,\frac{16}{3}\,y & +\,\frac{64}{27}\\
 & -\,4y^2 & -\,\frac{32}{3}\,y & -\,\frac{64}{9}\\
 & & +\,4y & +\,\frac{16}{3}\\
 & & & -\,6\\
\hline
y^3 & * & -\,\frac{4}{3}\,y & -\,\frac{146}{27} = 0.
\end{array}$$

161. Exterminatio ſecundi termini reſolutioni faciliori æquationum cubicarum & biquadraticarum inſervit. Hoc pacto quoque, æquatio quadratica affecta ſublato termino ſecundo ad normam quadraticæ puræ reduci poteſt. Sic in æquatione $x^2 - px + q = 0$, ſubſtituendo $y + \frac{1}{2}p$ pro x, prodit $y^2 - \frac{1}{4}p^2 + q = 0$, adeoque $y^2 = \frac{1}{4}p^2 - q$, & extracta radice $y = \pm \sqrt{\frac{p^2}{4} - q}$ adeoque $x = y + \frac{1}{2}p = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$, ut antea § 29.

162. 4. *Eadem methodo poteſt & tertius æquationis terminus tolli*, quod in conſtructionibus quibuſdem geometricis poſtulatur; inveniendo nempe valorem c qui coefficientem tertii termini æquationis transformatæ (n) dimenſionum deſtruet. Fingatur enim coefficientem tertii termini $n \cdot \frac{\overline{n-1}}{2} c^2 - p \cdot \overline{n-1}\, c + q = 0$, adeoque concinnando, $c^2 - \frac{2p}{n} c + \frac{2q}{n \cdot \overline{n-1}} = 0$. Et per reductionem, erit $c = + \frac{p}{n} \pm \sqrt{\frac{p^2}{n^2} - \frac{2q}{n \cdot \overline{n-1}}}$. Ut ſi tollendus ſit tertius terminus æquationis præcedentis $x^3 - 4x^2 + 4x - 6 = 0$, prodit $c = + \frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{8}{6}} = + \frac{4}{3} \pm \frac{2}{3} = + \frac{2}{3}$ vel $+ 2$, quamobrem vice $y + c$ ſcribi debet $y + \frac{2}{3}$ vel $y + 2$ pro x, et in utroque caſu tolletur tertius terminus; nam ſi

pro x ſcribatur $y + \frac{2}{3}$ orietur hæc æquatio $y^3 - y^2 * - \frac{137}{27} = 0$, ſin ſcribatur $y + 2$, orietur hæc $y^3 + 2y^2 * - 6 = 0$.

Simili ratione, tolli poſſit 4^{tus} terminus, inveniendo valorem idoneum c per reſolutionem æquationis cubicæ; quintus terminus per reſolutionem biquadraticæ; & ſic deinceps. Verùm hæ tranſmutationes ob calculum difficilem adhiberi non ſolent.

Cæterum obſervandum eſt, quod hac methodo plures terminos intermedios *ſimul* tollere non licet, cum ſingulis terminis tollendis proprius deſideratur valor c. Methodum generalem at valde operoſam, omnes terminos intermedios tollendi, tradit *Bezout*, *Mem. de l'Acad.* an. 1772.

163. III. *Radices æquationum per datos numeros multiplicare vel dividere.*

Si radices æquationis $x^n - px^{n-1} + qx^{n-2}$ &c. $+ u = 0$, multiplicandæ forent per numerum c, fiat $cx = y$, deinde ſubſtituendo $\frac{y}{c}$ pro x, proveniet æquatio $\frac{y^n}{c^n} - \frac{py^{n-1}}{c^{n-1}} + \frac{qy^{n-2}}{c^{n-2}}$ &c. $+ u = 0$. Sive $y^n - pcy^{n-1} + qc^2y^{n+2}$ &c. $+ c^n u = 0$, cujus radices ſunt c^{plo} majores quam ante. Et rurſus ſi dividendæ ſint radices per c, fiat $\frac{x}{c} = y$, deinde ſubſtituendo cy pro x, proveniet $y^n - \frac{py^{n-1}}{c} + \frac{qy^{n-2}}{c^2}$ &c. $+ \frac{u}{c^n} = 0$.

Hinc colligere licet, quod æquationis radices per datum aliquem numerum multiplicare vel dividere poſſumus, multiplicando vel dividendo ſeriem terminorum æquationis incipiendo a primo termino, per ſeriem numerorum continue proportionalium 1, c, c^2, c^3, &c. c^n, reſpective.

164. 1. *Hoc pacto, fractiones ſemper tollere poſſumus, & tamen efficere ut terminus altiſſimus æquationis reſultantis coefficiente careat.* Sic multiplicando radices æquationis $x^3 + \frac{x^2}{f} + \frac{x}{g} + \frac{u}{b} = 0$, per c, prodit $y^3 + \frac{cy^2}{f} + \frac{c^2y}{g} + \frac{c^3u}{b} = 0$, hoc eſt, ſubſtituendo fgb pro c, ſi modo f, g & b, ſint numeri ad invicem primi, $y^3 + gby^2 + f^2gb^2y + f^3g^3b^2u = 0$. Quod ſi f, g & b non ſint ad invicem primi, valorem depreſſiorem c uſurpare licet. Sic in æquatione præcedente $y^3 - \frac{4}{3}y - \frac{146}{27} = 0$, ſubſtituendo $\frac{1}{3}z$ pro y, provenit æquatio $\frac{z^3}{27} - \frac{4z}{9} - \frac{146}{27} = 0$, ſive $\frac{z^3 - 12z - 146}{27} = 0$, & rejecto terminorum communi denominatore, $z^3 - 12z - 146 = 0$, cujus radices ſunt triplo majores radicibus datæ æquationis. Atque ita in æquatione $y^4 - \frac{43}{8}y^2 + \frac{75}{8}y - \frac{851}{256} = 0$, ad

tollendas fractiones ſcribe $\frac{1}{4}z$ pro y, & orietur $z^4 - 86z^2 + 600z - 851 = 0$.

165. 2. *Et hoc pacto poſſunt radicales* aliquando *tolli*, ubi eadem quantitas radicalis terminos alternos, vel a primo termino vel a ſecundo incipientes, ingreditur. Sic in æquatione $x^3\sqrt{3} - 2x\sqrt{3} + 1 = 0$, ſcribendo $\frac{y}{\sqrt{3}}$ pro x, prodit $\frac{y^3\sqrt{3}}{3\sqrt{3}} - \frac{2y\sqrt{3}}{\sqrt{3}} + 1 = 0$, ſive $y^3 - 6y + 3 = 0$. Itemque in æquatione $x^3 - p\sqrt{a}\,x^2 + qx - r\sqrt{a} = 0$ ſubſtituendo $\frac{y}{\sqrt{a}}$ pro x, emergit $y^3 - pay^2 + qay - ra^2 = 0$.

Sic et in æquatione $x^3 - 2x + \sqrt{3} = 0$, ſcribendo $y\sqrt{3}$ pro x, prodit $3y^3\sqrt{3} - 2y\sqrt{3} + \sqrt{3} = 0$, ſeu $3y^3 - 2y + 1 = 0$.

166. IV. *Æquationis radices in earum reciprocas transmutare.*

Sic æquatio noviſſima $3y^3 - 2y + 1 = 0$, cujus prima radix eſt -1, & reliquæ $\frac{1 \pm \sqrt{-\frac{1}{3}}}{2} = \frac{1 \pm \frac{1}{3}\sqrt{-3}}{2} = \frac{3 \pm \sqrt{-3}}{6}$; ſcribendo $\frac{1}{z}$ pro y, evadit $\frac{3}{z^3} - \frac{2}{z^2} + 1 = 0$, ſive concinnando & ordinando, $z^3 - 2z^2 + 3 = 0$, ejuſque prima radix -1, & reliquæ $\frac{3 \pm \sqrt{-3}}{2} = \frac{6 \times \overline{3 \pm \sqrt{-3}}}{12} = \frac{6}{3 \mp \sqrt{-3}}$ nempe radicum præcedentium reciprocæ.

167. 1. *Hoc pacto radix minima in maximam converti poteſt, & maxima in minimam*; quod uſum nonnullum habere poteſt in limitibus radicum exquirendis.

168. 2. *Poteſt etiam æquationis terminus penultimus hac operatione tolli, ſi modo ſecundus prius tollatur*; ut factum vides in exemplo noviſſimo. Aut ſi antepenultimum tolli cupias, id fiet ſi modo tertium prius tollas.

169. V. *Invenire æquationem cujus radices ſint quadrata radicum datæ æquationis.*

Proponatur æquatio cubica $x^3 - Px^2 + Qx - R = 0$, cujus radices ſint a, b, c, maximam deſignante a; & mutando ſigna terminorum alternorum orietur æquatio $x^3 + Px^2 + Qx + R = 0$, cujus radices idcirco erunt $-a$, $-b$, $-c$. Deinde his æquationibus in ſe ductis, emerget æquatio 6 dimenſionum $(x^3 + Qx)^2 - (Px^2 + R)^2$ ſeu $x^6 + (2Q - P^2)x^4 + (Q^2 - 2PR)x^2 - R^2 = (x - a)(x + a)(x - b)(x + b)(x - c)(x + c)$ ſeu $(x^2 - a^2)(x^2 - b^2)(x^2 - c^2)$, cujus termini alterni deſunt; evaneſcentibus nimirum ſummâ radicum $a, -a, b, -b$, &c. ſummâ contentorum ſub ſingulis ternis, $-aab$, $+aab$, &c. ſub ſingulis quinis, $+aabbc$, $-aabbc$, &c. ob dimenſionem æquationis reſultantis ſemper parem. Minuendi gratiâ igitur ſubſtitue hinc inde y pro x^2, & deprimetur æquatio reſultans ad cubicam $y^3 + (2Q - P^2)y^2 + (Q^2 - 2PR)y - R^2 = (y - a^2)(y - b^2)(y - c^2)$, cujus radices ſunt a^2, b^2, c^2, ut manifeſtum eſt. Et ſic univerſaliter: Sit æquatio $x^n - Px^{n-1} + Qx^{n-2} - Rx^{n-3}$ + &c. $= 0$, eritque æquatio reſultans $y^n + (2Q - P^2)y^{n-1}$ $+ (2S - 2PR + Q^2)y^{n-2}$ + &c. $= 0$.

170. Porro, hanc operationem inſtituendo in æquationem *quadratorum*, tanquam datam, emerget æquatio cujus radices erunt *biquadrata* radicum æquationis datæ; eandemque iterando in æquationem proxime inventam, tanquam datam, æquationes *per ſaltum* invenire licet cujus radices ſint *poteſtates* $8^{\text{væ}}$, $16^{\text{mæ}}$, $32^{\text{mæ}}$, $64^{\text{mæ}}$, &c. radicum datæ æquationis, abſque tædio omnes poteſtates intermedias radicum inveniendi, per Formulas, § 79.

Sic æquatio propoſita, ſublato termino ſecundo, erit $x^3 + Qx - R = 0$, unde, evaneſcentibus terminis æquationum reſultantium quos intrant dimenſiones P, erunt *

$$y^3 + 2Q\,y^2 + Q^2 y - R^2 = 0, \text{ \textit{æquat. quadrat.}}$$

$$y^3 - 2Q^2 y^2 + (Q^4 + 4QR^2)\,y - R^4 = 0; \text{ \textit{æquat.} 4 \textit{poteſt.}}$$

$$\left.\begin{array}{l} y^3 - (2Q^4 - 8QR^2)\,y^2 + \\ \quad + (Q^8 + 8Q^5R^2 + 12Q^2R^4)\,y - R^8 \end{array}\right\} = 0, \text{ \textit{æq.} 8 \textit{poteſt.}}$$

$$y^3 - (2Q^8 - 48Q^5R^2 + 40Q^2R^4)\,y^2 + \&c. = 0, \text{ \textit{æq.} 16 \textit{pot.}}$$

$$\left.\begin{array}{l} y^3 - (2Q^{16} - 224Q^{13}R^2 + 2288Q^{10}R^4 \\ \quad - 4224Q^7R^6 + 1320Q^4R^8 - 32QR^{10} \end{array}\right\} \begin{array}{l} y^2 + \&c. = 0, \\ \text{\textit{æq.} 32 \textit{poteſt.}} \end{array}$$

$$\left.\begin{array}{l} y^3 - (2Q^{32} - 960Q^{29}R^2 + 58464Q^{26}R^4 \\ \quad - 1048320Q^{23}R^6 + 7104240Q^{20}R^8 \\ \quad - 19997120Q^{17}R^{10} + 23772800Q^{14}R^{12} \\ \quad - 11410944Q^{11}R^{14} + 1961256Q^8R^{16} \\ \quad - 93632Q^5R^{18} + 672Q^2R^{20} \end{array}\right\} \begin{array}{l} y^2 + \&c. = 0, \\ \text{\textit{æq.} 64 \textit{pot.}} \end{array}$$

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* Vide *Bertrand, Elem. Mathematiques; Geneva*, 1778, vol. II, p. 501.

Unde mutando ſigna terminorum ſecundorum æquationum reſultantium habebitur ſumma poteſtatum 2, 4, 8, 16, 32, 64, &c. radicum æquationis datæ.

171. *Et hoc pacto, radices impoſſibiles, ſiquæ ſunt, utplurimum detegere poſſumus.*

Nam ſi æquationis datæ nullæ radices impoſſibiles ſint, ſingulorum radicum quadrata erunt affirmativa, adeoque termini æquationis quadratorum, vicibus alternis affirmativi & negativi. Ubi igitur in æquatione quadratorum occurrit ſignorum ſimilium ſucceſſio, indicio eſt æquationem datam radices impoſſibiles ſortiri. Et hoc pacto ſemper detegetur impoſſibilitas radicum æquationis cujuſcunque ubi P deleto, Q ſit affirmativum. Sic habitâ æquatione $x^3 + 6x - 20 = 0$, prodit æquatio quadratorum $y^3 + 12y^2 + 36y - 400 = 0$, quæ duas habet ſignorum ſimilium ſucceſſiones. Ergo, &c. Quòd ſi Q negativum ſit, licet radices ſint impoſſibiles, nulla deprehendi poteſt ſignorum ſucceſſio in æquatione quadratorum; verùm operationem continuando, in cubicis ſaltem, impoſſibilitatem radicum ſemper deteget æquatio biquadratorum, ubi R^2 ſit majus quam $\frac{1}{4}Q^3$, evadente tertio termino hujus $(Q^4 + 4QR^2)\,y$ negativo. Et ſimiliter in æquationibus altioribus caſus determinare licet, ubi radicibus impoſſibilibus detegendis impar erit hæc methodus. Sic habitâ æquatione $x^3 - 2x + 4 = 0$, prodit æquatio quadratorum $y^3 - 4y^2 + 4y - 16 = 0$, quæ nullam habet ſignorum ſimilium ſucceſſionem; at in æquatione biquadratorum $y^3 - 8y^2 - 112y - 256 = 0$, duæ prodeunt ſucceſſiones. Ergo, &c.

172. *Numerus autem ſucceſſionum ſimilium ſignorum ſemper æqualis vel minor erit numero radicum impoſſibilium.*

Sint enim radices impoſſibiles $a + b\sqrt{-1}$ & $a - b\sqrt{-1}$, harum quadrata erunt $a^2 - b^2 + 2ab\sqrt{-1}$ & $a^2 - b^2 - 2ab\sqrt{-1}$ reſpective, ſive, (poſito $a^2 - b^2 = \text{A}$, & $2ab = \text{B}$) $\text{A} + \text{B}\sqrt{-1}$ & $\text{A} - \text{B}\sqrt{-1}$ *; ubi A erit affirmativum vel negativum, prout a^2 ſit majus minuſve quam b^2. At multiplicando factores $x \pm \text{A} + \text{B}\sqrt{-1}$ & $x \pm \text{A} - \text{B}\sqrt{-1}$ in ſe generabitur æquatio quadratica $x^2 \pm 2\text{A}x \begin{smallmatrix} + \text{A}^2 \\ + \text{B}^2 \end{smallmatrix} = 0$,

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* Radices binas impoſſibiles æquationis algebraicæ ad formam $\text{A} \pm \text{B}\sqrt{-1}$ reduci poſſe, primus, ex calculo integrali demonſtravit *D'Alembert*, *Mem. Acad. Berlin*, 1746. Hunc ſecutus eſt *Euler*, *Mem.* 749; & *Foncenex*, *Mem. Turin*, 1760; cujus demonſtrationem ex principiis trigonometricis petitam impugnat *La Grange*, *Mem. Berlin*, 1772, aliamque proponit; contra vero tuetur *Bertrand*, *Elemen. des Mathematiques*, vol. 2, p. 489. Hujus reductionis adeo decantatæ veritatem, ex principiis algebraicis ſic colligere licet.

Æquationis quadraticæ $x^2 - px + q = 0$, radices impoſſibiles ſunt $\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$. Abbreviandi gratiâ, ſit $\frac{1}{4}p^2 - q = -r$, eritque $\sqrt{\frac{1}{4}p^2 - q} = \sqrt{-r} = \sqrt{r} \cdot \sqrt{-1}$, adeoque poſito $\frac{1}{2}p = a$ & $\sqrt{r} = b$, radices quadraticæ impoſſibiles formam induent $a \pm b\sqrt{-1}$. Et quoniam æquationis cujuſcunque algebraicæ radices impoſſibiles conflari neceſſe eſt per additionem, ſubductionem, multiplicationem, diviſionem, involutionem aut evolutionem quantitatum impoſſibilium ſive ſimplicium ſive mixtarum; deſignantibus A & B quantitatum per haſce operationes reſultantium partes reales cum propriis ſignis, erit,

$$\text{I.}\quad \begin{array}{r} a + b\sqrt{-1} \\ \pm g + h\sqrt{-1} \end{array} = (a \pm g) + (b \pm h)\sqrt{-1} = \text{A} + \text{B}\sqrt{-1}.$$

$$\text{II.}\quad \begin{array}{r} a + b\sqrt{-1} \\ \times g + h\sqrt{-1} \end{array} = (ag - bh) + (bg + ah)\sqrt{-1} = \text{A} + \text{B}\sqrt{-1}.$$

$$\text{III.}\quad \frac{a + b\sqrt{-1}}{g + h\sqrt{-1}} = \frac{(a + b\sqrt{-1})\,(g - h\sqrt{-1})}{(g + h\sqrt{-1})\,(g - h\sqrt{-1})} = \frac{(ag + bh) + (bg - ah)\sqrt{-1}}{g^2 - h^2}$$

$$= \text{A} + \text{B}\sqrt{-1}.$$

in qua ſucceſſiones ſignorum ſimilium ſunt duæ vel nullæ; ergo, &c. Et ſimiliter in æquationibus altioribus demonſtrare licet.

IV. $$(a+b\sqrt{-1})^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n}a^{\frac{m}{n}-1}(b\sqrt{-1}) + \frac{\frac{m}{n}\cdot\frac{m}{n}-1}{2.}a^{\frac{m}{n}-2}(b\sqrt{-1})^2 + \frac{\frac{m}{n}\cdot\frac{m}{n}-1\cdot\frac{m}{n}-2}{2.\quad 3.}a^{\frac{m}{n}-3}(b\sqrt{-1})^3$$

$$+\frac{\frac{m}{n}\cdot\frac{m}{n}-1\cdot\frac{m}{n}-2\cdot\frac{m}{n}-3}{2.\quad 3.\quad 4.}a^{\frac{m}{n}-4}(b\sqrt{-1})^4 \text{ \&c.}$$

Hujus ſeriei terminos impares occupant poteſtates pares quantitatis $b\sqrt{-1}$, & vice verſa. Sed poteſtates pares evadunt $-b^2$, $+b^4$, $-b^6$, &c. omnes nimirum reales, impares vero evadunt $+b\sqrt{-1}$, $-b^3\sqrt{-1}$, $+b^5\sqrt{-1}$, &c. ſeu $(+b-b^3+b^5-\&c.)\sqrt{-1}$. Adeoque aggregatum terminorum ſeriei imparium $a^{\frac{m}{n}} - \frac{\frac{m}{n}\cdot\frac{m}{n}-1}{2}a^{\frac{m}{n}-2}b^2 + \&c.$ deſignare poteſt A; parium vero $\left(\frac{m}{n}a^{\frac{m}{n}-1}b - \frac{\frac{m}{n}\cdot\frac{m}{n}-1\cdot\frac{m}{n}-2}{2\quad 3}a^{\frac{m}{n}-3}b^3 + \&c.\right)\sqrt{-1}$, deſignare poteſt $B\sqrt{-1}$.

Radix igitur ex his operationibus reſultans ſemper prodibit formæ $A+B\sqrt{-1}$, cui germana exſurget formæ $A-B\sqrt{-1}$, per Cor. VII. Ergo, &c.

Eadem quoque reductio valet in impoſſibilibus *exponentialibus*, ſeu quarum indices ſunt irrationales vel impoſſibiles, velut $(a\pm b\sqrt{-1})^{g+h\sqrt{-1}}$. Hunc vero caſum prætermittimus, quippe qui in Elementis Algebræ parum ſit utilis, ejuſque demonſtratio non niſi ex principiis trigonometricis & logiſticis petenda videtur. Si quis tamen deſiderat, conſulat *Auct. cit.* aut *Fenn's Elements of Specious Arithm. Dublin*, 1772, *p.* 222.

173. VI. *Invenire æquationem cujus radices ſint quadrata differentiarum radicum datæ æquationis.*

Proponatur æquatio $x^3 - Px^2 + Qx - R = 0$, cujus radices ſint a, b, c; erunt differentiæ radicum $a - b$, $b - a$, $a - c$, $c - a$, $b - c$, $c - b$; ſeu abbreviandi gratiâ, $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$. Adeoque erit $(y - \alpha)(y + \alpha)(y - \beta)(y + \beta)(y - \gamma)(y + \gamma)$ ſeu $(y^2 - \alpha^2)(y^2 - \beta^2)(y^2 - \gamma^2) = y^6 - P'y^4 + Q'y^2 - R' = 0$, evaneſcentibus nimirum terminis alternis. Subſtitue iterum z pro y^2, & emerget $(z - \alpha^2)(z - \beta^2)(z - \gamma^2) = z^3 - P'z^2 + Q'z - R' = 0$, cujus radices ſunt α^2, β^2, γ^2, quadrata nempe differentiarum radicum datæ æquationis, ut manifeſtum eſt. Et ſic univerſaliter: æquationis $x^n - Px^{n-1} + Qx^{n-2} - \&c. = 0$, erit numerus differentiarum radicum $= n \times \overline{n-1}$, & proinde æquatio quadratorum differentiarum erit $n \cdot \frac{n-1}{2}$ dimenſionum.

Jam quo eruantur coefficientes, P', Q', R', erit

Summa radicum $(a - b)^2 + (a - c)^2 + (b - c)^2 =$

$$= 2(a^2 + b^2 + c^2) - 2(ab + ac + bc) =$$

Adeoque P' . . $= 2(P^2 - 2Q) - 2Q = 2P^2 - 6Q$.

Summa rect. ſub binis rad. $(a - b)^2 (a - c)^2 + (a - b)^2 (b - c)^2 + (a - c)^2 (b - c)^2$

$$= \begin{cases} a^4 - 2a^3(b + c) + 3a^2b^2 \\ b^4 - 2b^3(a + c) + 3a^2c^2 \\ c^4 - 2c^3(a + b) + 3b^2c^2 \end{cases}$$

1. $a^4 + b^4 + c^4 = P^4 - 4P^2Q + 4PR + 2Q^2.$

2. $-2\left(a^3 \times \overline{b+c} + b^3 \times \overline{a+c} + c^3 \times \overline{a+b}\right) =$

$= -2\left(a^3 \times \overline{P-a} + b^3 \times \overline{P-b} + c^3 \times \overline{P-c}\right) =$

$= -2\left(\overline{a^3+b^3+c^3} \times P - \overline{a^4+b^4+c^4}\right) = -2P^2Q + 2PR + 4Q^2.$

3. $+3\left(a^2b^2 + a^2c^2 + b^2c^2\right) = 3\left(\overline{ab+ac+bc}^2 - 2abc \times \overline{a+b+c}\right) =$

$= 3(Q^2 - 2PR) = 3Q^2 - 6PR.$

$$\text{Adeoque } Q' = \left\{\begin{array}{rrrr} P^4 - 4P^2Q & + 4PR & + 2Q^2 \\ -2P^2Q & + 2PR & + 4Q^2 \\ & - 6PR & + 3Q^2 \\ \hline P^4 - 6P^2Q & * & + 9Q^2. \end{array}\right.$$

Summa content. ſub ternis rad. $(a-b)^2\ (a-c)^2\ (b-c)^2.$

$$= \left\{\begin{array}{lllll} a^4b^2 & -2a^4bc & +2a^3b^2c & -2a^3b^3 & -6a^2b^2c^2. \\ a^4c^2 & -2b^4ac & +2a^3bc^2 & -2a^3c^3 & \\ b^4a^2 & -2c^4ab & +2b^3a^2c & -2b^3c^3 & \\ b^4c^2 & & +2b^3ac^2 & & \\ c^4a^2 & & +2c^3a^2b & & \\ c^4b^2 & & +2c^3ab^2 & & \end{array}\right.$$

1. $a^4 \times \overline{b^2+c^2} + b^4 \times \overline{a^2+c^2} + c^4 \times \overline{a^2+b^2} =$

$= a^4 \times \overline{P^2 - 2Q - a^2} + b^4 \times \overline{P^2 - 2Q - b^2} + c^4 \times \overline{P^2 - 2Q - c^2} =$

$= (a^4 + b^4 + c^4)\, P^2 - (a^4 + b^4 + c^4)\, 2Q - (a^6 + b^6 + c^6) =$

$= -2P^3R + P^2Q^2 + 4PQR - 2Q^3 - 3R^2.$

2. $— 2(a^3 + b^3 + c^3)\,abc = — 2(P^3 — 3PQ + 3R)\,R =$
$= — 2P^3R + 6PQR — 6R^2.$

3. $2abc\,(a^2 \times \overline{b+c} + b^2 \times \overline{a+c} + c^2 \times \overline{b+c}) =$
$= 2R\,(\overline{a^2+b^2+c^2} \times P — \overline{a^3+b^3+c^3}) = 2PQR — 6R^2.$

4. $— 2(a^3b^3 + a^3c^3 + b^3c^3) =$
$= — 2(\overline{ab+ac+bc}^3 — 3(a^2 \times \overline{b+c} + b^2 \times \overline{a+c} + c^2 \times \overline{a+b}) —$
$— 6a^2b^2c^2) = 6PQR — 2Q^3 — 6R^2.$

5. $— 6a^2b^2c^2 = — 6R^2.$

Adeoque harum ſumma ſeu $R' = — 4P^3R + P^2Q^2 + 18PQR — 4Q^3 — 27R^2$. Unde prodit æquatio quadr. different. $z^3 — (2P^2 — 6Q)\,z^2 + (P^4 — 6P^2Q + 9Q^2)z + 4P^3R — P^2Q^2 — 18PQR + 4Q^3 + 27R^2 = 0$.

174. Tollendo autem terminum ſecundum datæ æquationis, formam induet ſimpliciorem æquatio quadratorum differentiarum, nempe

$$z^3 + 6Qz^2 + 9Q^2z + 4Q^3 + 27R^2 = 0.$$

175. Proponatur æquatio biquadratica $x^4 * + Qx^2 — Rx + S = 0$, eritque æquatio reſultans $z^6 — P'z^5 + Q'z^4 — R'z^3 + S'z^2 — T'z + V' = 0$.

Jam quo eruantur hujus coefficientes, erit
$P' = (a—b)^2 + (a—c)^2 + (a—d)^2 + (b—c)^2 + (b—d)^2 + (c—d)^2$
$= 3(a^2 + b^2 + c^2 + d^2) — 2(ab + ac + ad + bc + bd + cd)$
$= 3(* — 2Q) — 2Q = — 8Q.$

$$Q' = (a - b)^2 (a - c)^2 + (a - b)^2 (a - d)^2 + \&c.$$
$$+ (a - c)^2 (b - c)^2 + \&c.\ \&c.$$

$$= \begin{cases} 3a^4 - 4a^3 \times \overline{b+c+d} + 8a^2b^2 - 2a^2 \times \overline{bc+bd+cd} + 12abcd, \\ 3b^4 - 4b^3 \times \overline{a+c+d} + 8a^2c^2 - 2b^2 \times \overline{ac+ad+cd} \\ 3c^4 - 4c^3 \times \overline{a+b+d} + 8a^2d^2 - 2c^2 \times \overline{ab+ad+bd} \\ 3d^4 - 4d^3 \times \overline{a+b+c} + 8b^2c^2 - 2d^2 \times \overline{ab+ac+bc} \\ \qquad\qquad\qquad\qquad + 8b^2d^2 \\ \qquad\qquad\qquad\qquad + 8c^2d^2 \end{cases}$$

1. $3 (a^4 + b^4 + c^4 + d^4) = 3 (2Q^2 - 4S) = 6Q^2 - 12S.$
2. $- 4 (a^3 + b^3 + c^3 + d^3) P - (a^4 + b^4 + c^4 + d^4)$
   $= 4 (2Q^2 - 4S) = 8Q^2 - 16S.$
3. $+ 8 (a^2b^2 + a^2c^2 + \&c.) = 8 (Q^2 + 2S) = 8Q^2 + 16S.$
4. $- 2a (abc + abd + acd) - 2b (abc + \&c.) - \&c.\ \&c.$
   $= - 2 (a + b + c + d) R + 8abcd = + 8S.$
5. $+ 12abcd = 12S.$ Adeoque $Q' = 22Q^2 + 8S.$

Et cæteris coefficientibus R', S', T', V' ſimiliter inveſtigatis, prodit æquatio quadratorum differentiarum,

$$z^6 + 8Qz^5 + (22Q^2 + 8S) z^4 + (28Q^3 + 26R^2 + 16QS) z^3$$
$$+ (17Q^4 + 48R^2Q - 112S^2 + 24Q^2S) z^2 + (4Q^5 + 18Q^2R^2$$
$$+ 32Q^3S - 192QS^2 + 216R^2S) z + 256S^3 - 128Q^2S^2$$
$$+ 144R^2QS + 16Q^4S - 4R^2Q^3 - 27R^4 = 0.$$

176. Si proponatur æquatio quinque dimenſionum $x^5 + qx^3 - rx^2 + sx - t = 0$, erit æquatio quadratorum differentiarum radicum,

$$\begin{gathered}
z^{10} + 10qz^9 + (39q^2 + 10s)z^8 + (80q^3 + 50qs + 25r^2)z^7 \\
+ (95q^4 + 124q^2s - 95s^2 + 92qr^2 + 200rt)z^6 + (66q^5 \\
- 360qs^2 + 196q^3s + 118q^2r^2 + 260r^2s + 625t^2 + 400qrt)z^5 \\
+ (25q^6 + 40s^3 - 53r^4 + 52q^3r^2 - 522q^2s^2 + 194q^4s + \\
708qr^2s + 240q^2rt + 1750qt^2 - 950srt)z^4 + (4q^7 + 106q^5s \\
- 80qs^3 - 308q^3s^2 - 102qr^4 - 7q^4r^2 + 570r^2s^2 + 612q^2r^2s \\
+ 700r^3t - 3750t^2s + 2500t^2q^2 + 80q^3rt - 2150qrst)z^3 + \\
(400s^4 - 360q^2s^3 - 15q^4s^2 + 24q^6s - 8q^5r^2 - 45q^2r^4 - \\
270r^4s + 140q^3r^2s + 960qr^2s^2 + 1875r^2t^2 + 1000rts^2 - \\
5000qst^2 + 1750q^3t^2 + 40q^4rt + 600qtr^3 - 1650q^2rst)z^2 + \\
(36q^5s^2 - 224q^3s^3 + 320qs^4 + 4q^3r^4 + 27r^6 - 40r^2s^3 + \\
434q^2r^2s^2 - 24q^4r^2s - 198qr^4s + 5000s^2t^2 - 450r^3st - \\
6250rt^3 + 675q^4t^2 - 3750q^2st^2 + 3000qr^2t^2 + 60q^2r^3t + \\
200qrts^2 - 330q^3rst)z + 3125t^4 - 3750qrt^3 + \overline{2000qs^2 + 2250r^2s} \\
\overline{- 900sq^3 + 825q^2r^2 + 108q^5} \times t^2 - \overline{1600rs^3 - 560q^2rs^2 - 16q^3r^3 +} \\
\overline{630qsr^3 + 72q^4rs - 108r^5} \times t + \overline{256s^3 - 128q^2s^2 + 144qr^2s + 16q^4s} \\
\overline{- 4q^3r^2 - 27r^4} \times s^2 = 0.
\end{gathered}$$

177. Et ſic univerſaliter: propoſita æquatione $x^n - Px^{n-1} + Qx^{n-2} - \&c. = 0$, æquationis reſultantis $z^{\cdot\frac{\overline{n-1}}{2}} - P'z^{\frac{n.\overline{n-1}}{2} - 1} + Q'z^{\frac{n.\overline{n-1}}{2} - 2} - \&c. = 0$, erunt coefficientes

$$P' = \left\{ \begin{array}{l} \overline{n-1}\,(a^2 + b^2 + c^2 + \&c.) = (n-1)\,\overline{P^2 - 2Q} \\ -\,2\,(ab + ac + ad + \&c.) = \qquad\qquad\qquad -\,2Q \end{array} \right\} = \overline{n-1}\,P^2 - 2nQ.$$

$$Q' = \left\{ \begin{array}{l} \overline{n-1\,.\,\frac{n-2}{2}}\,(a^4 + b^4 + c^4\ \&c.) = \overline{n-1\,.\,\frac{n-2}{2}}\,(P^4 - 4P^2Q + 4PR + 2Q^2 - 4S) \\ -\,2\,.\,\overline{n-2}\,(a^3 \times \overline{b+c+d\ \&c.} + b^3 \times \overline{a+c+d\ \&c.} + \&c.) = -\,2\,.\,\overline{n-2}\,(P^2Q - PR - 2Q^2 + 4S) \\ -\,2\,.\,\overline{n-3}\,(a^2 \times \overline{bc+bd+cd}\ \&c. + b^2 \times \overline{ac+ad+\&c.} + \&c.) = -\,2\,.\,\overline{n-3}\,(PR - 4S) \\ +\,n\,.\,n-2\,(a^2b^2 + a^2c^2 + a^2d^2 + \&c.) = n\,.\,\overline{n-2}\,(-\,2PR + Q^2 + 2S) \\ +\,12\,(abcd + abce + \&c.) = 12S. \end{array} \right.$$

$$= \overline{n-1}\,.\,\overline{\frac{n-2}{2}}\,P^4 - 2n\,.\,\overline{n-2}\,P^2Q - 2\,.\,\overline{n-3}\,PR + \overline{2n+3}\,.\,\overline{n-2}\,Q^2 + 2nS.$$

Et ſic deinceps.

178. 1. *Hac operatione radices impoſſibiles, ſiquæ ſunt, earumque numerum ſemper detegere poſſumus.*

Nam ſi omnes datæ æquationis radices ſint reales, neceſſe eſt ut quadrata differentiarum radicum ſint omnia affirmativa, adeoque æquationis reſultantis termini vicibus alternis affirmativi & negativi. Quòd ſi data æquatio radices impoſſibiles ſortiatur, cum differentia duarum correſpondentium radicum impoſſibilium $a + b\sqrt{-1}$ & $a - b\sqrt{-1}$, vel $+ b\sqrt{-1}$ & $- b\sqrt{-1}$, ſit $2b\sqrt{-1}$, erit hujus quadratum $- 4b^2$, reale nempe & negativum; unde æquationi reſultanti competit una ſaltem radix realis & negativa, et proinde una ſaltem ſucceſſio ſignorum ſimilium. Atque hinc numerus radicum impoſſibilium in data æquatione *duplum* radicum negativarum, ſeu ſucceſſionum ſignorum ſimilium in æquatione reſultante nunquam tranſcendere poteſt.

179. Ut ſi proponatur æquatio cubica $x^3 + qx - r = 0$, ubi q affirmativum eſt, vel ubi q negativum eſt, & $4q^3$ ſimul minus quam $27r^2$, emerget in æquatione reſultante § 174 ſucceſſio ſimilium ſignorum, adeoque æquationis datæ duæ erunt radices impoſſibiles, tertia autem realis & affirmativa erit ſi r ſit negativum, et vicè verſâ, per § 146. Sic habitâ æquatione $x^3 - 2x + 4 = 0$, erit æquatio quadratorum differentiarum $z^3 - 12z^2 + 36z + 300 = 0$, adeoque ob ſucceſſionem ſimilium ſignorum, datæ æquationis duæ erunt radices impoſſibiles, tertie autem negativa. In cubicis igitur ſemper definietur numerus radicum impoſſibilium, utpote qui binarium tranſcendere nequit.

180. Si proponatur æquatio biquadratica $x^4 + qx^2 - rx + s = 0$, ubi in æquatione reſultante prodit ſucceſſio ſignorum ſimilium, conſtat æquationem datam radices impoſſibiles ſortiri. Cæterum *numerum* quoque radicum impoſſibilium, terminis rite collatis, eruere licet. Fingatur enim omnes datæ æquationis radices eſſe impoſſibiles, eritque æquationis reſultantis ultimus terminus $256s^3 - 128q^2s^2 +$ &c. affirmativus, cum ex ſex radicibus ejus, binæ quæque correſpondentes radices in ſe ductæ factum referent affirmativum, formæ B^2 vel $A^2 + B^2$, adeoque factum ſub omnibus erit affirmativum. Et proinde ſi $256s^3 - 128q^2s^2 +$ &c. emergat negativus, æquationis datæ quatuor radices impoſſibiles eſſe nequeunt.

Hinc æquationis biquadraticæ radices impoſſibiles, ſiquæ hac operatione detegantur, erunt vel quatuor, ubi æquationis reſultantis terminus ultimus eſt affirmativus aut nihil; vel duæ tantum, ubi eſt negativus: & duæ quæ ſuperſunt radices reales, contrariis afficientur ſignis, ubi ultimus datæ æquationis terminus eſt negativus, per § 147.

181. Similiter, æquationis quintæ dimenſionis, quæ neceſſario unam habet radicem realem, erunt radices impoſſibiles quatuor, ubi æquationis reſultantis terminus ultimus eſt affirmativus aut nihil; vel duæ tantum, ubi eſt negativus. Atque iiſdem veſtigiis inſiſtendo, numerum radicum impoſſibilium æquationis cujuſcunque eruere licet.

182. Operoſa certe evadit hæc methodus eruendi radices impoſſibiles, præſertim in æquationibus dimenſionum ſuperiorum. At ubi faciliore uti non licet, difficilem faſtidire haud æquum eſt. Doctrinam radicum impoſſibilium ſiquis penitius excutere velit, auctorem ſupra laudatum,

at paulo quidem reconditiorem, *Waring*, conſulat, quem in hac re, potiſſimum, ut interpretes ſecuti ſumus. *Med. Algeb. Prob.* 5 & 11. Conſulat quoque *La Grange*, *Acad. Berlin*, *an.* 1767, 1768, & 1772.

183. 2. *Hac operatione quoque, ubi duæ æquationis radices ſunt æquales, detegere poſſumus.*

Nam ubi ultimus æquationis reſultantis terminus evaneſcit, neceſſe eſt ut una ex radicibus ejus evaneſcat, ſive ut quadratum differentiæ duarum radicum datæ æquationis evaneſcat, adeoque ut hæ duæ radices ſint inter ſe æquales. Defectus igitur termini ultimi æquationis reſultantis, indicio eſt æquationem datam duas radices inter ſe æquales ſortiri.

184. Facilius autem ope Tranſmutationis II, caſus detegere licet, ubi duæ, tres, pluresve datæ æquationis radices, ſint inter ſe æquales. Nam ubi ultimus transformatæ terminus evaneſcit, erit una radix ſeu valor $y = 0$, adeoque datæ æquationis una radix ſeu valor $x = \pm c$, ob $x \mp c = y = 0$. Ubi duo poſtremi transformatæ termini evaneſcunt, erunt duæ radices ſeu valores $y = 0$, adeoque datæ æquationis duæ radices $x = \pm c$, adeoque æquales inter ſe. Similiter, ubi tres poſtremi termini deſunt, erunt datæ æquationis tres radices æquales inter ſe. Et ſic deinceps.

185. Hinc ſi datæ æquationis duæ ſint radices inter ſe æquales, transformatæ coefficiens penultima, quæ in hoc caſu fit nihil, ſubſtituendo x pro c, æquationem conſtituet dimenſionis depreſſioris cui radix erit c, una nempe radicum æqualium datæ æquationis. Si datæ æquationis tres radices ſint inter ſe æquales, coefficiens transformatæ ante-

penultima æquationem præbebit dimenſionis adhuc depreſſioris, cui radix erit una ex tribus radicibus æqualibus datæ æquationis. Atque ſic deinceps.

His traditis de radicum natura, atque præcipuis affectionibus & tranſmutationibus æquationum, pergamus jam reſolutionem æquationum radices rationales habentium docere.

DE RESOLUTIONE ÆQUATIONUM RADICES RATIONALES HABENTIUM.

186. Quoniam æquationis cujuſcunque $x^n \,.\, Px^{n-1} \,.\, Qx^{n-2} \,.\, \&c.\ U = 0$, terminos nec fractos nec ſurdos habentis omnes radices non ſurdæ ſunt aliqui ex diviſoribus *integris* termini ultimi; per § 144, COR. VI. Subſtituendi ſunt in æquatione pro *x*, omnes diviſores ultimi termini tam ſimplices quam compoſiti ſigillatim, et ſi horum aliquis omnes æquationis terminos ſe mutuo deſtruere efficiet, radix æquationis erit. Quòd ſi diviſorum nullus hanc conditionem impleverit, patet nullam eſſe æquationis radicem niſi quæ ſit aut irrationalis aut impoſſibilis.

Invenientur autem omnes diviſores ultimi termini, *ſi quantitas ſimplex eſt, eam dividendo per minimum ejus diviſorem, & quotum per minimum diviſorem ejus, donec quotus reſtet indiviſibilis. Hi diviſores una cum quoto indiviſibili conſtituent omnes quantitatis diviſores primos ſeu ſimplices; et rectangula aut contenta ſub ſingulis binis, ternis, quaternis, &c. horum diviſorum, omnes ejus diviſores compoſitos.*

Sic in æquatione $x^3 - 4x^2 - 7x + 10 = 0$, diviſores termini ultimi prodeunt $1, -1, 2, -2, 5, -5, 10, -10$; ſcriptis autem ſucceſſive

$$\left.\begin{array}{r}1\\-1\\2\\-2\\5\end{array}\right\}\text{pro } x\text{, prodit}\left\{\begin{array}{l}1-4-7+10=0, \textit{ adeoq. } +1 \textit{ erit radix}\\ -1-4+7+10=12,\\ 8-16-14+10=-12,\\ -8-16+14+10=0, \textit{ adeoq. } -2 \textit{ erit radix}\\ 125-100-35+10=0, \textit{ adeoq. } +5 \textit{ erit radix}\end{array}\right.$$

&c. &c.

187. Inventis hac methodo omnibus radicibus æquationis *præter unam*, hæc facilius erui poteſt, dividendo terminum ultimum ſub ſigno mutato, per contentum radicum jam inventarum. Sic in hac æquatione, dividendo -10 per $+1 \times -2 = -2$, quotus $+5$ erit tertia radix.

Inventis omnibus radicibus *præter duas*, hæ erui poſſunt ſive ſint reales ſive impoſſibiles, deprimendo æquationem datam ad quadraticam, ope factorum ſimplicium vel compoſitorum ex radicibus jam inventis reſultantium, & hujus reductio dabit duas quæ ſuperſunt radices. Sic inventâ $+1$ radice æquationis cubicæ $x^3 - 4x^2 - 7x + 10 = 0$, dividendo per $x - 1 = 0$ reſultabit quadratica $x^2 - 3x - 10 = 0$, cujus radices ſunt $+5$ vel -2. Inventis item $+2$ & $+4$ radicibus biquadraticæ $x^4 - 4x^3 - 8x + 32 = 0$, dividendo per $x - 2 = 0$, et rurſus quotum per $x - 4 = 0$; vel ſimul dividendo per $(x - 2)(x - 4) = 0$, emerget æquatio quadratica $x^2 + 2x + 4 = 0$, cujus radices ſunt $-1 + \sqrt{-3}$ & $-1 - \sqrt{-3}$. Habentur itaque quatuor radices quæſitæ.

188. *Idem brevius abſolvere licet tollendo ſecundum terminum.*

Sic in æquatione noviſſimâ ſcribendo $y + 1$ pro x, prodit $y^4 - 6y^2 - 16y + 21 = 0$, & tentando diviſores termini ultimi 21, duæ prodeunt radices rationales $+ 1$ & $+ 3$, quarum ſumma eſt 4, adeoque ob defectum ſecundi termini, ſumma reliquarum erit $- 4$. Deſignet v unam ex his, alteram deſignabit $- 4 - v$, eritque harum factum $- 4v - v^2 = \frac{21}{3} = 7$, adeoque $v^2 + 4v = - 7$, & $v = - 2 + \sqrt{-3}$ & altera $- 4 - v = - 2 - \sqrt{-3}$. Prodeunt igitur quatuor radices æquationis novæ $+ 1$, $+ 3$, $- 2 + \sqrt{-3}$, $- 2 - \sqrt{-3}$, quæ unitate auctæ dabunt radices æquationis datæ.

189. Cæterum ubi diviſores ultimi termini ſunt permulti, ut in æquatione $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120 = 0$, cujus ultimi termini diviſores 32 ſunt, nempe 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120, $- 1$, $- 2$, &c. hoſce omnes tentare tædio foret. Laboris igitur minuendi gratiâ, primus excogitavit *De Beaune* methodum inveniendi limites inter quos conſiſtunt radices, quam poſtea perfecit *Newtonus*. Jam enim non opùs erit diviſores tentare niſi qui ſunt inter hos limites.

DE LIMITIBUS ÆQUATIONUM.

190. I. *Radicum affirmativarum limites invenire.*

Minuantur radices æquationis datæ terminis progreſſionis 1, 2, 3, &c. ſucceſſive, donec omnes novæ æquationis termini evadant affirmativi, erunt omnes radices hujus negativæ, adeoque & radices datæ æquationis ſic diminutæ; quarum idcirco decrementum in hoc caſu omnem radicem affirmativam tranſcendet, § 158, ſive erit limes.

Ut ſi proponatur æquatio $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120 = 0$, evadent coefficientes æquationes novæ, per *Reg.* § 157.

Si x ſit	1	2	3 &c.
$x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120$	-38	$+46$	
$5x^4 - 8x^3 - 30x^2 + 60x + 63$	$+90$	$+79$	
$10x^3 - 12x^2 - 30x + 30$	-2	$+2$	
$10x^2 - 8x - 10$	-8	$+14$	
$5x - 2$	$+3$	$+8$	
1	$+1$	$+1$	

Quoniam igitur ſubſtituendo 1 pro x coefficientes aliquæ prodeunt negativæ, nimirum ultima $= -38$, quarta $= -2$, tertia $= -8$, limes erit major quam 1; at ſubſtituendo 2 pro x, prodeunt omnes affirmativæ, unde erit 2 major quam radicum affirmativarum maxima. Et hic obſervandum eſt, quod ſi coefficiens aliqua evaſerit negativa reliquarum valores invenire non neceſſe erit. Sic in hoc exemplo, ſi ſcribatur 1 pro x, prodit tertia coefficiens $10x^2 - 8x - 10 = -8$ negativa, adeoque operationem ulterius proſequi non neceſſe erit. Sæpe quidem ſufficiet ſi ſeligamus coefficientem aliquam ubi termini negativi maxime prævalere videntur, reliquis omiſſis.

191. Atque ideo, ſi termini duo plureſve primi æquationis propoſitæ ſint affirmativi, non neceſſe erit omnes coefficientes novæ æquationis eruere; ſiſti poteſt operatio, ubi ad coefficientem aliquam pervenitur cujus termini omnes ſunt affirmativi præter ultimum. Sic habitâ æquatione $x^7 + 3x^6 + 5x^5 - 7x^4 + 9x^3 - 11x^2 + 13x - 15 = 0$, coefficientes novæ æquationis evadent

$$\begin{array}{l}
x^7 + 3x^6 + 5x^5 - 7x^4 + 9x^3 - 11x^2 + 13x - 15 \\
7x^6 + 18x^5 + 25x^4 - 28x^3 + 27x^2 - 22x + 13 \\
21x^5 + 45x^4 + 50x^3 - 42x^2 + 27x - 11 \\
35x^4 + 60x^3 + 50x^2 - 28x + 9 \\
35x^3 + 45x^2 + 25x - 7 \\
\text{\&c.} \qquad \text{\&c.}
\end{array}$$

Inventis autem his quinque coefficientibus ultimis æquationis novæ, operationem ultra continuare non opus, cum coefficientis proxime inſequentis $21x^2 + 18x + 5$, necnon reliquarum deinceps, termini omnes ſunt affirmativi.

192. II. *Radicum negativarum limites invenire.*

Augeantur radices datæ æquationis, &c, donec termini alterni tantum evadant negativi, erunt omnes radices datæ æquationis ſic auctæ, affirmativæ; quarum idcirco augmentum, &c. erit radicum negativarum limes.

Vel quod perinde eſt, mutentur ſigna terminorum alternorum, & inveniatur per *præced.* limes radicum affirmativarum æquationis reſultantis, hic enim mutato ſigno limes erit radicum negativarum datæ æquationis.

Sic mutatis ſignis terminorum alternorum, coefficientes æquationis propoſitæ evadent

$$\begin{aligned}
&x^5 + 2x^4 - 10x^3 - 30x^2 + 63x + 120\\
&5x^4 + 8x^3 - 30x^2 - 60x + 63\\
&10x^3 + 12x^2 - 30x - 30\\
&10x^2 + 8x - 10\\
&\&c.
\end{aligned}$$

Ex his ſeligo coefficientem aliquam ubi termini negativi maxime prævalere videntur, puta $5x^4 + 8x^3 - 30x^2 - 60x + 63$, & hic ſubſtituendo pro x numeros 1, & 2, prodeunt numeri negativi -14 & -33 reſpective. Unde limes erit major quam -2. Subſtituendo autem numerum 3 prodit numerus affirmativus 234. Et ſimiliter in cæteris coefficientibus ſubſtituendo numerum 3 pro x prodit ſemper numerus affirmativus. Id quod ex inſpectione ſola colligere licet. Quare numerus -3 tranſcendit omnes radices negativas. Atque ita habentur limites 2 & -3 inter quos radices omnes æquationis propoſitæ conſiſtunt. Jam enim non opus erit diviſores termini ultimi 120 tentare, niſi qui ſunt inter hos limites, nimirum diviſores 1, -1, & -2. Sed horum nullus radix eſt, unde certum eſt æquationem non habere radicem niſi quæ ſit ſurda.

193. Hæc limitum inventio uſui erit in reductione æquationum quarum radices inter anguſtos limites concludantur, ſeu ſint parum inæquales inter ſe. Quod ſi radicum una ad maximum termini ultimi diviſorem accedat, altera ad minimum, factores complures ultimi termini inter ipſos limites conſiſtere poſſunt; quo in caſu, parum ſubſidii ad laborem operationis minuendum conferet hæc methodus.

Aliam methodum ſeligendi diviſores ultimi termini ad æquationem dividendam idoneos, *Newtonianâ* in praxi ſæpe commodiorem adjungere libet. Hujus inventorem fuiſſe *Waeſſenaer* refert *Schooten*, *Com. in Geom. Cartes*, *lib.* 3. *p.* 307. *Edit.* 1659.

194. *Minuantur radices æquationis propoſitæ numero quodam dato, puta unitate* (unitatem, vel denarium ad id aſſumere commodiſſimum erit) *& radices æquationis novæ unitate minores evadent radicibus datæ. Dein augeantur unitate diviſores ultimi termini novæ æquationis, & ſi horum aliqui ſic aucti inter diviſores termini ultimi datæ æquationis reperiantur, talium præ cæteris ratio habenda eſt. Et rurſus, augeantur unitate radices, &c. Dein minuantur unitate diviſores, &c. Secunda hæc operatio plures adhuc diviſores ſuperfluos reſecabit; & aliquando omnes tollet; ubi nimirum radices nullæ ſint rationales.*

Sic in æquatione $x^3 - x^2 - 30x + 72 = 0$, ſubſtituendo + 1 pro x, fiet ultimus terminus novæ æquationis + 42, ejuſque diviſores nempe 1, 2, 3, 6, 7, 14, 21, 42, — 1, — 2, &c; unitate aucti evadent 2, 3, 4, 7, 8, 15, 22, 43, 0, — 1, — 2, — 5, — 6, &c. Quibus collatis cum diviſoribus ultimi termini æquationis datæ 72, nempe

1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72, — 1, — 2, &c. utrinque deprehendentur 2, 3, 4, 8, — 1, — 2, — 6. Et rurſus, ſubſtituendo — 1 pro x, fiet ultimus terminus novæ æquationis + 100, ejuſque diviſores nempe, 1, 2, 4, 5, 10, 20, 25, 50, 100, — 1, — 2, — 4, &c. unitate diminuti evadent, 1, 3′, 4′, 9, &c. — 2′, — 3, — 5, — 6′, &c. quorum + 3, + 4, — 2, — 6, cum radicibus per præcedentem operationem reſervatis conſentiunt: ergo ultimi termini diviſores ſimplices tentandi, ſunt + 3, + 4, — 2 & — 6, vel æquationis diviſores compoſiti $x - 3$, $x - 4$, &c. unde prodeunt radices + 3, + 4, & — 6.

Sic quoque in æquatione præcedente $x^5 - 2x^4$ &c. — 120 = 0 ſubſtituendo + 1 pro x, fit ultimus terminus novæ æquationis — 38, ejuſque diviſores, 1, 2, 19, 38, — 1, — 2, &c. unitate aucti evadent 2, 3, 20, 39, — 1, — 18, — 37, quorum 2, 3, 20, — 1 tantum inter diviſores ultimi termini 120 reperiuntur. Et rurſus, ſubſtituendo — 1 pro x, prodit, &c. — 146, ejuſque diviſores 1, 2, 73, 146, — 1, — 2, &c. unitate diminuti evadunt 1, 72, 145, — 2, — 3, &c. omnes nimirum a diviſoribus per præcedentem operationem reſervatis diverſi: unde colligitur æquationem nullam ſortiri radicem niſi quæ ſit ſurda.

Diviſores ultimi termini datæ æquationis per hanc regulam *rejectos*, inter radices datæ æquationis cenſeri non poſſe ex eo conſtat, quod ſi fingantur radices eſſe datæ æquationis, unitate auctæ vel diminutæ deberent eſſe ex diviſoribus ultimi termini novæ æquationis, contra hypotheſin. Ex diviſoribus quoque *reſervatis*, raro omnes munere radicum fungentur, verùm diviſores hujuſmodi inutiles, ex operatione iteratâ minuendi radices, &c. deprehendi poſſint.

DE INVENTIONE DIVISORUM COMPOSITORUM ÆQUATIONIS.

195. Ex hâc *Waeſſanaeri* methodo ſeligendi diviſores ultimi termini æquationis datæ quorum præ cæteris ratio habenda eſt, deduxiſſe videtur *Newtonus* Regulam præclaram pro *inventione diviſorum compoſitorum unius dimenſionis*. Quam ſic inveſtigare licet.

Finge $ax + b$ diviſorem eſſe æquationis $px^n + qx^{n-1} + rx^{n-2}$ &c. $+ u = 0$, ubi a, b, p, q, &c. deſignant numeros integros. Neceſſe eſt diviſoris partem priorem ax, atque idcirco coefficientem ejus a, eſſe diviſorem numeralem altiſſimi termini px^n; itemque partem poſteriorem b eſſe diviſorem ultimi termini u: nam ſi non, $ax + b$ totius æquationis non foret diviſor. Subſtitue jam pro x ſigillatim tres vel plures terminos progreſſionis arithmeticæ decreſcentis 3, 2, 1, 0, − 1, &c. et erunt valores reſultantes

x		$px^n + qx^{n-1} + rx^{n-2}$ &c. $+ u$	$ax + b$
2	-	$2^n p + 2^{n-1} q + 2^{n-2} r$ &c. $+ u$	$2a + b$
1	-	$p \quad + q \quad + r$ &c. $+ u$	$a + b$
0	-	$* \quad * \quad * \quad + u$	$* + b$
-1	-	$\mp p \quad \pm q \quad \mp r$ &c. $+ u$	$-a + b$
&c.		&c.	&c.

ubi obſervandum eſt, quod diviſoris valores reſultantes $2a + b$, $a + b$, $+ b$, &c. conſtituent alteram progreſſionem arithmeticam communi differentia a decreſcentem, cujus terminus quiſque erit diviſor valoris correſpondentis ex ſubſtitutione ejuſdem numeri pro x in æquatione datâ reſultantis; adeoque terminus b e regione zero, diviſor erit valoris correſpondentis u, affirmativè ponendus ſi deſcendat series diviſorum, negativè ſi aſcendat, utpote in progreſſione decreſcente conſtitutus. Hinc elicitur regula inſequens.

196. *In æquatione propoſitâ pro* x *ſubſtitue ſigillatim tres vel plures terminos progreſſionis arithmeticæ,* 3, 2, 1, 0, — 1, *&c. ac valores totidem reſultantes una cum omnibus ſuis diviſoribus tam affirmativis quam negativis, ſtatue e regione correſpondentium terminorum progreſſionis. Dein e regione etiam ſtatue progreſſiones arithmeticas decreſcentes quæ per omnium numerorum diviſores percurrunt, & quarum termini differunt vel unitate, vel numero aliquo qui dividit altiſſimum terminum propoſitæ æquationis. Si qua occurrit ejuſmodi progreſſio, iſte terminus ejus qui ſtat e regione zero in progreſſione prima, cum ſigno ſuo annexus quantitati incognitæ* x *in communem differentiam terminorum ductæ, componet quantitatem per quam diviſio tentanda eſt.*

Ut ſi æquatio ſit $x^3 - x^2 - 10x + 6 = 0$, pro x ſubſtituendo ſigillatim terminos progreſſionis $1, 0, -1$, orientur numeri $-4, +6, +14$, quos cum omnibus eorum diviſoribus colloca è regione terminorum progreſſionis, hoc modo.

$$\begin{array}{ccccc} & & & \text{Diviſores} & \text{Progr. ar.} \\ x = \left\{\begin{array}{r} +1 \\ 0 \\ -1 \end{array}\right\} & x^3 - x^2 - 10x + 6 = & \left\{\begin{array}{r} -4 \\ +6 \\ +14 \end{array}\right\} & \begin{array}{l} 1 . 2 . 4 . \\ 1 . 2 . 3 . 6 \\ 1 . 2 . 7 . 14 \end{array} & \left\{\begin{array}{r} +4 \\ +3 \\ +2 \end{array}\right. \end{array}$$

Dein quoniam altiſſimus terminus x^3 per nullum numerum præter unitatem diviſibilis eſt, quære in diviſoribus progreſſionem decreſcentem cujus termini differunt unitate, & hujuſmodi progreſſionis unice reſultantis $4, 3, 2$, ſelige terminum $+3$, qui ſtat e regione zero in progreſſione prima, & diviſionem tenta per $x + 3$, unde prodibit $x^2 - 4x + 2 = 0$ cujus radices ſunt $2 \pm \sqrt{2}$. Adeoque tres radices ſunt -3, $2 + \sqrt{2}$, $2 - \sqrt{2}$.

Et rurſus ſi æquatio ſit $25x^4 + 110x^3 - 161x^2 - 190x + 144 = 0$, § 94, erit

$$\begin{array}{ccccc} x = \left.\begin{array}{r} +1 \\ 0 \\ -1 \\ \&c. \end{array}\right\} & 25x^4 + 110x^3 \ \&c. & \left\}\begin{array}{r} -272 \\ +144 \\ +288 \\ \&c. \end{array}\right. & \left\{\begin{array}{l} 1, 2, 4, 8, 16, \&c. \\ 1, 2, 3, 4, 6, \&c. \\ 1, 2, 3, 4, 6, \&c. \\ \&c. \end{array}\right\} & \begin{array}{l} +4 +8\&c. -1\&c. \\ +3 +3\&c. -2\&c. \\ +2 -2\&c. -3\&c. \\ \&c. \end{array} \end{array}$$

Ex his progreſſionibus tres ſeliguntur diviſores tentandi, nempe $x + 3$, $5x + 3$, & $x - 2$, quorum ultimus ſolus ſuccedit, et quotientem dat $25x^3 + 160x^2 + 159x - 72 = 0$.

Atque ita ſi æquatio ſit $24a^5 - 50a^4 + 49a^3 - 140a^2 + 64a + 30 = 0$, operatio erit ut ſequitur.

$$a = \left\{\begin{array}{r} +2 \\ +1 \\ 0 \\ -1 \end{array}\right\} 24a^5 - 50a^4\,\&c. = \left\{\begin{array}{r} 42 \\ 23 \\ 30 \\ 297 \end{array}\right\} \begin{array}{ll} 1, 2, 3, 6, 7, & \&c. \\ 1, 23, & \\ 1, 2, 3, 5, 6, & \&c. \\ 1, 3, 9, 11, 27, & \&c. \end{array} \left\{\begin{array}{rrr} +3 & +3 & +7 \\ +1 & -1 & +1 \\ -1 & -5 & -5 \\ -3 & -9 & -11 \end{array}\right.$$

Tres occurrunt hic progreſſiones quæ tres dant diviſores tentandos $2a - 1$, $4a - 5$, & $6a - 5$, quorum ultimus ſolus ſuccedit, prodeunte $4a^4 - 5a^3 + 4a^2 - 20a - 6 = 0$.

Atque hic obſervandum, quod progreſſionem 2, 1, 0, — 1, &c. continuando, facilius detegentur diviſores inutiles. Sic in exemplo noviſſimo, ſi quintus accedat terminus progreſſionis ſubſtituendo — 2 pro a, prodit — 2618, cujus ex diviſoribus 1, 2, 7, 11, 17, &c. — 17 ultimam progreſſionem 7, 1, — 5, — 11, continuabit, cæteris interruptis.

198. Si nullus occurrit hac methodo diviſor, vel nullus qui dividit propoſitam æquationem, concludendum erit æquationem illam non admittere diviſorem *unius* dimenſionis. Poteſt tamen fortaſſe, ſi plurium ſit quam trium dimenſionum, diviſorem admittere *duarum*. Et ſi ita, diviſor ille eruetur ope methodi inſequentis, quam ſic inveſtigare licet.

Finge $ax^2 + bx + c$ diviſorem eſſe æquationis $px^n + qx^{n-1}$ &c. $+ u = 0$. Neceſſe eſt ut a ſit diviſor numeralis termini altiſſimi px^n, aliter enim totam æquationem non divideret. Subſtitue jam ſigillatim pro x quatuor vel plures terminos progreſſionis arithmeticæ 3, 2, 1, 0, — 1, — 2, et erunt valores reſultantes

x	$px^{n} + qx^{n-1} + rx^{n-2} + \&c. + u$	$ax^{2} + bx + c$
3	$3^{n}p + 3^{n-1}q + 3^{n-2}r + \&c. + u$	$9a + 3b + c$
2	$2^{n}p + 2^{n-1}q + 2^{n-2}r + \&c. + u$	$4a + 2b + c$
1	$p + q + r + \&c. + u$	$a + b + c$
0	$* \quad * \quad * \quad \&c. + u$	$* \quad * + c$
-1	$\mp p \pm q \mp r \pm \&c. + u$	$a - b + c$
-2	$\mp 2^{n}p \pm 2^{n-1}q \mp 2^{n-2}r \pm \&c. + u$	$4a - 2b + c$

Ubi obſervandum eſt, quod ſi de valoribus ſucceſſivis diviſoris generalis ſubducantur primi horum termini $9a$, $4a$, a, &c. reſpective, reſidua $3b + c$, $2b + c$, $b + c$, &c. conſtituent alteram progreſſionem arithmeticam cujus differentia communis eſt b, & cujus terminus c e regione zerò in progreſſione prima erit ex diviſoribus termini correſpondentis u. Unde elicitur regula inſequens.

198. *In æquatione propoſita ſubſtitue pro* x, *quatuor vel plures terminos progreſſionis hujus* 3, 2, 1, 0, — 1, — 2, *& de diviſoribus omnibus numerorum reſultantium tam affirmativis quam negativis, ſubduc quadrata correſpondentium terminorum progreſſionis illius, in diviſorem aliquem numeralem termini altiſſimi æquationis ducta; & reſidua propriis cum ſignis e regione progreſſionis colloca. Dein progreſſiones omnes collaterales tam creſcentes quam decreſcentes nota quæ per iſta reſidua percurrunt. Et*

erit diviſor ille duarum dimenſionum tentandus, cujus primi termini coefficiens eſt prædictus diviſor numeralis termini altiſſimi; ſecundi coefficiens, differentia communis progreſſionis illius collateralis cum ſigno affirmativo ſi decreſcat progreſſio, negativo ſi creſcat; & e tertius terminus, iſte progreſſionis collateralis terminus qui ſtat regione zero in progreſſione primâ.

Invenietur autem differentia communis, ſubducendo terminum iſtiuſmodi progreſſionis e regione zero in progreſſione prima, de termino ejuſdem progreſſionis proxime ſuperiore, eritque differentia affirmativa aut negativa prout progreſſio decreſcat vel creſcat.

Ut ſi æquatio propoſita ſit $x^4 - x^3 - 5x^2 + 12x - 6 = 0$, ſubſtituendo pro x, ſucceſſive 3, 2, 1, 0, — 1, — 2, numeros prodeuntes 39, 6, 1, — 6, — 21, — 26, una cum eorum diviſoribus tam affirmativis quam negativis ſic diſpone

3	39	1,3,13,39,—1,—3,&c.	—8,—6,4,30,—10,—12,&c.	+4	—6
2	6	1,2, 3, 6,—1,—2,&c.	—5,—2,—1,2,—5,— 6,&c.	+2	—3
1	1	1, — 1,	0,—2,	+0	+0
0	— 6	1,2, 3, 6,—1,—2,&c.	1, 2, 3, 6,—1,—3,&c.	—2	+3
—1	—21	1,3, 7,21,—1,—3,&c.	0, 2, 6,20,—2,—4,&c.	—4	+6
—2	—26	1,2,13,26,—1,—2,&c.	—3,—2, 9,22,—5,—6,&c.	—6	+9

Dein de his diviſoribus ſigillatim ſubduc quadrata terminorum progreſſionis primæ ducta in diviſorem numeralem termini x^4, qui unitas eſt, *viz.* terminos 9, 4, 1, 0, 1, 4, & prodibunt termini — 8, — 6, &c. — 3, — 2, &c. e latere pariter diſpoſiti; qui duas præbent progreſſiones arithmeticas + 4 + 2, + 0, &c. et — 6, — 3, + 0, &c.

collaterales, Unde prodeunt diviſores tentandi $x^2 + 2x - 2$ & $x^2 - 3x + 3$, per quorum utrumque res ſuccedit. At factoris $x^2 + 2x - 2 = 0$ radices ſunt $-1 \pm \sqrt{-3}$, irrationales, et factoris $x^2 - 3x + 3 = 0$ radices $\frac{+3 \pm \sqrt{-3}}{2}$ ſunt impoſſibiles. Hæc regula igitur aliquando utilis erit in radicibus æquationum ſurdis & impoſſibilibus eruendis.

199. Si nullus inveniri poteſt hoc pacto diviſor qui ſuccedit, concludendum eſt æquationem propoſitam non admittere diviſorem duarum dimenſionum. Hinc quoque concludere licet, æquationem non admittere diviſorem trium dimenſionum, ſi modo quintum non tranſcendat gradum. Sin æquatio ad ſextum gradum aut ultra aſſurgat, poteſt forſan nullum admittere diviſorem niſi qui ſit trium pluriumve dimenſionum. Poſſet quidem eadem methodus extendi ad inventionem hujuſmodi diviſorum quærendo in prædictis reſiduis, progreſſiones non arithmeticas, ſed alias quaſdam quarum terminorum differentiæ primæ, ſecundæ, tertiæ, &c. ſunt in arithmetica progreſſione: at in his tyro non eſt detinendus. Siquis tamen hæc deſiderat, adeat *M^cLaurin*, *Algeb. Part.* II. § 70.

DE INVENTIONE DIVISORUM ÆQUATIONUM LITERALIUM QUARUM TERMINI SUNT HOMOGENEI.

200. Proponatur æquatio $x^n - ax^{n = 1} - 5a^2x^{n-2} + 12a^3x^{n-3}$ &c. $- 6a^n = 0$. Et ſi dividantur radices

hujus per a, ſubſtituendo ay pro x emerget $y^n - y^{n-1} - 5y^{n-2} + 12y^{n-3}$ &c. $- 6 = 0$. Deinde per regulas præcedentes quære hujus diviſorem, & reſtituendo in diviſore prodeunte $\frac{x}{a}$ pro y, & tollendo fractiones, habebitur datæ æquationis diviſor. Hinc elicitur regula inſequens.

201. *Ubi in æquatione propoſita duæ ſunt literæ, & omnes ejus termini ad dimenſiones æque altas aſcendunt; pro una iſtarum literarum pone unitatem, dein per regulas præcedentes quære diviſorem, ac diviſoris hujus comple deficientes dimenſiones reſtituendo literam illam pro unitate.*

Ut ſi æquatio ſit $6x^4 - ax^3 - 21a^2x^2 + 3a^3x + 20a^4 = 0$ ponendo 1 pro a, evadit $6x^4 - x^3 - 21x^2 + 3x + 20 = 0$, cujus eruetur diviſor $3x + 4$, & completa deficiente dimenſione poſterioris termini per dimenſionem a, fit $3x + 4a$ diviſor quæſitus. Ita ſi æquatio ſit $x^4 - bx^3 - 5b^2x^2 + 12b^3x - 6b^4 = 0$ poſito 1 pro b, & æquationis reſultantis $x^4 - x^3 - 5x^2 + 12x - 6 = 0$ invento diviſore $x^2 + 2x - 2$, completis ejus deficientibus dimenſionibus per dimenſiones b, habebitur diviſor quæſitus $x^2 + 2bx - 2b^2$.

Poſſet eadem methodus extendi ad inventionem diviſorum ubi in æquatione propoſitâ tres vel plures ſunt literæ, idque ſive termini ſint æquialti ſive non; cæterum hiſce non eſt immorandum. Conſulat lector horum ſtudioſus *Clairaut*, *Algeb.* 4ta edit. 1768, p. 184, &c.

DE RESOLUTIONE ÆQUATIONUM DUAS PLURESVE RADICES INTER SE ÆQUALES HABENTIUM.

202. Ubi ex § 183, vel aliunde compertum eſt æquationem binas pluresve radices æquales ſortiri, conferendo hanc cum æquatione dimenſionis inferioris inde eliciendâ, per § 185, facile inveniri poteſt valor unius radicum æqualium.

203. Ut ſi proponatur æquatio cubica $x^3 - px^2 + qx - r = 0$, duas habens radices inter ſe æquales, inde elicietur per § 185, æquatio quadratica $3x^2 - 2px + q = 0$, radicem habens unam radicum æqualium. Collatis autem hiſce æquationibus inter ſe, per Reg. § 70, reſultabit quadratica $px^2 - 2qx + 3r = 0$, et quadraticis iterum collatis, reſultabit ſimplex æq. $(2p^2 - 6q)\,x - pq + 9r = 0$, unde prodit $x = \frac{pq - 9r}{2p^2 - 6q}$.

Hunc valorem x unam radicum æqualium exponere, per compoſitionem æquationum conſtabit. Sint enim radices a, a, b, erit $p = 2a + b$, $q = a^2 + 2ab$, & $r = a^2b$. Adeoque his ſubſtitutis, prodit $x = \frac{pq - 9r}{2p^2 - 6q} = \frac{2a^3 - 4a^2b + 2ab^2}{2a^2 - 4ab + 2b^2} = a$ *. Sic habitâ æquatione $x^3 + 5x^2 - 32x + 36 = 0$,

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* Vide *Diſſert.* D. *Hellins*, *Phil. Tranſ.* an. 1782, p. 417.

fiat $p = -5$, $q = -32$, $r = -36$, erit $x = \frac{160 + 324}{50 + 192}$ $= \frac{284}{242} = 2$. Adeoque dividendo æquationem datam per $(x - 2)^2$ eruetur quotiens $x + 9 = 0$. Ergo tertia radix erit -9. Habitâ quoque æquatione $x^3 - \frac{3}{4}x - \frac{1}{4} = 0$, erit $x = \frac{-9r}{-6q} = \frac{-\frac{9}{4}}{\frac{18}{4}} = -\frac{1}{2}$. Adeoque tertia radix $+1$.

204. Si proponatur æquatio biquadratica $x^4 - px^3 + qx^2 - rx + s = 0$, duas habens radices reales, harum una erit radix æquationis cubicæ $4x^3 - 3px^2 + 2qx - r = 0$. Et collatis his æquationibus, emerget cubica $px^3 - 2qx^2 + 3rx - 4s = 0$. Et collatis cubicis, emerget quadratica $(3p^2 - 8q)\,x^2 + (12r - 2pq)\,x + pr - 16s = 0$, ſeu abbreviandi gratiâ, $Ax^2 + Bx + C = 0$. Et collatis iterum cubicis, per Reg. § 73, emerget altera quadratica $(pr - 16s)\,x^2 + (12ps - 2qr)\,x + 3r^2 - 8qs = 0$, ſeu $Cx^2 + Dx + E = 0$. Dein collatis quadraticis, emerget $(AD - BC)\,x + AE - C^2 = 0$, adeoque $x = \frac{C^2 - AE}{AD - BC}$. Reſtitutis autem valoribus A, B, C, &c. prodit

$$x = \frac{6p^2qs - 2p^2r^2 - 8prs + 6qr^2 - 16q^2s + 64s^2}{9p^3s - p^2qr - 3pr^2 - 32pqs + 4q^2r + 48rs}.$$

Deleto autem termino ſecundo æquationis, erit $x = \frac{6qr^2 - 16q^2s + 64s^2}{4q^2r + 48rs}$ $= \frac{(3r^2 - 8qs)\,q + 32s^2}{(q^2 + 12s)\,2r}$.

Sic habita æquatione $x^4 - 11x^2 + 18x - 8 = 0$, ſtatuendo $q = -11$, $r = -18$, $s = -8$, prodit $x = \frac{-(972 - 704)\,11 + 2048}{-(121 - 96)\,36} = \frac{-900}{-900} = +1$. Et dividendo æquationem per $(x - 1)^2$ prodit quotiens $x^2 + 2x - 8 = 0$, cujus reductio dabit reliquas radices $+2$ & -4.

Si tres biquadraticæ radices ſint inter ſe æquales, harum duæ erunt radices æquationis cubicæ $4x^3 - 3px^2 + 2qx - r = 0$, et una, quadraticæ $6x^2 - 3px + q = 0$. Et his collatis, eruetur $x = \frac{pq - 6r}{3p^2 - 8q}$. Ut ſi æquatio ſit $x^4 - x^3 - 3x^2 + 5x - 2 = 0$, erit $x = \frac{-3 + 30}{3 + 24} = +1$, adeoque dividendo per $(x - 1)^3$ eruetur quotiens $x + 2 = 0$, adeoque quarta radix erit -2.

Iiſdem veſtigiis inſiſtendo valores radicum æqualium, & inde reliquarum, in æquationibus dimenſionum ſuperiorum eruere licet.

205. Hactenus reductionem æquationum tradidimus quæ rationales diviſores admittant. Deficientibus autem hujuſmodi diviſoribus, aliquando reduci poteſt æquatio quatuor, ſex aut plurium dimenſionum, per ſurdum aliquem diviſorem. Methodum quidem tales diviſores inveniendi tradit *Newtonus*, quærendo nimirum quantitatem aliquam complexam quæ quadratum utriuſque partis æquationis propoſitæ compleat, quâ additâ, per extractionem

ſurdæ radicis quadraticæ utrobique res ſuccedit. Has autem reductiones prætereo, quarum utpote raro poſſibilium, perexiguus ſit uſus, ipſo fatente *Newtono*, "qui rei poſſi-"bilitatem potius quam praxim commodiſſimam voluerit "exponere." De his conſulendus eſt *M^cLaurin*, &c. et præcipue *Simpſon*, *Mathematical Tracts*.

Hanc analyſeos partem haud minore ſtudio excoluiſſe videtur *Leibnitz*, *Newtoni* æmulus, licet meditationes ejus in lucem non prodierunt. In epiſtolis ſuis ad *Oldenburg*, an. 1676, ita ſcribit.

"De æquationum radicibus ſurdis generalibus inveni-"endis, ſive quod idem eſt, tollendis æquationum poteſ-"tatibus intermediis, multa & ego meditatus ſum; & "jam vere anni ſuperioris ſpecimina *Hugenio* communica-"veram regularum *Cardanicis* ſimilium. Seriem enim ha-"bebam ejuſmodi regularum in infinitum euntem, in "quibus & *Cardanica* continebatur. Sed ultra gradum "cubicum non erant generales. Perſpexi tamen inde me-"thodum veram progrediendi longius.——Sed ſiquis la-"borem non ſubterfugeret, eum docere poſſum methodum "analyticam generalem infallibilem, per quam omnium "æquationum radices generales invenire liceret." Vide *Commerc. Epiſtol.* N° 53.

Merito tamen dubitare licet, annon promiſſa tam ingentia celeberrimi ſcriptoris fiduciam fefelliſſent, ſimodo calculis ſuis ultimum manum imponere voluerit. Multa enim ineunte calculo facilia fore fidimus, quæ temere aſſumpta & prorſus impoſſibilia arguet matura inveſtigatio.

Pergamus jam æquationum cubicarum & biquadraticarum reſolutiones generaliores docere.

DE RADICIBUS ÆQUATIONUM CUBICARUM.

206. Quo melius inſtituatur diſquiſitio de radicum cubicarum natura, commodum erit æquationem quamcunque cubicam, delendo terminum ſecundum, ad formam ſimpliciorem reducere $x^3 * . qx = . r$; ubi variantibus ſignis q & r, quatuor diſtinguuntur caſus, quorum tres priores tantùm (ut in quadraticis) pro legitimis habuerunt analyſtæ vetuſtiores.

I. $x^3 + qx = + r$

II. $x^3 - qx = + r$

III. $x^3 - qx = - r$

IV. $x^3 + qx = - r$.

207. Deſignante p, primam radicem quoquo modo inveniendam, quæ ſemper realis erit, dividatur æquatio $x^3 + qx - r = 0$, per factorem ſimplicem $x - p = 0$, et (evaneſcente reſiduo $p^3 + pq - r$) emergit quotiens $x^2 + px + p^2 + q = 0$. Et extracta hujus radice quadratica, prodit $x = -\frac{1}{2}p \pm \sqrt{-\frac{3}{4}p^2 - q}$. Æquationis itaque $x^3 + qx = r$, tres radices ſunt p, $-\frac{1}{2}p + \sqrt{-\frac{3}{4}p^2 - q}$, vel $-\frac{1}{2}p - \sqrt{-\frac{3}{4}p^2 - q}$. *Et ſic univerſaliter : æquationis* $x^3 . qx = r$, *ſi una radix ſit* p, *erunt reliquæ* $\frac{1}{2}p \pm \sqrt{-\frac{3}{4}p^2 . q}$, *ubi* $\frac{1}{2}p$ & q *cum ſignis mutatis, ſed* $\frac{3}{4}p^2$ *ſemper negativè ponendum.* Sunt igitur duæ

poſteriores radices ſemper impoſſibiles ubi q in data æquatione affirmativum eſt, adeoque $\sqrt{-\frac{3}{4}p^2-q}$ quantitas impoſſibilis; aut ubi q negativum eſt & ſimul minus quam $\frac{3}{4}p^2$, adeoque $\sqrt{-\frac{3}{4}p^2+q}$ impoſſibilis: at ubi q negativum eſt ac ſimul æquale aut majus quam $\frac{3}{4}p^2$, ſunt ſimul reales, aut evaneſcente, aut poſſibili jam factâ $\sqrt{-\frac{3}{4}p^2+q}$.

Ibi itaque verſabitur limes radicum realium & impoſſibilium ubi q in data æquatione negativum eſt & æquari primùm incipit $\frac{3}{4}p^2$. In ipſo autem limite, prodit $\overline{\frac{1}{3}q}|^3 = \overline{\frac{1}{4}p^2}|^3 = \frac{1}{64}p^6$. Itemque ſubſtituendo $\frac{3}{4}p^2$ pro q in æquationem $p^3 - pq = . r$, prodit $\frac{1}{4}p^3 = . r$, adeoque $\overline{\frac{1}{2}r}|^2 = \overline{\frac{1}{8}p^3}|^2 = \frac{1}{64}p^6$. Ergo, $\overline{\frac{1}{3}q}|^3 = (\frac{1}{64}p^6 =) \overline{\frac{1}{2}r}|^2$.

Ubi vero q in data æquatione negativum eſt ac ſimul minus quam $\frac{3}{4}p^2$, prodit $\overline{\frac{1}{3}q}|^3$ minus quam $\overline{\frac{1}{4}p^2}|^3$ ſeu quam $\frac{1}{64}p^6$. Sed in hoc caſu, $r\ (= p^3 - pq)$ fit majus quam $\frac{1}{4}p^3$, adeoque $\overline{\frac{1}{2}r}|^2$ majus quam $\overline{\frac{1}{8}p^3}|^2$ ſeu quam $\frac{1}{64}p^6$. Ergo, *a fortiori*, prodit $\overline{\frac{1}{3}q}|^3$ minus quam $\overline{\frac{1}{2}r}|^2$. Contra ſi q ſit majus, &c.

268. Hinc partim ex ſigno quo afficitur q in data æquatione, partim ex magnitudine ejus reſpectu r, innoteſcit geneſis radicum realium & impoſſibilium in æquatione $x^3 . qx = . r$. Nam ubi q affirmativum eſt, qualiſcunque

ſit ejus magnitudo, ſemper duæ ex radicibus æquationis ſunt impoſſibiles; idem quoque fiet, ubi q negativum eſt & $\frac{q^3}{27}$ ſimul minus quam $\frac{1}{4}r^2$. At ubi q negativum eſt & $\frac{q^3}{27}$ ſimul æquale aut majus quam $\frac{1}{4}r^2$, ſunt omnes radices reales. Sicut aliunde collegimus, § 179, 183.

Hanc methodum deducendi affectiones radicum æquationum cubicarum termino ſecundo carentium immediate ex ipſa æquationum natura, ſuggeſſit diſſertatio viri in hiſce verſatiſſimi, *Hutton*, *on Cubic Æquations*, & *infinite Series* *.

De resolutione æquationum cubicarum.

209. Proponatur æquatio cubica cujus ſecundus terminus deeſt, $x^3 * + qx = r$. Ad hanc enim formam æquationem omnem cubicam reduci poſſe conſtat ex præcedentibus. Finge x duas in partes dividi, quarum ſit major y & minor z. Et ſubſtituendo $y + z$ pro x in hac æquatione, prodit $y^3 + z^3 + 3yz\ (y + z) + q\ (y + z) = r$. Finge iterum $3yz$ æquari q cum ſigno mutato, et terminis $3yz$ $(y + z)$ atque $q\ (y + z)$ hoc pacto ſe deſtruentibus, manebit $y^3 + z^3 = r$, ſive, ſubſtituendo $\frac{-q^3}{27y^3}$ pro z^3, (ob $3yz$

* *Phil. Tranſ.* an. 1780, p. 387.

$= - q$, adeoque $z = \frac{-q}{3y}$) erit $y^3 - \frac{q^3}{27y^3} = r$, ſeu $y^6 = ry^3 + \frac{q^3}{27}$. Adeoque per reductionem, $y^3 = \frac{1}{2}r + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ & $y = \sqrt[3]{:\ \frac{1}{2}r + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}$. Itemque $z^3 =$ $(r - y^3 = r - \frac{1}{2}r - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} =)\ \frac{1}{2}r - \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ & $z = \sqrt[3]{:\ \frac{1}{2}r - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}$ *. *Et ſic univerſaliter: Si ſit* $x^3 . qx = . r$, *erit* $x = \sqrt[3]{:\ \frac{1}{2}r + \sqrt{\frac{r^2}{4} . \frac{q^3}{27}}} + \sqrt[3]{:\ \frac{1}{2}r - \sqrt{\frac{r^2}{4} . \frac{q^3}{27}}}$. *Ubi* $\frac{1}{2}r$ & $\frac{q^3}{27}$ *iiſdem ſignis ac* r & q *in æquatione priore afficienda ſunt, ſed* $\frac{1}{4}r^2$ *ſemper affirmativè ponendum.*

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* Hæc formula appellatur *Cardani regula*, licet inventionem ejus *Scipioni Ferreo* Bononienſi tribuit ipſe *Cardanus*. Diſcipulo ſuo *Antonio M. Florido* formulam tradidit *Ferreus*. At *Florido* cum in certamen veniſſet cum *Nicoluo Tartalea* Brixellenſi, occaſionem dedit ut *Tartalea*, rem diu ſecum meditatus extuderit, cæteroſque formulæ caſus ſimul invenerit: et adverſarii problematis 30 ſpatio duarum horarum (ut ipſe ſcribit) reſolutis, alia viciſſim iſti objecit ex cæteris caſibus, quibus ſolvendis omnino impar erat. Formulam ipſam *Cardano* roganti, multis precibus exoratus impertivit *Tartalea*, ſuppreſsâ tamen demonſtratione, eumque juramento adſtrinxit, ſe hanc neutiquam vulgaturum nec niſi cifris commiſſurum. Verùm *Cardanus* promiſſorum immemor, hanc publici juris fecit in opere quodam *De Arte magna*, an. 1545. Unde ingens controverſia exarſit, reclamante *Tartalea* fidem fregiſſe *Cardanum*, regerente *Cardano*, ſe regulam novis inventis auctam & demonſtratione inſuper munitam, quaſi propriam jure edidiſſe, necnon ſcrupulum injiciente an *Tartalea* inventum ſibi recte vindicaverit. Nec extinctæ ſunt lites niſi cum ipſo *Tartalea*, qui an. 1557 obiit.

Quanti hanc formulam fecit *Cardanus*, ex ipſius verbis conſtabit. " Temporibus noſtris *Scipio Ferreus* Bononienſis capitulum cubi & rerum numero æqualium invenit. Rem ſane pulchram & admirabilem, cum omnem humanam ſubtilitatem, omnis ingenii mortalis claritatem ars hæc longe ſuperet, Inventâ primâ radice $x = y + z$, innoteſcent reliquæ. Quoniam enim unitati competit radix cubica triplex, nempe,

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" donum profecto cœleſte, experimentum autem virtutis animorum, atque adeo " illuſtre, ut qui hæc attigerit nihil non intelligere poſſe ſe credat." *Cardan. Opera.* Tom IV, p 222. *Cardanum* vero fefellit radices reales, in *Caſu irreducibili*, ſub ſpecie impoſſibili adumbrari. Primus hoc exploratum habuit *Bombellius*.

Hujus regulæ hiſtoriam concludemus cum verſibus Italicis quibus primi caſus ſolutionem expoſuit *Tartalea.*

Quando che il cubo con le coſe appreſſo
S' agguaglia a qualche numero diſcreto,
Trovami due altri differenti in eſſo
Dapoi terrai queſto per conſueto
Ch' il lor producto ſempre ſi eguale
Al terzo cubo delle coſe netto;
El reſiduo poi tuo generale
Delli lor lati cubi ben ſottratto
Verra la tua coſa principale.

Hoſce verſus ſic iterpretari licet.

Quando *cubus* cum *rebus* adjectis æquatur *numero*, i. e. $x^3 + qx = r$, inveniantur duo alii numeri (y & z) quorum differentia ſit r, & productum (yz) æquale cubo tertiæ partis coefficientis rerum $\left(\frac{q^3}{27}\right)$. Sed per primam conditionem, prodit $r + z = y$ & $y - r = z$, adeoque per ſecundam conditionem, $y^2 - ry = \frac{q^3}{27}$, & per reductionem, $y = \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} + \frac{1}{2}r$. Itemque, $rz + z^2 = \frac{q^3}{27}$, & per reductionem, $z = \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} - \frac{1}{2}r$. Deinde radicibus y & z cu-

1, $\frac{-1+\sqrt{-3}}{2}$, vel $\frac{-1-\sqrt{-3}}{2}$, coefficiente triplici exponendæ ſunt y & z, ſcilicet

$$1\,y,\ \text{vel}\ \frac{-1+\sqrt{-3}}{2}\,y,\ \text{vel}\ \frac{-1-\sqrt{-3}}{2}\,y,$$

$$\&\ 1\,z,\ \text{vel}\ \frac{-1-\sqrt{-3}}{2}\,z,\ \text{vel}\ \frac{-1+\sqrt{-3}}{2}\,z.$$

Ubi $\sqrt{-3}$ contrariis ſignis afficiendum in valoribus correſpondentibus y & z, ut productum ex his in unoquoque caſu ſit $yz = \frac{-q}{3}$. Quod hoc pacto ita ſe habet, nam

$$\frac{-1+\sqrt{-3}}{2}\,y \times \frac{-1-\sqrt{-3}}{2}\,z = \frac{1+3}{4}\,yz = yz.$$

Prodeunt igitur tres radices,

1. $y + z$,

2. $\frac{-1+\sqrt{-3}}{2}\,y + \frac{-1-\sqrt{-3}}{2}\,z =$

$$= \frac{-y-z}{2} + \frac{y-z}{2}\sqrt{-3} = -\frac{y+z}{2} + \frac{y-z}{2}\sqrt{-3}.$$

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bice extractis, & ſubductâ majore radice cubicâ a minore, prodit *res* ſeu $x = \sqrt[3]{:\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}+\frac{1}{2}r} - \sqrt[3]{:\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}-\frac{1}{2}r}$, ſeu quod eodem redit, $= \sqrt[3]{:\frac{1}{2}r+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}} + \sqrt[3]{:\frac{1}{2}r-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}$. Vide *Montucla, Hiſt. de Mathem.* Vol. I, p. 480. Et *Diſſert. De Gua. Mem. de l'Acad.* an. 1741.

3. $\frac{-1-\sqrt{-3}}{2}y + \frac{-1+\sqrt{-3}}{2}z =$

$$= \frac{-y-z}{2} + \frac{-y+z}{2}\sqrt{-3} = -\frac{\overline{y+z}}{2} - \frac{\overline{y-z}}{2}\sqrt{-3}.$$

Ubi obſervandum eſt, quod ſi q ſit negativum & $\frac{q^3}{27}$ ſimul æquale $\frac{1}{4}r^2$, evaneſcente $\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}$ fiet tam y quam z $= \sqrt[3]{\frac{1}{2}r}$, adeoque prima radix $y+z = 2\sqrt[3]{\frac{1}{2}r}$, reliquæ autem, evaneſcente $\frac{y-z}{2}\sqrt{-3}$, evadent $-\frac{\overline{y+z}}{2} = -y$ reſpective, adeoque coaleſcent in $-\sqrt[3]{\frac{1}{2}r}$.

210. Et hoc modo erui poſſunt radices omnium cubicarum æquationum, ubi q affirmativum eſt; vel ubi q negativum eſt & $\frac{q^3}{27}$ ſimul minus quam $\frac{1}{4}r^2$, adeoque $\sqrt{\frac{1}{4}r^2-\frac{q^3}{27}}$ quantitas realis, id eſt, ubi duæ ex radicibus æquationis ſunt impoſſibiles; vel etiam ubi q negativum eſt & $\frac{q^3}{27}$ ſimul æquale $\frac{1}{4}r^2$, id eſt, ubi duæ ex radicibus ſunt equales. At ubi q negativum eſt & $\frac{q^3}{27}$ ſimul majus quam $\frac{1}{4}r^2$, fit $\sqrt{\frac{1}{4}r^2-\frac{q^3}{27}}$ quantitas impoſſibilis, adeoque tres radices ſpeciem ementiuntur impoſſibilem, ubi revera ſunt omnes reales. Vide § 208.

211. Hic caſus, qui vulgo appellatur *irreducibilis*, plus quam 200 annos analyſtarum ingenia exercuit. De hoc ita *Schootenus* *, " Ubi $\frac{1}{4}r^2$ ſit minus quam $\frac{-q^3}{27}$ ibi præ- " dicta regula non habet locum, nec ejuſdem beneficio " radix ullo modo intelligibili explicari poteſt. Quæ qui- " dem res olim multæ fuit caliginis, & ut ſcribit *Albertus* " *Girardus* in libello cui titulus, *Invention nouvelle en l'Al-* " *gebra*, qui anno 1629 prodiit: *Hoc eſt in quo auctores hac-* " *tenus fuerunt valde intricati, & ut verum fatear, in re quam* " *maxime difficili.*" Nec minus in hoc etiamnum hærent Analyſtæ quam in circuli quadraturâ Geometræ †. Latet certe vitium in altera conditionum ad reſolutionem datæ æquationis adſcitarum. Nam licet quantitatem x duas in partes y & z dividere *ſemper* poſſibile ſit, tamen partium rectangulum yz æquari $\frac{1}{3}q$ cum ſigno mutato *aliquando* impoſſibile erit. Ubi quidem q affirmativum eſt, adeoque yz $(= -\frac{1}{3}q)$ negativum, contraria ſortientur ſigna y & z, atque idcirco creſcente minore quantitate z a nihilo ad infinitum, ſi ita ſimul creſcat y ut exceſſus ejus ſupra z ſit ſemper æqualis x, creſcet quoque rectangulum $-yz$ a nihilo ad infinitum, atque ideo alicubi inter creſcendum fiet æquale $-\frac{1}{3}q$. Ubi vero q negativum eſt, evadet yz $(= -\frac{1}{3}q)$ affirmativum, ſignis igitur ſimilibus afficientur y & z, affirmativis utique ſi x affirmativum ſit, negativis ſi ſecus. Cæterum yz rectangulum ſub partibus x tranſcen-

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* Vide *Schooteni Comment. in opera Cartes*, vol. 1, p. 345.

† Prodiit anno ſuperiore *Padue* liber cujus titulus ſaltem, deſideratum hoc tandem ſuppleri poſſe expectationem præbet, *Della poſſibilita della reale ſolutione analitica del caſo irreducibile*, auct. *Giambat. Nicolai*, Algeb. Profeſſ. At nondum ad manus pervenit.

dere nequit quadratum ſemiſſis x, ſeu $\frac{1}{4}x^2$. § 46. Et proinde conditio $yz = -\frac{1}{3}q$ plane impoſſibilis eſt ubi $-\frac{1}{3}q$ ſit majus quam $\frac{1}{4}x^2$ ſeu (quod eodem redit) $\frac{1}{4}p^2$, adeoque $-\frac{q^3}{27}$ majus quam $\frac{1}{4}r^2$. per § 207. Et ex hypotheſi impoſſibili concluſionem impoſſibilem colligi mirum eſſe non debet.

212. Pergamus jam hujuſce Regulæ uſum exemplis illuſtrare.

Ex. 1. Sit æquatio reſolvenda $x^3 + 30x = 117$. Erit $\frac{1}{2}r = \frac{117}{2}$, $\frac{1}{3}q = 10$, $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}} = \sqrt{\frac{13689}{4} + 1000} = \sqrt{\frac{17689}{4}}$ $= \frac{133}{2}$, $y = \sqrt[3]{\frac{117}{2} + \frac{133}{2}} = \sqrt[3]{\frac{250}{2}} = \sqrt[3]{125} = 5$, & $z =$ $\sqrt[3]{\frac{117}{2} - \frac{133}{2}} = \sqrt[3]{\frac{-16}{2}} = \sqrt[3]{-8} = -2$. Ergo $y + z = 5$ $- 2 = 3$, et reliquæ radices $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} =$ $= \frac{-3 \pm 7\sqrt{-3}}{2}$.

Ex. 2. Proponatur æquatio $u^3 + 3u^2 + 9u = 13$. Deleto termino ſecundo, ſubſtituendo nimirum $x - 1$ pro u, erit $x^3 + 6x = 20$. Adeoque $\frac{1}{2}r = 10$, $\frac{1}{3}q = 2$, $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ $= \sqrt{100 + 8} = 10{,}3923$. Ergo, $y + z = \sqrt[3]{:10 + 10{,}3923}$

$+ \sqrt[3]{10 - 10{,}3923} = 2{,}732 - 0{,}732 = 2$. Et $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = -1 \pm 1{,}732\sqrt{-3}$. Hinc $u = (x - 1 =)$ 1, vel $-2 \pm 1{,}732\sqrt{-3}$.

Ex. 3. Sit æquatio $u^6 - 3u^4 - 2u^2 = 8$. Depreſsâ æquatione et ſublato termino ſecundo, ſubſtituendo utique $x + 1$ pro u^2, prodit $x^3 - 5x = 12$. Adeoque $\frac{1}{2}r = 6$, $\frac{1}{3}q = \frac{5}{3}$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{36 - \frac{125}{27}} = 5{,}6009$. Ergo $y + z = \sqrt[3]{6 + 5{,}6009} + \sqrt[3]{6 - 5{,}6009} = 2{,}26376 + {,}73624 = 3$. Et proinde $u^2 = (x + 1 =) 4$. Reliquæ radices ut in præcedentibus exemplis prodeunt impoſſibiles.

213. Aliquando binomiæ quantitatis (qualem parit hæc æquationis cubicæ reſolutio) ex una parte rationali (ſive hæc integra fuerit ſive fracta) et alterâ irrationali, vel ex duabus irrationalibus ad invicem incommenſurabilibus conflatæ, radix cubica per binomiam ſimpliciorem deſignari poteſt; ut olim expoſuit *Bombellius*. Ex methodis autem in hunc ſinem excogitatis, *Walliſianam* * licet tentando ſit, in praxi commodiſſimam reperi. Hanc idcirco in compendium redactam promotamque jam tradam.

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* Vide *Wallis*, *Opera Mathemat.* vol. 2, p. 188, & ſeq.

INVESTIGATIO RADICIBUS BINOMII CUBI.

I.

214. Sit quantitas $A \pm \sqrt{B^2Q}$ ex unâ parte integrâ A, et alterâ irrationali $\sqrt{B^2Q}$ conflata. Reducatur $\sqrt{B^2Q}$ ad formam ſimpliciſſimam $B\sqrt{Q}$ extrahendo quicquid eſt rationale, ſi modo B deſignet numerum ab unitate diverſum. Sit radix quæſita $x \pm b\sqrt{y}$, et manifeſtum eſt, quod $b\sqrt{y}$ debet eſſe pars aliquota $B\sqrt{Q}$, adeoque b pars aliquota B et $\sqrt{y}$ ipſi $\sqrt{Q}$ æqualis; niſi enim hæc ſurda $\sqrt{Q}$ ingrederetur radicem ingredi non poſſet cubum. Erit igitur $x^3 \pm 3x^2b\sqrt{y} + 3xb^2y \pm b^3y\sqrt{y} = (\overline{x \pm b\sqrt{y}}^3 =)\ A \pm B\sqrt{Q}$. Adeoque pars integra cubi $x^3 + 3xb^2y = A$, & pars irrationalis $3x^2b\sqrt{y} + b^3y\sqrt{y}$ ſeu $(3x^2b + b^3y)\sqrt{y} = B\sqrt{Q}$. Unde dividendo per $\sqrt{y} = \sqrt{Q}$, prodit $3x^2b + b^3y = B$, adeoque per reductionem, $x^2 = \frac{B}{3b} - \frac{b^2y}{3}$, et $x = \pm \sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$. Sigillatim jam ſubſtitue pro b, integros 1, 2, 3, &c. aut fractos $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ &c. donec $\sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$ emergat rationalis. Deinde ſubſtituto valore rationali x, ſi $x^3 + 3xb^2y$ evadit æqualis A, fiet quæſitum, et innoteſcet utraque pars radicis x & $b\sqrt{y}$, ſi modo radix extrahi poteſt.

Integros 1, 2, 3, &c. aut integrorum ſemiſſes $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. ſubſtituendos eſſe pro b, quò partibus radicis in ſe ductis emergat A integer, ita demonſtrare licet.

Sit radix quæſita $\frac{x}{m} + \frac{b\sqrt{y}}{n}$, deſignantibus m et n integros ad x & $b\sqrt{y}$ reſpective primos; eritque $\frac{x^3}{m^3} + \frac{3x^2b\sqrt{y}}{m^2n} + \frac{3xb^2y}{mn^2} + \frac{b^3y\sqrt{y}}{n^3}$ radicis cubus, ubi $\frac{x^3}{m^3} + \frac{3xb^2y}{mn^2}$ ſeu A, necnon $\frac{3x^2b}{m^2n} + \frac{b^3y}{n^3}$ ſeu B, debet eſſe integer. Ponatur $\frac{3x^2b}{m^2n} + \frac{b^3y}{n^3} = c$ int. Et concinnando erit $y = \frac{n^3c}{b^3} - \frac{3n^2x^2}{m^2b^2}$ integer, adeoque $\frac{3n^2}{m^2b^2}\, x^2$ integer, adeoque coefficiens $\frac{3n^2}{m^2b^2}$ integer, et n ex multiplis mb. Pone igitur $n = mb \times d$. Et ſubſtituto hoc valore n, prodit $y = m^3cd^3 - 3x^2d^2$. Atque ſubſtitutis iterum valoribus n & y jam inventis, prodit $\frac{x^3}{m^3} + \frac{3xb^2y}{mn^2} = 3xcd - \frac{8x^3}{m^3}$ integer, adeoque $\frac{8x^3}{m^3}$ integer, adeoque coefficiens $\frac{8}{m^3}$ integer. Et proinde $m^3 = 1$ vel 8, et $m = 1$ vel 2. Cæterum fuit $y = d^2 \times \overline{m^3cd - 3x^2}$, adeoque $\sqrt{y} = d\sqrt{m^3cd - 3x^2}$, adeoque altera pars radicis $\frac{b\sqrt{y}}{n} = \frac{bd\sqrt{m^3cd - 3x^2}}{mbd} = \frac{\sqrt{m^3cd - 3x^2}}{m}$. Ergo, utraque pars radicis communem habet denominatorem $m = 1$ ubi x & b ſunt integri, vel $= 2$ ubi ſunt fracti.

215. Porro, ne operam fruſtrà inſumamus tentando ſucceſſive valores inutiles b, obſervandum eſt quod creſcente b, decreſcet $x = \sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$; aut vice verſâ. Et proinde ſi juſto majus aſſumatur b, juſto minus emerget x, adeoque $x^3 + 3b^2y$ minus quam A; aut vice verſâ.

Secundum Methodum *Walliſianam* radicibus exquirendis ſolummodo adhibentur valores affirmativi x; at ope valorum negativorum x aliæ inſuper radices explorari poſſunt.

216. *Ex.* 1. Si quæratur radix cubica quantitatis $10 + \sqrt{108}$ ſeu $10 + 6\sqrt{3}$; ſubſtituto 1 pro b, prodit $x = \sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$ $= \sqrt{\frac{6}{3} - \frac{3}{3}} = 1$. Deinde ſubſtituto 1 pro x in æquationem $x^3 + 3xb^2y = A$, prodit $1 + 9 = 10$, ut oportet. Ergo radix tentanda $1 + \sqrt{3}$. Hujus autem cubus per multiplicationem dat $10 + \sqrt{108}$; itaque fiet quæſitum.

Ex. 2. Sit radix cubica extrahenda ex $\sqrt{968} + 25$ ſeu $22\sqrt{2} + 25$. Subſtituto 1 pro b, prodit $x = \sqrt{\frac{22}{3} - \frac{2}{3}}$ $= \sqrt{\frac{20}{3}}$. Res igitur non ſuccedit, emergente x irrationali.

Subſtituatur iterum 2 pro b, et prodit $x = \sqrt{\frac{22}{6} - \frac{8}{3}} =$ $\sqrt{\frac{6}{6}} = 1$. Deinde, ſubſtituto 1 pro x, æquatio $x^3 + 3xb^2y$

$= A$, evadit $1 + 24 = 25$, ut oportet. Ergo radix tentanda $2\sqrt{2} + 1$.

Ex. 3. Si radix cubica extrahenda ſit ex $2 + \sqrt{5}$, ſubſtituto 1 pro b prodit $x = \sqrt{\frac{1}{3} - \frac{5}{3}} = \sqrt{-\frac{4}{3}}$ impoſſibilis, quod indicio eſt b non ſortiri valorem integrum. Subſtituatur itaque $\frac{1}{2}$ pro b, et prodit $x = \sqrt{\frac{2}{3} - \frac{5}{12}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, quo ſubſtituto, &c. evadit $\frac{1 + 15}{8} = 2$; ut oportet. Adeoque radix tentanda $\frac{1}{2} + \frac{1}{2}\sqrt{5}$. Atque hic quidem notandum, quod radicibus eruendis ubi x & b ſunt fracti, per ſe impares ſunt methodus *Newtoniana* & *Clairautiana*, atque idcirco ulteriore egent artificio quo hujuſmodi exemplis accommodentur.

217. Aliquando formam irrationalem induunt x & b. Nam

Ex. 4. Si radix cubica extrahenda ſit ex $68 - \sqrt{4734}$ ſeu, $68 - 27\sqrt{6}$, valoribus x & b rationalibus fruſtra tentatis, radicem nihilominus eruere licet multiplicando quantitatem datam per $\frac{2}{2}$, unde prodit $\frac{136 - 54\sqrt{6}}{2}$, idem ſcilicet quantitatis valor. Dein extractâ ſeorſim numeratoris ac denominatoris radice cubicâ invenietur radix quæſita, $\frac{4 - \sqrt{6}}{\sqrt[3]{2}}$.

II.

218. Si quantitatis $A \pm B\sqrt{Q}$ pars A ſit fractio, reducentur partes ad communem denominatorem. Dein ut antea, extractâ ſeorſim numeratoris ac denominatoris radice cubica orietur radix quæſita, ſi modo extrahi poteſt.

Ex. 5. Si ex $\sqrt{242} - 12\frac{1}{2}$ radix cubica extrahenda ſit; reductis partibus ad communem denominatorem fiet $\frac{\sqrt{968} - 25}{2}$. Dein extractâ radice cubica, &c. orietur $\frac{2\sqrt{2} - 1}{\sqrt[3]{2}}$ radix quæſita.

III.

219. Quantitatis $A \pm B\sqrt{-Q}$ ſit una pars A rationalis, altera $B\sqrt{-Q}$ impoſſibilis. Deſignante $x \pm b\sqrt{-y}$ radicem, multiplicando erit $x^3 \pm 3x^2b\sqrt{-y} - 3xb^2y \mp b^3y\sqrt{-y} = A \pm B\sqrt{-Q}$. Adeoque $x^3 - 3xb^2y = A$, & $3x^2b - b^3y = B$, et $x = \pm\sqrt{\frac{B}{3b} + \frac{b^2y}{3}}$. Deinde operando ut antea, eruetur radix, ſi modo extrahi poteſt *.

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* His veſtigiis inſiſtendo methodum proferre poſſumus extrahendi radicem quadraticam binomii $A \pm B\sqrt{Q}$, ex parte integra A & irrationali $B\sqrt{Q}$ conflati,

Ex. 6. Sit radix cubica extrahenda ex $-5 \pm \sqrt{-2}$. Ponatur $b = 1$, eritque $x = \sqrt{\frac{1}{3} + \frac{2}{3}} = 1$. Quo valore x ſubſtituto prodit $x^3 - 3xb^2y = 1 - 6 = -5$, ut oportet. Ergo radix tentanda $1 \pm \sqrt{-2}$, quæ utique per multiplicationem dat $-5 \pm \sqrt{-2}$.

220. Miror itaque unde *Saunderſon* ad *De Moivre* ſcribens, profitetur, " Se nihil ad rem deprehendiſſe, de ex-
" trahenda radice cubica binomii impoſſibilis, inſtar
" $-5 + \sqrt{-2}$ vel $-5 - \sqrt{-2}$, ne quidem apud ipſum
" *Wallis* qui hoc aggreditur." Magiſque miror unde in

Newtonianâ § 52, in praxi commodiorem. Deſignante enim $x \pm b\sqrt{y}$ radicem, erit $x^2 \pm 2xb\sqrt{y} + b^2y = A \pm B\sqrt{Q}$, adeoque $x^2 + b^2y = A$; $2xb = B$, et inde $x = \frac{B}{2b}$. Deinde ſigillatim ſumendo $b = 1, 2, 3$, &c. ſubſtituantur valores x reſultantes, donec $x^2 + b^2y$ evadat æqualis A; et fiet quæſitum, ſi modo radix extrahi poteſt. Ut ſi extrahenda ſit radix ex $3 + \sqrt{8}$ ſeu $3 + 2\sqrt{2}$; ſume $b = 1$, eritque $x = \frac{2}{2} = 1$, quo ſubſtituto prodit $x^2 + b^2y = 1 + 2 = 3$, ut oportet. Ergo radix tentanda $1 + \sqrt{2}$.

Quòd ſi $B\sqrt{-Q}$ ſit impoſſibilis, erit $x^2 - b^2y = A$, cæteris manentibus. (Radicem, in hoc caſu, deſignante $x \pm b\sqrt{-y}$.) Ut ſi quæratur radix quadratica quantitatis impoſſibilis $1 \pm 4\sqrt{-3}$; ſume $b = 1$, eritque $x = \frac{4}{2} = 2$. Quo ſubſtituto, prodit $x^2 - b^2y = 4 - 3 = 1$, ut oportet. Ergo radix tentanda $2 \pm \sqrt{-3}$.

regulam *Walliſianam* animadvertit. *De Moivre*, quaſi " *peti-* " *tio principii* foret, aut ſaltem tentamen quod radice eruen- " dâ ſæpe deficeretur." In Logicis certe parum verſatum fuiſſe videtur *De Moivre*, aut hanc regulam ad *petitionem principii* vix retulerit; et æquo ſaltem jure, in regulam *pro inventione Diviſorum*, adeo decantatam, animadvertere potuerit, cum hæc quoque non niſi tentando ſuccedit. Vide *Append. ad Saunderſon*, *Algeb. vol.* II. *p.* 743.

Ex. 7. Si quæratur $\sqrt[3]{:}\ 81 + \sqrt{-2700} = \sqrt[3]{:}\ 81 + 30\sqrt{-3}$; ponendo $b = 1$, erit $x = \sqrt{11}$. Res igitur non ſuccedit, emergente x irrationali. Rurſus, ponendo $b = 2$, erit $x = \pm\sqrt{9} = \pm 3$. Si ſubſtituatur $+ 3$ pro x, prodit $x^3 - 3xb^2y = 27 - 108 = -81$, adeoque res non ſuccedit, quoniam ſigno affirmativo affici debet 81. At ſi ſubſtituatur -3 pro x, prodit $-27 + 108 = +81$; ut oportet. Ergo radix tentanda $-3 + 2\sqrt{-2}$.

Altera quoque radix erui poteſt ſumendo $b = \frac{1}{2}$, unde erit $x = \sqrt{20 + \frac{1}{4}} = \frac{9}{2}$, adeoque $x^3 - 3xb^2y = \frac{729}{8} - \frac{81}{8} = 81$; ut oportet. Ergo radix tentanda $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$.

221. Cæterum in hujuſmodi operationibus ſi cubi partes communem habent diviſorem cubicum, ſatius erit radices factorum ſeorſim extrahere. Sic in noviſſimo exemplo, $81 + \sqrt{-2700} = 27 \times \overline{3 + \sqrt{-\frac{100}{27}}}$, radicibus factorum

ſeorſim extractis, valet radix quæſita $3\times\overline{-1+2\sqrt{-\frac{1}{3}}}$ $=-3+2\sqrt{-\frac{9}{3}}=-3+2\sqrt{-3}$, ut in priore caſu; et altera radix $3\times\overline{\frac{3}{2}+\frac{1}{2}\sqrt{-\frac{1}{3}}}=\frac{9}{2}+\frac{1}{2}\sqrt{-3}$, ut in poſteriore.

IV.

222. Sit quantitas $A\sqrt{P}\pm B\sqrt{Q}$, cujus utraque pars irrationalis eſt. Deſignante $a\sqrt{x}\pm b\sqrt{y}$ radicem, multiplicando erit $a^3x\sqrt{x}\pm 3a^2xb\sqrt{y}+3ab^2y\sqrt{x}\pm b^3y\sqrt{y}$ $=A\sqrt{P}\pm B\sqrt{Q}$; adeoque, $a^3x+3ab^2y=A$ & $3a^2xb+b^3y=B$, unde prodit $a=\sqrt{\frac{B}{3xb}-\frac{b^2y}{3x}}$. Pro x ſubſtitue jam valorem ſuum P; et ſubſtitue ſigillatim pro b integros 1, 2, 3, &c. donec eruatur valor a rationalis; quo ſubſtituto, ſi a^3x+3ab^2y prodeat æqualis A, fiet quæſitum.

Ex. 8. Si radix cubica extrahenda eſt ex $\sqrt{5408}+\sqrt{5400}$ ſeu $52\sqrt{2}\pm 30\sqrt{6}$; ſume $b=1$, evadetque $a=\sqrt{5-1}$ $=2$. Et hoc valore a ſubſtituto, prodit $a^3x+3ab^2y=16$ $+36=52$, ut oportet. Ergo radix tentanda $2\sqrt{2}\pm\sqrt{6}$.

DE RESOLUTIONE ÆQUATIONUM CUBICARUM PER EXTRACTIONEM RADICIS BINOMII CUBI.

223. Aliquando radices in *caſu irreducibili* exprimere poſſibile erit, ubi ſcilicet radices valorum y & z ope regulæ jam allatæ cubice extrahere licet. Nam quod ſingulis membris videtur impoſſibile, mutua connexione deſtruetur *.

Ex. 1. Proponatur æquatio $x^3 - 15x = 4$. Erit $\frac{1}{2}r = 2$, $\frac{1}{3}q = 5$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{4 - 125} = \sqrt{-121} = 11\sqrt{-1}$, adeoque $y + z = \sqrt[3]{:2 + 11\sqrt{-1}} + \sqrt[3]{:2 - 11\sqrt{-1}}$ $= \overline{2 + \sqrt{-1}} + \overline{2 - \sqrt{-1}} = 4$. Itemque $- \frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = -2 \pm \sqrt{-1}\,.\,\sqrt{-3} = -2 \pm \sqrt{3}$. In hoc exemplo igitur prima radix 4 eſt maxima.

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* *Symbola imaginaria* proprie quidem calculum reſpuunt, cum ſint tantum impoſſibilitatis ſigna; veruntamen ubi quantitates reales deſignare ponuntur (ut in hoc caſu) quod mirum & paradoxi inſtar videatur, ex earum ſummis, productis & rationibus ſecundum regulas arithmeticas computatis, veræ & legitimæ ſequuntur concluſiones. Ita ut Geometria, etſi calculum quantitatum impoſſibilium proprie non agnoſcit, tamen concluſiones inde elicitas demonſtrationibus accuratiſſimis munit atque ſtabilit. Cæterum ſignis imaginariis quantitates adumbrari reales ibi tantum conceditur, ubi quæſtio ſit de menſuris angulorum aut rationum, idque propter arctum nexum & affinitatem inter arcus circulares & ſectores hyperbolicos intercedentem. Atque hinc oritur quod concluſiones his propriæ ad illos per analogiam tranferuntur. Sic iidem prodeunt valores rectangulorum ſub ſinubus & coſinubus ſectorum circulorum, atque ſub ordinatis & abſciſſis ſectorum hyperbolicorum; calculo in utroque caſu ſimiliter inſtituto, niſi quod in priore, mediantibus ſignis imaginariis, in poſteriore, ſignificantibus ad eandem concluſionem devenitur. At hoc tantum hic loci monuiſſe fas eſt. Adeat lector horum ſtudioſus, *Playfair, on the arithmetic of impoſſible quantities*, *Phil. Tranſact.* an. 1778, p. 318.

Ex. 2. Sit $x^3 - 6x = 4$. Erit $\frac{1}{2}r = 2$, $\frac{1}{3}q = 2$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{4 - 8} = \sqrt{-4} = 2\sqrt{-1}$. Adeoque $y + z$ $= \sqrt[3]{:2 + 2\sqrt{-1}} + \sqrt[3]{:2 - 2\sqrt{-1}} = -1 + \sqrt{-1}$ $-1 - \sqrt{-1} = -2$. Itemque $- \frac{y+z}{2} + \frac{y-z}{2}$ $\sqrt{-3} = 1 \pm \sqrt{-1}\,.\,\sqrt{-3} = 1 \pm \sqrt{3}$, ubi prima radix -2 eſt media.

Ex. 3. Sit $x^3 - 9x = -10$. Erit $\frac{1}{2}r = 5$, $\frac{1}{3}q = 3$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{25 - 27} = \sqrt{-2}$. Adeoque $y + z =$ $\sqrt[3]{:-5 + \sqrt{-2}} + \sqrt[3]{:-5 - \sqrt{-2}} = 1 + \sqrt{-2} + 1 -$ $\sqrt{-2} = 2$. Itemque $- \frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = -1$ $\pm \sqrt{-2}\,.\,\sqrt{-3} = -1 \pm \sqrt{6}$. Ubi prima radix 2 eſt media.

Ex. 4. Sit $x^3 - 15x = 22$. Erit $\frac{1}{2}r = 11$, $\frac{1}{3}q = 5$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{121 - 125} = \sqrt{-4}$. Adeoque $y + z = \sqrt[3]{11 + \sqrt{-4}} + \sqrt[3]{11 - \sqrt{-4}} = -1 + \sqrt{-4} - 1 - \sqrt{-4} = -2$. Itemque $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = 1 \pm \sqrt{-4} \cdot \sqrt{-3} = 1 \pm \sqrt{12}$. Ubi prima radix -2 eſt minima.

Hinc liquet, primam radicem hac methodo inventam trium radicum realium unamquamque indifferenter deſignare poſſe, non maximam ſolùm, ut vulgo perhibent.

In *caſu reducibili* quoque aliquando facilius erui poſſunt radices ope regulæ allatæ. Ut ſi repetatur æquatio $x^3 + 6x = 20$, prodit $y + z = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = 1 + \sqrt{3} + 1 - \sqrt{3} = 2$; ut antea, § 212.

224. Cæterum deficiente *Cardani* regula ubi binomiarum impoſſibilium inde reſultantium radices cubicæ accurate extrahi nequeunt, ad theorema binomium confugiendum erit, cujus ope harum radices *ſemper* per ſeries infinitas proxime exponi poſſunt, ſecundum methodum quam primus excogitaſſe videtur *Leibnitz*, § 205, tradiderunt verò *Ronayne*, *Algebra*, 8vo Lond. 1717, & Edit. 2da 1727, et *Nicole*. *Mem. de l'Acad. Paris*, 1738.

DE RESOLUTIONE ÆQUATIONUM CUBICARUM PER SERIES INFINITAS.

225. Abbreviandi gratiâ ponatur $\frac{1}{2}r = a$, & $\sqrt{\frac{r^2}{4} \pm \frac{q^3}{27}} = \sqrt{\pm c}$, ſcilicet $= \sqrt{c}$ in caſu reducibili, et $\sqrt{-c}$ in caſu irreducibili. Per. theorema binomium, erit in priore caſû,

I.

$$
\begin{aligned}
y &= (a + \sqrt{c})^{\frac{1}{3}} = \sqrt[3]{a} \times : 1 + \frac{\sqrt{c}}{3a} - \frac{2c}{3.6a^2} + \frac{2.5c\sqrt{c}}{3.6.9a^3} - \frac{2.5.8c^2}{3.6.9.12a^4} \\
z &= (a - \sqrt{c})^{\frac{1}{3}} = \sqrt[3]{a} \times : 1 - \frac{\sqrt{c}}{3a} - \frac{2c}{3.6a^2} - \frac{2.5c\sqrt{c}}{3.6.9a^3} - \frac{2.5\,8c^2}{3\,6.9.12a^4} \\
\therefore\ y + z &= 2\sqrt[3]{a} \times : 1 \quad * \quad - \frac{2c}{3.6a^2} \quad * \quad - \frac{2.5.8c^2}{3.6.9.12a^4} \\
&\qquad \&c.
\end{aligned}
$$

Exponet itaque ſeries $2\sqrt[3]{a} \times : 1 - \frac{2c}{3.6a^2} -$ &c. primam radicem, qualem invenerunt *Ronayne* & *Nicole*. Erit quoque

$$
\begin{aligned}
y - z &= 2\sqrt[3]{a} \times : \quad * + \frac{\sqrt{c}}{3a} \quad * \quad + \frac{2.5c\sqrt{c}}{3.6.9a^3} \quad * \quad \&c. \\
&= 2\sqrt[3]{a} \times \frac{\sqrt{c}}{a} \times : \frac{1}{3} \quad * \quad + \frac{2.5c}{3.6.9a^2} \quad * \quad \&c. \\
&= \frac{2\sqrt[3]{a} \times \sqrt{c}}{\sqrt[3]{a^3}} \text{ ſeu } \frac{2\sqrt{c}}{\sqrt[3]{a^2}} \times : \frac{1}{3} + \frac{2.5c}{3.6.9a^2} + \frac{2.5.11c^2}{3.6.9.12.15a^4} \\
&\qquad \&c.
\end{aligned}
$$

Adeoque reliquæ radices, quales expoſuit *Clairaut*,

$$\left.\begin{matrix} -\dfrac{y+z}{2} \\ \pm\dfrac{y-z}{2}\sqrt{-3} \end{matrix}\right\} = \left\{\begin{matrix} -\sqrt[3]{a}\times:1-\dfrac{2c}{3\cdot6a^2}-\dfrac{2\cdot5\cdot8c^2}{3\cdot6\cdot9\cdot12a^4}\ \&c. \\ \pm\dfrac{\sqrt{-3c}}{\sqrt[3]{a^2}}\times:\dfrac{1}{3}+\dfrac{2\cdot5c}{3\cdot6\cdot9a^2}+\dfrac{2\cdot5\cdot8\cdot11c^2}{3\cdot6\cdot9\cdot12\cdot15a^4}\ \&c. \end{matrix}\right.$$

II.

Et in poſteriore caſu, erunt radices,

$$y+z=2\sqrt[3]{a}\times:1+\frac{2c}{3\cdot6a^2}-\frac{2\cdot5\cdot8c^2}{3\cdot6\cdot9\cdot12a^4}+\&c.$$

$$\text{et}\ \left.\begin{matrix} -\dfrac{y+z}{2} \\ \pm\dfrac{y-z}{2}\sqrt{-3} \end{matrix}\right\} = \left\{\begin{matrix} -\sqrt[3]{a}\times:1+\dfrac{2c}{3\cdot6a^2}-\dfrac{2\cdot5\cdot8c^2}{3\cdot6\cdot9\cdot12a^4}+\&c. \\ \pm\dfrac{\sqrt{3c}}{\sqrt[3]{a^2}}\times:\dfrac{1}{3}-\dfrac{2\cdot5c}{3\cdot6\cdot9a^2}+\dfrac{2\cdot5\cdot8\cdot11c^2}{3\cdot6\cdot9\cdot12\cdot15a^4}-\&c. \end{matrix}\right.$$

Mutatis ſcilicet ſignis terminorum alternorum a ſecundo incipientium, utpote qui poteſtates impares quantitatis jam negativæ $-c$ involvunt.

226. Si $-\frac{q^3}{27}$ ſit duplum $\frac{1}{4}r^2$, ſimplicius efferri poſſunt hæ ſeries. Nam ob $\sqrt{c}$ in hoc caſu æquale a, erit $\frac{\sqrt{c}}{a}=1$.

Adeoque, prima radix,

$$y + z = 2\sqrt[3]{a} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} - \&c.$$

$$\text{Et reliquæ,} = \begin{cases} -\sqrt[3]{a} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \&c. \\ \pm\sqrt[3]{a} . \sqrt{-3} \times : \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} - \&c. \end{cases}$$

227. Cæterum hæ ſeries radicibus eruendis *directe* inſerviunt tantùm ubi convergunt, hoc eſt, ſi $-\frac{q^3}{27}$ non ſit duplo majus quàm $\frac{1}{4}r^2$, adeoque $\sqrt{c}$ non majus quam c. Et quo proprius $-\frac{q^3}{27}$ accedit ad $\frac{1}{4}r^2$, eo minus evadet $\sqrt{c}$ reſpectu a, et proinde eo citius converget ſeries ex evolutione $(a \pm \sqrt{c})^{\frac{1}{3}}$ oriunda. Quod ſi $\frac{q^3}{27}$ affirmativum ſit, aut ſi negativum ſit & ſimul duplo majus quam $\frac{1}{4}r^2$, foret $\sqrt{c}$ majus quam a, unde divergeret ſeries. In hoc caſu igitur *invertendi* ſunt termini a & $\sqrt{c}$; nam ex evolutione $(\sqrt{c} \pm a)^{\frac{1}{3}}$ prodibit ſeries convergens ejuſdem valoris. Adeoque in caſu reducibili erit

III.

$$y = \sqrt[3]{:\sqrt{c} + a}, \& \; z = \sqrt[3]{: - \sqrt{c} + a} = - \sqrt[3]{: \sqrt{c} - a}.$$

$$\therefore y = (\sqrt{c} + a)^{\frac{1}{3}} = \sqrt[3]{c^{\frac{1}{2}}} \times : 1 + \frac{a}{3\sqrt{c}} - \frac{2a^2}{3.6c} + \frac{2.5a^3}{3.6.9c\sqrt{c}} - \&c.$$

$$z = -(\sqrt{c} - a)^{\frac{1}{3}} = \sqrt[3]{c^{\frac{1}{2}}} \times : -1 + \frac{a}{3\sqrt{c}} + \frac{2a^2}{3.6c} + \frac{2.5a^3}{3.6.9c\sqrt{c}} + \&c.$$

Et $y + z = 2\sqrt[3]{c^{\frac{1}{2}}} \times : * + \frac{a}{3\sqrt{c}} * + \frac{2 \cdot 5a^3}{3 \cdot 6 \cdot 9c\sqrt{c}} * + \&c.$

$$= 2\sqrt[3]{c^{\frac{1}{2}}} \cdot \frac{a}{\sqrt{c}} \text{ ſeu } \frac{2a}{\sqrt[6]{c^2}} \text{ ſeu } \frac{2a}{\sqrt[3]{c}} \times : \frac{1}{3} + \frac{2 \cdot 5a^2}{3 \cdot 6 \cdot 9c} + \frac{2 \cdot 5 \cdot 8 \cdot 11a^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^2} + \&c.$$

exponet primam radicem, qualem invenit *Clairaut*.

Itemque,

$$\left.\begin{array}{l} -\frac{y+z}{2} \\ \pm\frac{y-z}{2}\sqrt{-3} \end{array}\right\} = \left\{\begin{array}{l} -\frac{a}{\sqrt[3]{c}} \times : -\frac{1}{3} + \frac{2 \cdot 5a^2}{3 \cdot 6 \cdot 9c} + \frac{2 \cdot 5 \cdot 8 \cdot 11a^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^2} + \&c. \\ \pm\sqrt[3]{c^{\frac{1}{2}}} \cdot \sqrt{-3} \times : 1 - \frac{2a^2}{3 \cdot 6c} - \frac{2 \cdot 5 \cdot 8a^4}{3 \cdot 6 \cdot 9 \cdot 12c^2} - \&c. \end{array}\right.$$

reliquas, quales invenit *Hutton*.

IV.

Et in caſu irreducibili erit prima radix

$$= \frac{-2a}{\sqrt[3]{c}} \times : \frac{1}{3} - \frac{2 \cdot 5a^2}{3 \cdot 6 \cdot 9c} + \frac{2 \cdot 5 \cdot 8 \cdot 11a^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c} - \&c.$$

Et reliquæ,

$$= \left\{\begin{array}{l} \frac{a}{\sqrt[3]{c}} \times : \frac{1}{3} - \frac{2 \cdot 5a^2}{3 \cdot 6 \cdot 9c} + \&c. \\ \pm\sqrt[3]{c^{\frac{1}{2}}} \cdot \sqrt{3} \times : 1 + \frac{2a^2}{3 \cdot 6c} - \frac{2 \cdot 5 \cdot 8a^4}{3 \cdot 6 \cdot 9 \cdot 12c^2} + \&c. \end{array}\right.$$

228. Pergamus jam harum ſerierum uſum exemplis illuſtrare.

Ex. 1. Sit æquatio $x^3 - 12{,}03x = 16$. Erit $a = 8$, $-c = -{,}481201$ adeoque ob c negativum & multò minus quam a^2, ad hibendæ ſeries II. citiſſimè convergentes. Erit igitur

$$\frac{c}{a^2} = \frac{{,}481201}{64} = {,}007518 \,.\, \frac{c^2}{a^4} = {,}0000565$$

$$\frac{1c}{9a^2} = \frac{{,}007518}{9} = {,}000835 \,.\, \frac{10c^2}{243a^4} = \frac{{,}000565}{243} = {,}0000023$$

$$\begin{array}{r} 1{,}000835 \\ -\ {,}000002 \\ \hline \end{array} \qquad \frac{\sqrt{3c}}{\sqrt[3]{a^2}} = \frac{{,}12015}{4}$$

$$\begin{array}{r} 1 + \frac{c}{9a^2} - \frac{10c^2}{243a^4} = 1{,}000833 \\ \times\ 2\sqrt[3]{a} = 4 \\ \hline \end{array} \qquad \frac{\sqrt{3c}}{\sqrt[3]{a^2}} \times \frac{1}{3} = \frac{{,}03003}{3} = {,}010013.$$

Prima radix, $4{,}003332$

$$-\sqrt[3]{a} \times : 1 + \frac{c}{9a^2} = -\ 2{,}001666$$

$$\pm \frac{\sqrt{3c}}{\sqrt{a^2}} \times \frac{1}{3} \quad - \quad - \quad = \pm\ {,}010013$$

Media radix, - - $\mp 2{,}011679$

Ultima, - - - - $-1{,}991653$.

Ex. 2. Sit æquatio $x^3 - 50x = 120$. Erit $a = 60$, $c = -1029{,}629629$. Adeoque adhibendæ ſeries II. Erit igitur,

$$\frac{c}{a^2} = ,286008 \,.\, \frac{c^2}{a^4} = ,081800 \,.\, \frac{c^3}{a^6} = ,023395$$

$$\frac{1c}{9a^2} = ,031778 \,.\, \frac{10c^2}{243a^4} = ,003353 \,.\, \frac{154c^3}{6561a^6} = ,000549$$

$$\begin{array}{r} 1{,}031778 \\ +\ 000549 \\ \hline 1{,}032327 \\ -\ ,003353 \\ \hline 1{,}028974 \\ \times\ 2 \\ \hline 2{,}057948 \\ \times \sqrt[3]{a} = 3{,}914867 \\ \hline \text{Prim. rad.} = 8{,}056592 \ * \\ \left\{ \begin{array}{r} -\ 4{,}028296 \\ \pm\ \ ,096647 \end{array} \right. \\ \hline \text{Media,} \ -\ 4{,}124943 \\ \text{Ultima,} \ -\ 3{,}931649. \end{array}$$

$$\frac{\sqrt{c}}{\sqrt[3]{a^2}} = \frac{,555777}{15{,}3263} = ,036262$$

$$\frac{10c}{162a^2} = \frac{2{,}86008}{162} = ,017654$$

$$\begin{array}{r} \frac{1}{3} - \frac{10c}{162a^2} = +\ ,333333 \\ -\ ,017654 \\ \hline +\ ,315679 \\ \times \ \frac{\sqrt{c}}{\sqrt[3]{a^2}} = ,036262 \\ \hline ,096647 \end{array}$$

---

* Si ex diametro tribuſque circuli chordis contiguis factis fiat trapezium circulo inſcriptum, datis chordis *a*, *b*, *c* erui poteſt diameter *x* per reſolutionem æquationis $x^3 - (a^2 + b^2 + c^2)\,x = 2abc$, quam ex mutuis harum relationibus vario artificio educit *Newtonus*, *Arith. Univerſ.* Ed. 2, p. 101, &c. Ut ſi chordas reſpective deſignent numeri 3, 4, 5, reſolutio æquationis reſultantis $x^3 - 50x = 120$, dabit $x = 8{,}056$ &c. reliquæ verò radices — 4,124 &c. et — 3,931 &c. utpote negativæ aliò referuntur, nec ſi evaderent affirmativæ diametrum deſignare poſſent, ob limitationem in ſchemate, cum rectarum circulo inſcriptarum maximus prodit diameter. § 134. Vide *Maſeres*, *Diſſert. on Cardan's rule*, *Phil. Tranſ.* an. 1778.

229. Cæterum ubi plures ſerierum termini radicibus deſignandis requiruntur, in numeros profuſiores excurret calculus, plurimum vero minuetur labor combinando ſerierum terminos inter operandum inventos, & adhibendo logarithmos; ut in exemplis ſequentibus.

Ex. 3. Sit æquatio $x^3 - 15x = 22$. Erit $a = 11$ & $-c = -4$. Adeoque adhibendæ ſeries II. Erit quoque $\frac{c}{a^2} = \frac{4}{121} = \frac{1}{30,25}$. Et combinando terminos inter operandum inventos, erit

$$\begin{array}{lr|l}
A = & = 1,0000000 & \\
B = \frac{2c}{3.6a^2}\,A = & 36731 & C = \frac{5.8c}{9.12a^2} = -\,,0000450. \\
D = \frac{11.14c}{15.18a^2}\,C = & 8 & \\
\cline{2-2}
 & +\,1,0036739 & \\
 & -\,0,0000450 & \\
\cline{2-2}
\text{Prima ſeries} & = 1,0066289 & \\
 & \times\ 2 & \\
\cline{2-2}
 & 2,0072578 & \text{- - - log. } 0,3026031 \\
 & \times\ \sqrt[3]{11} & \text{- - - - } 0,3471309 \\
\cline{2-3}
\text{Prima radix} & = 4,4641016 & \text{- - - - } 0,6497340.
\end{array}$$

Itemque,

$\alpha = \frac{1}{3} A =$,3333333	$\beta = \frac{5}{9} B =$,0020406
$\gamma = \frac{11}{15} C =$ 330	$\delta = \frac{17}{21} D =$ 7
+ ,3333663	— ,0020413
— ,0020413	
,3313250	
× 2	
,6626500 - -	log. $\bar{1}$,8212842
$\times \sqrt{3}$ - - -	0,2385606
$\times \sqrt[3]{121}$ - - - -	0,6942618
Poſtrema ſeries ± ,2320508 - - - -	$\bar{1}$,3655830

— $\frac{1}{2}$ pri. rad. — 2,2320508

Media rad. = — 2,4641016

Ult. radix = — 2,0000000

Ex. 4. In æquatione $x^3 - 9x = -10$, prodit $a = -5$, $c = 25 - 27 = -2$. Adhibendæ igitur ſeries II. Et exiſtente $\frac{c}{a^2} = \frac{2}{25} = \frac{8}{100} = ,08.$ erit

$A = \quad = 1$

$B = \frac{2c}{3.6a^2} A = 0,0088889$ | $C = \frac{5.8c}{9.12a^2} B = ,0002634$

$D = \frac{11.14c}{15.18a^2} C = \quad 120$ | $E = \frac{17.20c}{21.24a^2} D = \quad 7$

$+ 1,0089009$ | $- ,0002641$

$- 0,0002641$

$+ 1,0086368$

2

$2,0172736$ - - - log. $0,3047649$

$\sqrt[3]{a} = -\sqrt[3]{5}$ - - - - $0,2329900$

Prima rad. $- 3,44948974$ - - - - $0,5377549$

Itemque,

$$\begin{array}{ll|ll} \alpha = \frac{1}{3} & A = ,3333333 & \beta = \frac{5}{9} & B = ,0049383 \\ \gamma = \frac{11}{15} & C = 1931 & \delta = \frac{17}{21} & D = 97 \\ \varepsilon = \frac{23}{27} & E = 6 & & \overline{ -,0049480} \end{array}$$

$$\begin{array}{lll} \overline{+ ,3335270} & & \\ -,0049480 & & \\ \overline{,3285790} & - \; - & \log.\ \bar{1},5166398 \\ \frac{\sqrt{3c}}{\sqrt[3]{a^{2}b}} = \frac{\sqrt{6}}{\sqrt[3]{25}} & - \; - \; - & \bar{1},9230956 \\ \hline \end{array}$$

$$\begin{array}{lll} \text{Altera series} & \pm ,27525513 & \bar{1},4397354 \\ -\frac{1}{2}\ \text{pri. rad.} & + 1,72474487 & \\ \hline \text{Media radix} & 2,00000000 & \\ \text{Ultima radix} & 1,44948974. & \end{array}$$

Ex. 5. Sit æquatio $x^3 - 6x = 4$. Prodit $a = 2$, $\sqrt{c} = \sqrt{4-8} = \sqrt{-4}$; adeoque ob $\sqrt{c} = a$, adhibendæ ſeries § 226. His autem tardius convergentibus, tot proferre terminos quot radicibus proxime deſignandis poſtulantur moleſtum foret. Hujus radices § 223 ſupra, prodierunt -2, & $1 \pm \sqrt{3}$.

Ex. 6. Sit æquatio $x^3 - 15x = 4$. Erit $a = 2$, $\sqrt{c} = \sqrt{4 - 125} = \sqrt{-121}$, adeoque ob $\sqrt{c}$ majus quam a, adhibendæ ſeries IV. Exiſtente igitur $\frac{a^2}{c} = \frac{4}{121}$. Erit,

+	—
$A = \frac{1}{3} = {,}3333333$	$B = \frac{2 \cdot 5a^2}{6 \cdot 9c} A = {,}0020406$
$C = \frac{8 \cdot 11a^2}{12 \cdot 15c} B = 330$	$D = \frac{14 \cdot 17}{18 \cdot 21} C = 7$
$+ {,}3333663$	$- {,}0020413$
$- {,}0020413$	

Prima ſeries	- ${,}3313250$ - - -	log. $\bar{1}{,}5202543$
	$\frac{4}{\sqrt[3]{121}} = \sqrt[3]{\frac{64}{121}}$ - - -	$\bar{2}{,}9077982$
Prima radix	$- {,}2679492$ - - -	$\bar{1}{,}4280525$

Itemque,

$\alpha = \frac{3}{1}\ A = 1$

$\beta = \frac{9}{5}\ B = 0{,}0036731$ | $\gamma = \frac{15}{11}\ C = -{,}0000450$

$\delta = \frac{21}{17}\ D = \quad 8$

$+\ 1{,}0036739$

$-\quad 0000450$

$1{,}0036289$ - - log. $0{,}0015732$

$\sqrt[3]{c_{\frac{1}{2}}} = \sqrt[3]{11}$ - - - $0{,}3471309$

$\sqrt{3}$ - - - - - - $0{,}2385606$

$\pm\ 3{,}8660254$ - - - - $0{,}5872647$

$-\frac{1}{2}$ primæ rad. $+\ 0{,}1339746$

Secunda radix $+\ 4{,}0000000$

Tertia radix $-\ 3{,}7320508.$

230. Poſſint quoque ſeries utriuſque claſſis tam directæ quam inverſæ, in præcedentibus adhibitæ in alias quæ citius convergant tranſmutari. Verum de his diſſerere non patitur noſtri inſtituti ratio. Conſulat lector harum ſtudioſus auctorem ſupra laudatum, *Hutton*, qui rem totam penitius explorat, fuſiuſque explicat.

DE ÆQUATIONUM BIQUADRATICARUM RESOLUTIONE.

229. Æquationum biquadraticarum reſolutionem primus invenit *Ferrarius*. Docuit vero *Carteſius* methodo inſequente radices æquationum biquadraticarum mediantibus cubicis eruere & exprimere.

Tollendus eſt primum ſecundus æquationis terminus. Sit æquatio reſultans $x^4 + Qx^2 + Rx + S = 0$. Pone hanc multiplicatione duarum quadraticarum $x^2 + ex + f = 0$, & $x^2 - ex + g = 0$ generari, id eſt, eandem eſſe cum hac $x^4 * + (f + g - e^2)\, x^2 + (eg - ef)\, x + fg = 0$, ubi neceſſe eſt ut quadraticarum termini ſecundi ſint iidem & contrariis ſignis affecti, quo evaneſcat ſecundus terminus biquadraticæ reſultantis. Et collatis biquadraticarum terminis homologis, fiet $f + g - e^2 = Q$, $eg - ef = R$, & $fg = S$. Quare erit

$$f + g = e^2 + Q$$

$$f - g = -\frac{R}{e}$$

Adeoque $2f * = e^2 + Q - \frac{R}{e}$

Et - $* \, 2g = e^2 + Q + \frac{R}{e}$

Adeoque $4fg = e^4 + 2Qe^2 + Q^2 - \frac{R^2}{e^2} = 4S$

Sive - - - $e^6 + 2Qe^4 + (Q^2 - 4S)\, e^2 - R^2 = 0$

Et poſito $y = e^2$, $y^3 + 2Qy^2 + (Q^2 - 4S)\, y - R^2 = 0$.

Æquationis cubicæ jam reſultantis extrahatur radix. Dein habitâ illa radice regrediendum erit, ponendo $\sqrt{y} = e$, $\dfrac{e^2 + Q - \frac{R}{e}}{2} = f$, $\dfrac{e^2 + Q + \frac{R}{e}}{2} = g$, & æquationes duæ $x^2 + ex + f = 0$, & $x^2 - ex + g = 0$, extractis earum radicibus dabunt quatuor radices æquationis biquadraticæ $x^4 + Qx^2 + Rx + S = 0$.

Ut ſi proponatur æquatio $x^4 - x^3 - 5x^2 + 12x - 6 = 0$. Tolle ſecundum terminum ſcribendo $v + \frac{1}{4}$ pro x, & orietur $v^4 - \frac{43}{8} v^2 + \frac{75}{8} v - \frac{851}{256} = 0$. Ad tollendas fractiones ſcribe $\frac{1}{4}z$ pro v, & orietur $z^4 - 86z^2 + 600z - 851 = 0$. Hic eſt $-86 = Q$, $600 = R$, & $-851 = S$, adeoque $y^3 + 2Qy^2 + (Q^2 - 4S)\,y - R^2 = 0$, ſubſtitutis æquipollentibus fiet $y^3 - 172y^2 + 10800y - 360000 = 0$. Unde per inventionem diviſorum, prodibit $y = 100$. Dein habito y, radix ejus 10 erit e, $\dfrac{-86 + 100 - 60}{2}$ ſeu -23 erit f, & $+37$ erit g. Unde prodeunt æquationes quadraticæ $z^2 + 10z - 23 = 0$, & $z^2 - 10z + 37 = 0$, adeoque per reductionem, $z = -5 \pm 4\sqrt{3}$, & $z = +5 \pm 2\sqrt{-3}$. Ergo erit $x = \left(v + \frac{1}{4} = \frac{z+1}{4} =\right) -1 \pm \sqrt{3}$, & $x = \dfrac{3 \pm \sqrt{-3}}{2}$, quæ erunt quatuor radices æquationis ſub initio propoſitæ $x^4 - x^3 - \text{\&c.} = 0$.

230. Aliam methodum eruendi radices æquationum biquadraticarum mediantibus cubicis, aliquando concinniorem tradit *Euler*, quam ſic inveſtigare licet. Vide *Euler*, *Elem. Algebræ*, *Tom.* 1; aut *Martin*, *Elem. de Mathem. Toulouſe*, 1781.

Aſſumamus æquationem cubicam $y^3 - P'y^2 + Q'y - R' = 0$ cujus radices ſint a, b, c. Sitque $x = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Quadrando erit

$$x^2 = a + b + c + 2\,(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$$

Seu - $x^2 = P' + 2\,(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$

Adeoque $x^2 - P' = 2\,(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$

Et quadr. $x^4 - 2P'x^2 + P'^2 = 4(ab + ac + bc) + 8\sqrt{abc}\,(\sqrt{a} + \sqrt{b} + \sqrt{c})$.

Seu - $x^4 - 2P'x^2 + P'^2 = 4Q' + 8\sqrt{R'}\,x$

Ergo - $x^4 - 2P'x^2 - 8\sqrt{R'}x + P'^2 - 4Q' = 0$.

Hujus itaque æquationis biquadraticæ una radix neceſſario erit $x = \sqrt{a} + \sqrt{b} + \sqrt{c}$, ut ex compoſitione manifeſtum eſt. Jam quo eruantur coefficientes æquationis cubicæ adſcititiæ $y^3 - P'y^2 + Q'y - R' = 0$, collatis inter ſe terminis homologis æquationis biquadraticæ noviſſime reſultantis & datæ $x^4 + Qx^2 + Rx + S = 0$, erit

$-2P' = Q$, adeoque $P' = -\frac{1}{2}Q$

$-8\sqrt{R'} = R$, adeoque $R' = +\frac{1}{64}R^2$

$P'^2 - 4Q' = S$, adeoque $Q' = \frac{1}{4}P'^2 - \frac{1}{4}S = \frac{1}{16}Q^2 - \frac{1}{4}S$.

Quibus valoribus P', Q', R' ſubſtitutis, emerget æquatio cubica $y^3 + \frac{1}{2}Qy^2 + (\frac{1}{16}Q^2 - \frac{1}{4}S)\,y - \frac{1}{64}R^2 = 0$, cujus radices ſunt a, b, c, ut conſtat.

Porro, quoniam ex tribus radicalibus $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, unaquæque poteſt eſſe vel affirmativa vel negativa, octo exorientur harum combinationes reſpectu ſignorum, quarum quatuor exponet x ubi datæ æquationis terminus penultimus eſt affirmativus, reliquas ubi eſt negativus. Et quo diſtinguantur in utroque caſu combinationes idoneæ, obſervandum eſt, quod ſi R affirmativum ſit, ob $R = -8\sqrt{R'}$, erit $\sqrt{R'}$ ſive $\sqrt{a} \times \sqrt{b} \times \sqrt{c}$, negativum, unde neceſſe eſt ut ex tribus radicalibus una ſit negativa & reliquarum rectangulum affirmativum, quò contentum ſub omnibus evadat negativum. In hoc caſu igitur prodeunt 4 radices ſeu valores x,

$$x = \begin{cases} -\sqrt{a} + \sqrt{b} + \sqrt{c} \\ -\sqrt{a} - \sqrt{b} - \sqrt{c} \\ +\sqrt{a} + \sqrt{b} - \sqrt{c} \\ +\sqrt{a} - \sqrt{b} + \sqrt{c}. \end{cases}$$

Quod ſi R negativum ſit, erit $\sqrt{a} \times \sqrt{b} \times \sqrt{c}$ affirmativum, adeoque

$$x = \begin{cases} +\sqrt{a} + \sqrt{b} + \sqrt{c} \\ +\sqrt{a} - \sqrt{b} - \sqrt{c} \\ -\sqrt{a} + \sqrt{b} - \sqrt{c} \\ -\sqrt{a} - \sqrt{b} + \sqrt{c} \end{cases}$$

Ut ſi repetatur æquatio $z^4 * - 86z^2 + 600z - 851 = 0$, æquatio cubica $y^3 + \frac{1}{2}Q\,y^2 + (\frac{1}{16}Q^2 - \frac{1}{4}S)\,y - \frac{1}{64}R^2 = 0$, ſubſtitutis æquipollentibus fiet $y^3 - 43y^2 + 675y - 5625$

$= 0$, cujus reſolutio dabit $y = 25$, & dividendo per $y - 25 = 0$, eruetur quadratica $y^2 - 18y + 225 = 0$, cujus reductio dabit reliquas radices $y = 9 \pm 12\sqrt{-1}$. Dein extractis omnium radicibus quadraticis, nempe 5, necnon & $2\sqrt{3} \pm \sqrt{-3}$, per § 52; ob terminum penultimum $600z$ affirmativum, erit

$$z = \begin{cases} -5 + (2\sqrt{3} + \sqrt{-3}) + (2\sqrt{3} - \sqrt{-3}) = -5 + 4\sqrt{3} \\ -5 - (2\sqrt{3} + \sqrt{-3}) - (2\sqrt{3} - \sqrt{-3}) = -5 - 4\sqrt{3} \\ +5 + (2\sqrt{3} + \sqrt{-3}) - (2\sqrt{3} - \sqrt{-3}) = +5 + 2\sqrt{-3} \\ +5 - (2\sqrt{3} + \sqrt{-3}) + (2\sqrt{3} - \sqrt{-3}) = +5 - 2\sqrt{-3} \end{cases}$$

Ut antea. Ergo erit $x =$ &c. Cæterum facilius inventæ ſunt radices æquationis biquadraticæ propoſitæ per *Newtoni* methodum inveniendi diviſores, ſupra p. 176. Adeoque, non niſi deficientibus diviſoribus rationalibus ad haſce confugiendum.

Notandum reſtat, quod minuendo quadruplâ ratione radices æquationis cubicæ methodo *Carteſianâ* adſcitæ $y^3 + 2Qy^2 + (Q^2 - 4S)\,y - R^2 = 0$, immediate elicietur *Euleri* æquatio cubica $y^3 + \frac{1}{2}Qy^2 + (\frac{1}{16}Q^2 - \frac{1}{4}S)\,y - \frac{1}{64}R^2 = 0$.

Aliam inſuper methodum, & elegantem quidem (ut ſemper) educendi radices biquadraticarum mediantibus cubicis docet *Simpſon*, *Algeb.* p. 150, &c. ubi varia proponit artificia ad laborem calculi quibuſdam in caſibus particularibus minuendum. Verum de his ſatis.

231. Methodus generalis reſolvendi æquationes quintæ altiorumve dimenſionum hactenus deſideratur; idque forſan haud magno analyſtarum detrimento, ſi modo ingentium calculorum in hiſce neceſſario inſtituendorum habeatur ratio. Pergamus itaque modos quamproxime eruendi radices æquationum irrationales docere. At priùs de inventione limitum inter quos conſiſtunt ſingulæ radices agendum.

DE LIMITIBUS RADICUM SINGULARUM ÆQUATIONIS.

232. Si in æquatione quacunque, binis quibuſdam numeris pro x ſucceſſive ſubſtitutis, numeri reſultantes contrariis afficiantur ſignis, æquatio unam *ſaltem* radicem realem habebit inter hoſce numeros ſubſtitutos interjectam. Poteſt vero habere plures, quarum numerus ſemper erit impar.

233. Si in æquatione quacunque unam aut plures radices inæquales habente, ſucceſſive ſubſtituantur pro x duo numeri quorum unus eſt major, alter minor unâ radicum realium, & quorum differentia ſimul ſit minor differentiâ inter hanc radicem & reliquarum proximam intercedente, numeri reſultantes neceſſario contrariis afficientur ſignis, & unica tantum radix realis inter hoſce numeros ſubſtitutos jacebit.

234. Quo eruatur autem numerus minimâ differentiarum inter binas quaſque radices intercedente minor; primo inveniatur æquatio quadratorum differentiarum radicum, $z^r - P'z^{r-1} + \&c. = 0$, per § 177. Dein ſub-

ſtituendo $\frac{1}{u}$ pro z, inveniatur æquatio $u^{r} - P''u^{r-1} +$ &c. $= 0$, cujus radices erunt præcedentium reciprocæ. Quæratur limes radicum hujus affirmativarum per § 190, puta l, eritque $\frac{1}{l}$ minor radice minimâ affirmativâ $\frac{1}{u}$ ſeu z; adeoque $\frac{1}{\sqrt{l}}$ neceſſario erit minor quâlibet differentiarum radicum realium & inæqualium datæ æquationis. Et ſi ſumatur δ integer æqualis aut proxime major quam $\sqrt{l}$, *a fortiori*, erit $\frac{1}{\delta}$ minor quâlibet differentiarum, &c. Denique ſubſtituendo ſucceſſive in æquatione datâ terminos progreſſionis $0, \frac{1}{\delta}, \frac{2}{\delta}, \frac{3}{\delta}$, &c. pro x, numeri reſultantes tot exhibebunt variationes ſignorum quot habet æquatio radices reales & affirmativas, & unaquæque radicum inæqualium inter limites $\frac{m}{\delta}$ & $\frac{m+1}{\delta}$ jacebit, ſi modo numerorum reſultantium ſigna prodibunt contraria. Sive, quod eodem redit, & in praxi facilius erit, multiplicentur radices datæ æquationis per δ, § 163, et in æquatione reſultante ſubſtituantur ſucceſſive termini progreſſionis 0, 1, 2, 3, &c. uſque ad extremos limites radicum affirmativarum. Et hoc pacto invenientur limites inter ſingulas æquationis reſultantis radices, qui utique per δ diviſi dabunt limites inter ſingulas radices inæquales affirmativas datæ æquationis. Et ſimiliter, limites inter ſingulas radices negativas invenire licet. Quod ſi termini ita ſubſtituti dant zero, erunt ipſæ radices æquationis.

Cor. Hinc, ſi $\sqrt{l}$ prodeat unitate minor, erit $\frac{1}{\sqrt{l}}$ unitate major, adeoque æquatio aut unicam ſortietur radicem realem affirmativam, aut ſi plures, erunt binarum quarumque differentiæ unitate majores; & converſim: in hoc caſu uſurpare licet $\delta = 1$. Quòd ſi $\sqrt{l}$ prodeat major unitate, æquatio habere poteſt radices quarum differentiæ ſint unitate minores.

Hujus methodi auctor eſt *La Grange*, cæterum in æquationibus ſuperiorum dimenſionum parum utilis evadit, cum quantitatem $\frac{1}{\sqrt{l}}$ in hiſce eruere valde operoſum foret. In praxi commodior erit methodus inſequens.

235. Primo, minuantur radices datæ æquationis quantitate quadam cognitâ, § 156, & ſucceſſive ſubſtitutis pro x terminis progreſſionis 0, 1, 2, &c. quærantur ex variatione ſignorum coefficientium novæ æquationis limites integri quorum differentia eſt 1. Dein, dupla ratione augeantur radices novæ æquationis, & quærantur limites radicum æquationis jam reſultantis quorum differentia eſt 1, nam inde innoteſcent limites arctiores radicum datæ æquationis, quorum nempe differentia eſt $\frac{1}{2}$. Atque iterum augeantur dupla ratione radices æquationis noviſſimæ, & quærantur hujus limites, inde vero innoteſcent limites radicum datæ æquationis adhuc arctiores, quorum nempe differentia eſt $\frac{1}{4}$. Et ſic deinceps.

Ut ſi proponatur æquatio $x^6 - 5x^5 + 2x^4 + 15x^3 - 10x^2 - x + 1 = 0$, erit

	$x=$	$+4$	$+3$	$+2$	$+1$	0
$x^6 - 5x^5 + 2x^4 + 15x^3 - 10x^2 - x + 1$		$+$	$-$	$+$	$+$	$+$
$6x^5 - 25x^4 + 8x^3 + 45x^2 - 20x - 1$		$+$	$+$	$-$	$+$	$-$
$15x^4 - 50x^3 + 12x^2 + 45x - 20$		$+$	$+$	$-$	$+$	$-$
$20x^3 - 50x^2 + 8x + 15$		$+$	$+$	$-$	$-$	$+$
$15x^2 - 25x + 2$		$+$	$+$	$+$	$-$	$+$
$6x - 5$		$+$	$+$	$+$	$+$	$-$
1		$+$	$+$	$+$	$+$	$+$

Ubi obſervandum, quod ſubſtituto 4 pro x, coefficientes omnes novæ æquationis emergunt affirmativæ; ergo, $+ 4$ erit maximæ radicis a limes. Subſtituto autem $+ 3$, prodit una variatio ſignorum; ergo maxima radix a inter limites 4 & 3 conſiſtet. Et ſubſtituto $+ 2$, prodeunt duæ variationes ſignorum, ergo ſecunda radix b, inter limites 3 & 2 conſiſtet. Et ſubſtituto $+ 1$, prodeunt duæ variationes, ſubſtituto autem 0, quatuor; ergo tertia & quarta radix c & d inter limites 1 & 0 conſiſtent.

236. Jam quo eruantur limites radicum negativarum, mutatis ſignis terminorum alternorum, fiet

	$x=$	0	-1	-2
$x^6 + 5x^5 + 2x^4 - 15x^3 - 10x^2 + x + 1$		$+$	$-$	$+$
$6x^5 + 25x^4 + 8x^3 - 45x^2 - 20x + 1$		$+$	$-$	$+$
$15x^4 + 50x^3 + 12x^2 - 45x - 20$		$-$	$+$	$+$
$20x^3 + 50x^2 + 8x - 15$		$-$	$+$	$+$
&c.		$+$	$+$	$+$

&c.

Ubi obſervandum, quod ſubſtituto o pro x, prodeunt duæ ſignorum variationes, ſubſtituto autem + 1, una tantum; ergo radix e inter limites o & — 1, conſiſtet. Et ſimiliter, radix f, inter limites — 1, & — 2.

237. Quod ſi deſiderentur limites radicum arctiores, multiplicando radices æquationis transformatæ per binarium, fiet,

	+7	+6	+5	+4	+3	+2	+1	0
$x^6 - 10x^5 + 8x^4 + 120x^3 - 160x^2 - 32x + 64$	+	—	+	+		+	—	+
$6x^5 - 50x^4 + 32x^3 + 360x^2 - 320x - 32$	+	+	—	—		+	—	—
$15x^4 - 100x^3 + 48x^2 + 360x - 160$	+	+	—	—		+	+	—
$20x^3 - 100x^2 + 32x + 120$	+	+	+	—		—	+	+
$15x^2 - 50x + 8$	+	+	+	+		—	—	+
$6x - 10$	+	+	+	+		+	—	—
1	+	+	+	+		+	+	+

Ex ſignorum reſultantium inſpectione conſtat, quod radix $2a$ conſiſtet inter limites 7 & 6, adeoque a inter limites $3\frac{1}{2}$ & 3; $2b$ inter limites 6 & 5, adeoque b inter limites 3 & $2\frac{1}{2}$; $2c$ inter limites 2 & 1, adeoque c inter limites 1 & $\frac{1}{2}$; $2d$ inter limites 1 & o, adeoque d inter limites $\frac{1}{2}$ & o. Et ſimiliter, radicum negativarum conſiſtet e inter limites arctiores o & — $\frac{1}{2}$; & f inter limites — $\frac{1}{2}$ & — 1. Et iiſdem veſtigiis inſiſtendo, invenire licet limites radicum adhuc arctiores. Prodeunt autem æquationis datæ radices,

$a = 3,303$ $e = -0,303$
b $2,618$ f $-1,618$.
c $0,618$
d $0,383$

Ut ex compoſitione manifeſtum erit.

Atque hic obſervandum eſt, quod inventis ſignis coefficientium æquationis transformatæ principis $x^6 - 5x^5$ &c. $= 0$, ex ſubſtitutione 0, 1, 2, 3, &c. ſucceſſive pro x, oriundis, ſimul determinantur ſigna coefficientium ſecundæ $x^6 - 10x^5$ &c. $= 0$, ex ſubſtitutione 0, 2, 4, 6, &c. oritura, utpote quæ neceſſario eadem erunt reſpective. Adeo ut hac methodo requiritur ſolummodo operatio ſubſtituendi in æquatione poſteriore numeros intermedios 1, 3, 5, &c.

238. Hac methodo inveniri poſſunt limites radicum realium inæqualium æquationis, non obſtantibus radicibus impoſſibilibus. Sic habitâ æquatione $x^4 - x^3 - 5x^2 + 12x - 6 = 0$, erit

	x	2	1	0
$x^4 - x^3 - 5x^2 + 12x - 6$		+	+	—
$4x^3 - 3x^2 - 10x + 12$ - -		+	+	+
$6x^2 - 3x - 5$ - - - -		+	—	—
$4x - 1$ - - - - - -		+	+	—
1 - - - - - - - - -		+	+	+

Ex hac operatione conſtat quod una ſaltem erit radix affirmativa inter 0 & 1, & quod duæ inter limites 1 & 2 conſiſtere poſſunt. Erit quoque

	x	0	-1	-2	-3
$x^4 + x^3 - 5x^2 - 12x - 6$		$-$	$-$	$-$	$+$
$4x^3 + 3x^2 - 10x - 12$ - -		$-$	$+$	$+$	$+$
$6x^2 + 3x - 5$ - - - -		$-$	$+$	$+$	$+$
&c.		$+$	$+$	$+$	$+$

Unde conſtat, quod una ſaltem erit radix negativa inter limites — 2 & — 3, & quod tres eſſe poſſint affirmativæ; at hujus radices ſupra prodierunt $x = -1 \pm \sqrt{3}$, & $x = \frac{3 \pm \sqrt{-3}}{2}$: duæ igitur radices impoſſibiles in utroque caſu inter affirmativas latent. Hinc ſi operatione continuatâ quærantur limites arctiores, & inter limites binos ſucceſſivos, binæ ſemper exſurgent radices affirmativæ aut binæ negativæ, indicio eſt æquationem radices æquales aut impoſſibiles ſortiri.

239. Inventis hoc modo limitibus unius radicum inæqualium, facile determinari poteſt annon limes alteruter parte ſui $\frac{1}{10}$ a verâ radice differt. Primo enim, loco x in æquationem ſubſtituatur limes inferior, iterumque hic limes parte ſui $\frac{1}{10}$ auctus; ſi numeri reſultantes prodeunt diverſis affecti ſignis, indicio eſt limitem inferiorem parte ſui $\frac{1}{10}$ a verâ radice non diſcrepare: contra, ſi numerorum reſultantium ſigna ſint eadem. Et ſimiliter, ſubſtituatur

pro x limes ſuperior, atque iterum hic limes parte ſui $\frac{1}{10}$ diminutus; ſi numeri reſultantes, &c. Hæc ita ſe habebunt ubi ipſorum limitum differentia major ſit parte $\frac{1}{10}$ limitis inferioris in priore caſu, ſuperioris autem in poſteriore; nam ſi minor ſit parte decimâ &c. *a fortiori*, erit differentia inter alterutrum limitem & radicem adhuc minor parte decima totius limitis.

DE RESOLUTIONE ÆQUATIONUM PER APPROXIMATIONEM.

240. Reſolutionem æquationum numeralium per approximationem primus inſtituit *Vieta*, at methodo valde operoſa. Pars hæc doctrinæ æquationum in praxi utiliſſima, *Newtono* plurimum debet, qui duas tradidit methodas quamproxime eruendi radices æquationum numeralium in *Arithmetica Univerſali*, Cap. *De limitibus æquationum*; et in tractatu *De analyſi per æquationes infinitas*, Cap. 4. Harum priorem auctam promotamque jam proferemus.

241. Quoniam radicum ſingularum æquationis nullas radices impoſſibiles habentis quadrata ſunt affirmativa, quadratorum ſumma affirmativa erit, ideoque quadrato maximæ radicis major. Et eodem argumento, ſummæ biquadratorum, bicuborum pariumve deinceps poteſtatum radicum omnium tranſcendent poteſtates conſimiles radicis maximæ. Quoniam vero poteſtates altiores maximæ radicis longe majore ratione creſcunt quam conſimiles poteſ-

tates reliquarum, (ſi modo inter radices ipſas notabile intercedat diſcrimen reſpectu magnitudinis) quo altius involvantur eo propius ad rationem æqualitatis inter ſe accedent poteſtates conſimiles maximæ radicis & ſummæ radicum omnium. Ut ſi æquatio ſit $x^3 - 19x + 30 = 0$, cujus radices ſunt 2, 3 & — 5; erit

$$5^2 : 5^3 + 3^2 + 2^2 :: 25 : 38 :: 1 : 1{,}52$$

$$5^4 : 5^4 + 3^4 + 2^4 :: 625 : 722 :: 1 : 1{,}1552$$

$$5^8 : 5^8 + 3^8 + 2^8 :: 390625 : 397442 :: 1 : 1{,}01745 \text{ \&c.}$$

$$5^{16} : 5^{16} + 3^{16} + 2^{16} :: 152587890625 : 152631002882$$

$$:: 1 : 1{,}000028 \text{ \&c.}$$

$$5^{32} : 5^{32} + 3^{32} + 2^{32} :: 1 : 1 + \frac{185302448381913\,7}{2328306436538696289062 5} ::$$

$$:: 1 : 1{,}00000008, \text{ \&c.}$$

Unde elicitur methodus inſequens quamproxime eruendi radicem æquationis maximam ſeu a nihilo remotiſſimam.

242. *Quære ſummam quadratorum radicum & extrahe ejus radicem quadraticam; hæc enim radix major erit quam radix maxima æquationis. Sed ad radicem maximam propius accedes, ſi quæras ſummam biquadratorum & extrahas ejus radicem biquadraticam. Et adhuc propius, ſi quæras poteſtatem* 8^{vam} *ſummæ radicum & extrahas ejus radicem* 8^{vam}. *Atque ita deinceps in infinitum.*

Sic in præcedente æquatione erit radix quadratica ſummæ quadratorum $\sqrt{38} = 6{,}164$&c. $\sqrt[4]{722} = 5{,}86$&c.

$\sqrt[8]{397442} = 5,018$&c. $\sqrt[16]{152631002882} = 5,000028$&c. $\sqrt[32]{33283068\text{c.}} = 5,00000001$&c. Liquet igitur quod in æquationibus radices inæquales habentibus, ſurdæ radices ſummarum poteſtatum ſucceſſivarum radicum, decreſcendo ad verum valorem maximæ radicis cito convergunt, & quo major ſit inæqualitas radicum, eo citius.

243. Quod ſi duæ pluresve æquationis radices ſint fere inter ſe æquales, ita ut radix maxima cum alterâ ejuſdem circiter magnitudinis conſocietur, ſurdæ radices, &c. ſucceſſivæ ad verum valorem maximæ radicis tardius convergent; adeoque neceſſe erit ſummas altiorum poteſtatum radicum quærere, earumque ſurdas radices extrahere quo eruatur maximæ radicis valor quamproxime verus.

Ut ſi proponatur æquatio $8x^3 - 6x + 1 = 0$, ad tollendam coefficientem primi termini, poſito $\frac{1}{2}y$ pro x, exſurget $y^3 - 3y + 1 = 0$. Et poſito $Q = -3$ & $R = -1$, per § 170, prodit ſumma quadratorum radicum hujus $-2Q = 6$; ſumma biquadratorum $2Q^2 = 18$; ſumma 8 poteſt. $2Q(Q^3 - 4R^2) = 186$. At quoniam quadratum 6 ſeu 36 ſit ad 18 in ratione 2 ad 1; hujus vero quadratum ſeu 324 ad 186 in ratione paulo minore tantum, conſtat radices ſurdas ſummarum poteſtatum hactenus inventarum non niſi tarde ad verum valorem maximæ radicis convergere. Quæratur itaque ſumma 16 poteſtatum radicum $2Q^2(Q^6 - 24Q^3R^2 + 20R^4) = 25146$, cujus radix 16ma prodit 1,8837&c. Quæratur iterum ſumma 32 poteſtatum $2Q(Q^{15} - 112Q^{12}R^2 + 1144Q^9R^4 -$

$2112Q^6R^6 + 660Q^3R^8 - 16Q^{10}) = 6 \times 97945135$, hujus autem radix 32^{ma} prodit 1,879478&c. Conſentientibus igitur hiſce valoribus maximæ radicis ſolummodo in primo decimalium loco, colligere licet valorem ex ſumma 16 poteſtatum reſultantem, ultra primam decimalem 8 non eſſe verum. Adeoque ſi maximæ radicis valor accuratius deſideretur, neceſſe erit operationem continuando ſummam 64 poteſtatum radicum eruere, nempe 18×1913111928-2396037, cujus radix 64^{ma} prodit 1,87938530&c. Conſentientibus igitur hiſce valoribus in tribus decimalium locis, colligere licet radicem ſummæ 32 poteſtatum maximam radicem vere exponere ad uſque tertiam decimalem 9. Quanto propius ad veram radicem accedit radix ſummæ 64 poteſtatum, iiſdem veſtigiis inſiſtendo detegere licet; facilius vero methodo inſequenti.

244. Quoniam differentia radicum ſummæ 32 & 64 poteſtatum, nempe 0,0000847&c. prodit pars 22000^{ma} circiter poſterioris (cum hæc differentia toties ſaltem in 1,87938 &c. continetur) colligere licet priorem excedere maximam radicem parte ſui 22000^{ma} circiter. Unde, ſi tres radices deſignent a, b, c, quorum maxima ſit a, habetur $(a^{32} + b^{32} + c^{32})^{\frac{1}{32}} = a + \frac{1}{22000}a$. Et quoniam duæ radices in hac æquatione fere æquales neceſſario erunt a & b, vel b & c; ſit 1^{mo} a ferme æqualis b. In hoc caſu, b (ob defectum termini ſecundi) tantum exſuperabit c, ut c^{32} quam minimum erit reſpectu $a^{32} + b^{32}$, adeoque tuto negligi poteſt. Erit igitur

$$(a^{32} + b^{32} *)^{\frac{1}{32}} = a + \frac{1}{22000} a$$

$$\text{Et } a^{32} + b^{32} = a^{32} + \frac{32}{22000} a^{32} + \frac{32 \cdot 31}{2 \cdot 22000^2} a^{32} + \&c.$$

$$\text{Itemque } b^{32} = \frac{32}{22000} a^{32} \text{ fere}, = \frac{2}{11 \times 5^3} a^{32}$$

$$\text{Adeoque } b^{64} = \frac{4}{11^2 \times 5^6} a^{64}$$

$$\text{Et } (a^{64} + b^{64})^{\frac{1}{64}} = \left(a^{64} + \frac{4}{11^2 \times 5^6} a^{64}\right)^{\frac{1}{64}}$$

$$= a\left(1 + \frac{4}{11^2 \times 5^6}\right)^{\frac{1}{64}} = a\left(\frac{1890629}{1890625}\right)^{\frac{1}{64}}$$

Unde liquet radicem 64^{am}, 1,8793853 valorem maximæ radicis ad uſque 6^{tum} decimalium locum & ultra, vere deſignare.

2do. Sit b ferme æqualis c. In hoc caſu erit

$$(a^{32} + 2b^{32})^{\frac{1}{32}} = a + \frac{1}{22000} a$$

$$\text{Et } a^{32} + 2b^{32} = a^{32} + \frac{32}{22000} a^{32} + \&c.$$

$$\text{Itemque } 2b^{32} = \frac{32}{22000} a^{32} = \frac{2}{11 \times 5^3} a^{32}$$

$$\text{Adeoque } 4b^{64} = \frac{4}{11^2 \times 5^6} a^{64}, \ \& \ 2b^{64} = \frac{2}{11^2 \times 5^6} a^{64}$$

$$\text{Et } (a^{64} + 2b^{64})^{\frac{1}{64}} = \left(a^{64} + \frac{2}{11^2 \times 5^6} a^{64}\right)^{\frac{1}{64}} = a\left(\frac{1890627}{1890625}\right)^{\frac{1}{64}}$$

Adeoque in hoc caſu, *a fortiori* liquet, radicem ad uſque 6^{tam} decimalem & ultra, vere deſignari.

Radici maximæ y ſic inventæ 1,879385 &c. præponendum ſignum —, ſubſtituto enim — 1,879385 pro y, accedit $y^3 - 3y + 1$ propius ad nihil; nempe $-$ 6,636293 &c. $+$ 5,638155 &c. $+ 1 = 0{,}001$ &c.

245. Inventâ maximâ radice, invenientur reliquæ, deprimendo æquationem cubicam ad quadraticam ope diviſoris inde reſultantis, cujus reductio dabit reliquas radices.

Aut ſimili methodo erui poteſt radix minima, inveniendo maximam radicem æquationis $z^3 - 3z^2 * + 1 = 0$, cujus radices ſunt præcedentium reciprocæ, ut patebit ſubſtituendo $\frac{1}{z}$ pro y; nam inventâ maximâ hujus radice $\frac{1}{c}$, per § 169, aut per formulas defectui tertii termini accommodandas, habetur $c = 0{,}347296$. Habitis igitur radicibus extremis, habetur media radix $b = 1{,}532088$ &c. utpote reliquarum complementum ad nihil. Ergo ob $x = \frac{1}{2}y$, prodeunt radices datæ æquationis $8x^3 - 6x + 1 = 0$,

$-$ 0,939692 &c. $+$ 0,766044 &c. $+$ 0,173648 &c.

246. Ubi vero duæ pluresve æquationis radices ſunt æquales, haſce, ſi iiſdem ſignis afficiuntur, facilius eruere licet per methodum § 202, &c. Et ſi contrariis, licet per iſtam methodum immediate eruendæ non ſunt, tamen quærendo æquationem quadratorum radicum, quæ tot ſortietur radices æquales quot erant in datâ, atque ſignis affirmativis affectas, haſce per iſtam methodum eruere licet, et extractis harum radicibus quadraticis innoteſcent radices æquales datæ æquationis.

247. Quod ſi ſummarum poteſtatum ſucceſſivarum hac methodo inveniendarum, prodit aliqua negativa, aut quadratum alicujus minus ſummâ poteſtatum proxime inſequente; indicio eſt æquationem radices impoſſibiles ſortiri. Veruntamen non obſtant radices impoſſibiles quo minus eruantur radices reales, ſi modo ex ſummis poteſtatum ſucceſſive reſultantibus quadratum cujuſque præcedentis ad inſequentem propius ſemper accedet.

Sic habita æquatione $x^3 - 5x^2 + 4x - 3 = 0$, quærendo æquationes quarum radices ſunt quadrata, biquadrata, &c. radicum datæ æquationis per § 169; ſummas quadratorum, biquadratorum, $8p$, $16p$, reſpective exponent 17, 317, 100709, 1014238111189. Et cum $17^2 = 289$ prodit minus quam 317, indicio eſt æquationem radices impoſſibiles ſortiri; quoniam vero haud multùm deficit a 317, hujus verò quadratum 100489 propius adhuc accedit ad 100709, non neceſſe erit ſummas poteſtatum ultra quærere ad radicem proxime eruendam. Erit igitur

$$(100709)^{\frac{1}{8}} = 4{,}2195, \text{ et } (1014238111189)^{\frac{1}{16}} = 4{,}220690;$$

quarum prior valorem radicis vere exprimet ad uſque primam decimalem 2 tantùm, poſterior vero ad uſque quintam 9 incluſive; ſicuti per methodum præcedentem colligere licet.

Habita quoque æquatione $y^3 - 2y - 5 = 0$, ſummas quadratorum, biquadratorum, $8p$, $16p$, $32p$, radicum exponent 4, 8, 432, 138912, 18833326272 reſpective. Et cum 8^2 ſeu 64 prodit minus quam 432, indicio eſt æquationem radices impoſſibiles ſortiri; quoniam vero quadratum hujus 186624 jam proprius accedit ad 138912, at intervallo majore quidem quam in exemplo noviſſimo, extrahenda erit radix $32p$. Hæc autem prodit 2,09457 affirmative ponenda per § 148, quæ utique radicem ad uſque 4^{tam} decimalem 5, vere exprimet.

248. Hæc methodus radicibus maximis & minimis tantum æquationis educendis immediate inſervit, idque calculo ſatis operoſo, ubi duæ plureſve radices ſint inter ſe æquales aut accurate aut quamproxime; tardius nimirum ad verum radicis valorem convergentibus ſurdis radicibus ſummarum poteſtatum ſucceſſivarum: & interdum, perturbantibus radicibus impoſſibilibus, deficit. Methodus altera *Newtoni* jam explicanda præ cæteris ſimplex & uſui accommodata videtur: Æquationes in quibus aliqui vel nulli termini deſint, eâdam fere facilitate tractantur; et ſingulis æquationum radicibus educendis æque reſpondet.

METHODUS ALTERA.

249. Sit æquatio $x^3 + ax^2 + bx + c = 0$ reſolvenda: inveniatur per § 239, aut per methodum præcedentem, k limes qui minus quam parte ſui decimâ differt a radice quæſitâ. 1^{mo}. Sit p cum proprio ſigno hujus complementum ad verum radicis valorem, & ſubſtituto $k + p$ pro x in æquationem, prodit per § 156, nova $(k^3 + ak^2 + bk + c) + (3k^2 + 2ak + b)\,p + (3k + a)\,p^2 + p^3 = 0$ (incipiendo nimirum a termino ultimo). Sive, deſignantibus A, B, C, D numeros coefficientes reſpectivè, $A + Bp + Cp^2 + Dp^3 = 0$. Et ſic univerſaliter: habitâ æquatione $x^n + ax^{n-1}$

$+ bx^{n-2} + \&c. = 0$, ſubſtituendo $k + p$ pro x, erunt coefficientes novæ æquationis,

$$\begin{array}{llllll} A = k^{n} & + ak^{n-1} & + bk^{n-2} & + ck^{n-3} & + \&c. \\ B = nk^{n-1} & + n-1 . ak^{n-2} & + n-2 . bk^{n-3} & + \&c. & \\ C = n . \frac{n-1}{2} k^{n-2} & + n-1 . \frac{n-2}{2} ak^{n-3} & + \&c. & & \\ \&c. & \&c. & & & \end{array}$$

Æquationis igitur reſultantis $A + Bp + Cp^{2} + Dp^{3} + \&c. = 0$, radix p exquirenda eſt, ut quotienti k addatur; nempe (neglectis terminis Cp^{2}, Dp^{3}, &c. ob parvitatem) $A + Bp = 0$, ſive $p = \frac{-A}{B}$ fere. 2^{do}. Sit q cum proprio ſigno hujus complementum ad verum valorem p, & perinde ſubſtituto $\frac{-A}{B} + q$ pro p, in æquationem reſultantem, prodit nova $A' + B'q + C'q^{2} + D'q^{3} + \&c. = 0$, et neglectis terminis $C'q^{2} + D'q^{3} + \&c.$ ut antea, erit $q = \frac{-A'}{B'}$ fere. 3^{tio}. Sit r hujus complementum ad verum valorem q, & ſubſtituto $\frac{-A'}{B'} + r$ pro q, in æquationem ultimo reſultantem, prodit $A'' + B''r + C''r^{2} + D''r^{3} + \&c. = 0$, adeoque $r = \frac{-A''}{B''}$ fere. Et ſic operatione quouſque placuerit continuatâ, tandem emergit $x = k - \frac{A}{B} - \frac{A'}{B'} - \frac{A''}{B''} - \&c.$ quam proxime. Quo propius autem accedit k ad verum radicis valorem, eo pauciora complementa adhibere opus erit ad radicem quamproxime eruendam.

250. Emergente, per primam operationem, $p = \frac{-A}{B} - \frac{C}{B}p^2 - \frac{D}{B}p^3 -$ &c. quot ad loca decimalium primus hujus terminus $\frac{-A}{B}$ valorem p fere deſignabit, cæteris neglectis; ſeu, quouſque continuanda eſt diviſio numeratoris A per denominatorem B, antequam reliqui termini $\frac{C}{B}p^2$, $\frac{D}{B}p^3$ valorem quotientis reſultantis afficiant; ſic inveſtigare licet.

Quoniam p, utpote per hypotheſin minus quam $\frac{1}{10}k$, eſt fractio decimalis unitate minor, ſi habeat cifras aliquas decimales primæ ejus figuræ ſignificanti præfixas, poteſtates ejus p^2, p^3, p^4, &c. ſaltem duplo, triplo, quadruplo, &c. plures cifras decimales præfixas habebunt reſpective. Adeoque 1mo. Si coefficientes $\frac{C}{B}$, $\frac{D}{B}$, &c. ſint quoque decimales unitate minores, & quæ cifras aliquas decimales habeant præfixas, terminus quiſque $\frac{C}{B} \times p^2$, &c. tot ſaltem habebit cifras decimales præfixas quot ſunt in utroque factore ſimul. Et 2do. Si coefficientes $\frac{C}{B}$, &c. ſint unitate majores, erit numerus cifrarum decimalium termino cuique $\frac{C}{B}p^2$, &c. præfixarum æqualis exceſſui cifrarum decimalium in p^2, &c. præ locis integris in $\frac{C}{B}$, &c. reſpective:

ut ex multiplicatione decimalium conſtat. Unde in omni caſu, cognito numero cifrarum decimalium complemento p præfixarum, innoteſcet numerus cifrarum decimalium cuique terminorum inſequentium poſt primum, ſcil. $\frac{C}{B}p^2$, $\frac{D}{B}p^3$, &c. præfigendarum; atque inde regrediendo, facile conſtabit, quod primus terminus $\frac{-A}{B}$ valorem p ad tot loca decimalium fere deſignabit, quot habet cifras decimales præfixas ſecundus terminus $\frac{C}{B}p^2$, &c. Itemque, quod duorum aggregatum $\frac{-A}{B} - \frac{C}{B}p^2$ tot ad loca decimalium valorem p fere deſignabit, quot habet cifras decimales præfixas tertius terminus $\frac{D}{B}p^3$. Et ſic deinceps. Ac ſimiliter, determinare licet ad quot loca decimalium termini $\frac{-A'}{B'}$ & $\frac{-A''}{B''}$ complementa ſucceſſiva q & r reſpective, fere exponent.

251. Cæterum ſi quæras an valores modo inventi $\frac{-A}{B}$, $\frac{-A'}{B'}$, &c. veris valoribus p, q, &c. reſpective majores vel minores ſint; hoc inveniri poteſt, ſubſtituendo $\frac{-A}{B}$ pro p in æquationem $A + Bp + Cp^2$ &c. $= 0$, nam ſi numerus reſultans affirmativus ſit, $\frac{-A}{B}$ major erit veritate, ſi negativus ſit, minor. Et ſic in cæteris. Idem quoque facilius

innoteſcet ex operatione proximâ, ſubſtituendi nimirum $\frac{-A}{B} + q$ pro p; nam ſi ex æquatione reſultante prodeat q affirmativus, indicio eſt quod $\frac{-A}{B}$ eſt minor, ſin q ſit negativus, major quam verus valor p.

152. *Ex.* Repetatur æquatio $y^3 - 2y - 5 = 0$, per methodum priorem reducta, § 247. Et ſi ab initio auſpicari cum *Newtono* placuerit, inveniatur per § 239, limes inferior 2, qui minus quam decimâ ſui parte differt a radice quæſitâ. 1mo. Subſtituatur $2 + p$ pro y in æquationem, & prodibit nova $- 1 + 10p + 6p^2 + p^3 = 0$, ſive (neglectis $6p^2 + p^3$ ob parvitatem) $- 1 + 10p = 0$, et $p = \frac{1}{10} = 0,1$ fere. 2do. Subſtituatur jam $0,1 + q$ pro p in æquationem reſultantem, & emerget $+ 0,061 + 11,23q + 6,3q^2 + q^3 = 0$, ſive, neglectis, ut antea, terminis in quibus q ultra primam dimenſionem aſſurgit, $q = \frac{-0,061}{11,23} = - 0,0054$ fere (dividendo nempe ad uſque quatuor decimalium loca, cum q duas habet cifras decimales primæ ſuæ figuræ ſignificanti 5 præfixas, adeoque q^2 necnon & $\frac{6,3q^2}{11,23}$ quatuor habebit, per § 250. Et proinde hic terminus $\frac{6,3q^2}{11,23}$ valorem p ex ſecundâ operatione reſultantem non afficiet intra limites quatuor locorum decimalium.) 3^{tio}. Subſtituatur iterum $- 0,0054 + r$ pro q in æquationem noviſſimam, & emerget $0,000541550536 + 11,16204748r + 6,2838r^2$

$+ r^3 = 0$, ſive $r = \frac{-0{,}000541550536}{11{,}16204748} = -\,0{,}00004851$ fere (dividendo nempe donec octo eliciantur decimalium loca, cum r habet 4 cifras decimales primæ ſuæ figuræ ſignificanti 4 præfixas, adeoque r^2 necnon & $\frac{6{,}2838r^2}{11{,}16204748}$ octo habebit.) Prodit igitur

$$\begin{array}{ll} \left.\begin{array}{l} k = +\,2 \\ p = +\,0{,}1 \end{array}\right\} = \;-\;- & +\,2{,}10000000 \\ \left.\begin{array}{l} q = -\,0{,}0054 \\ r = -\,0{,}00004851 \end{array}\right\} = & -\,0{,}00544851 \\ \hline \therefore\, y = \;-\;-\; = & +\,2{,}09455149. \end{array}$$

Ad minuendum ſubſtituendi laborem in tertia operatione, terminum altiſſimum q^3 æquationis ſecundò reſultantis propter exilitatem ſuam negligit *Newtonus*, unde ſubrepit error in ultimum figuram 3 valoris r hoc pacto eliciti, 0,00004853. Minui quoque poteſt dividendi labor, quoniam quatuor tantum deſiderantur figuræ decimales ſignificantes in quotiente, abſcindendo figuras ſignificantes numeratoris ultra quatuor primas, itemque denominatoris (ob parvitatem figurarum ab initio) ultra quinque primas: erit enim $\frac{-0{,}0005415}{11{,}162} = -\,0{,}00004851$ ut antea.

Obſervandum eſt, quod qualibet operatione duplicantur figuræ decimales radici adjiciendæ. Atque in hac methodo ita ſemper ſe res habebit, niſi forſan ubi $\frac{C}{B}$ ſit unitate major, quo in caſu, figurarum radicis ultimarum aliquot per terminum $\frac{C}{B}p^2$ affici poſſunt. Quoad materiam, vix differt hæc methodus ab illâ quam tradidit *Ralphſon*, in *Analyſi Æquationum Univerſali*, an. 1690, quæ utique ſimiliter procedit aſſumendo duos terminos ultimos tantum æquationis ex quaque operatione reſultantis.

253. Accuratius autem eruetur valor p, aſſumendo tres terminos ultimos æquationis reſultantis, nempe $A + Bp + Cp^2 = 0$, ſeu $A + (B + Cp)\,p = 0$; unde prodit $p = \frac{-A}{B+Cp}$, ſeu, ſubſtituendo $\frac{-A}{B}$ in denominatorem pro p, $= \frac{-A}{B - \frac{CA}{B}}$, vel $= \frac{-AB}{B^2 - AC}$; quarum prior formula praxi magis accommodata videtur, re nimirum per unam multiplicationem & duas diviſiones peractâ, quæ in poſteriore per tres multiplicationes ac unam diviſionem procedit. Et hoc eſt theorema *rationale Halleianum*, *Phil. Tranſ.* an. 1694, N° 210 *. Aut aliquantò accuratius, per extractionem radicis ſurdæ quadraticæ, unde prodit

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* *Analyſin per æquationes infinitas* excogitavit *Newtonus* ſaltem ante annum 1665. Vid. p. 33. Hanc a *Newtono* acceptam cum *Johanne Collins* communicavit *Barrow*, anno 1669. Eandemque in ſcriniis *Johannis Collins* repertam primus edidit *Jones*, an. 1711. Methodi *Newtonianæ* proxime eruendi radices æquationum numeralium & literalium, duobus terminis tantum æquationis reſultantis adhibitis, exemplum dedit *Wallis* in *Algebra*, an. 1683, *Anglice* editâ. De quo ita *Halley*, loc. cit. " Qualia in hoc negotio [reſolvendi æquationes numerales] præſtiterit ſagacisſima ingenii *Newtoniani* vis, ex contractiore ſpecimine a clariſſimo *Walliſio*, " Cap. 94, Algebræ ſuæ edito, potius conjecturâ aſſequi quam pro certo comperiri licet. Ac dum obſtinata auctoris modeſtia amicorum precibus devicta " cedat, inventaque hæc ſua pulcherrima in lucem promere dignetur, expectare " cogimur."

$p = \frac{-\frac{1}{2}B \pm \sqrt{\frac{1}{4}B^2 - AC}}{C}$, quarum ſeligenda eſt minor radix $\frac{-\frac{1}{2}B + \sqrt{\frac{1}{4}B^2 - AC}}{C}$, (in quâ utique partes $\frac{\frac{1}{2}B}{C}$ & $\frac{\sqrt{\frac{1}{4}B^2 - AC}}{C}$ contrariis ſignis afficiuntur) tum ob parvitatem ipſius p, tum quod hæc radix ad valorem prius inventum $\frac{-A}{B}$ propius accedit. Et hoc eſt theorema *irrationale Halleianum*, loco. cit. Sic in præcedente exemplo, per theorema *rationale*, prodit $p = \frac{1}{10 + \frac{6}{10}} = \frac{1}{10,6} = 0,0943$&c. et per theorema *irrationale*, prodit $p = \frac{-5 + \sqrt{31}}{6} = 0,9462$&c. quorum prior valor conſentit cum valore $p + q$, per *duas* operationes methodi præcedentis elicito, in duobus locis figurarum decimalium; poſterior autem, in tribus. Atque hinc forſan theorema *irrationale* tantùm, utpote accuratius, commendat *Newtonus*: utrumque tamen adhibere haud importunum erit, uno nimirum corrigente alterum.

254. Quoniam verò in hujuſmodi approximationibus maximi momenti eſt determinatio primæ figuræ decimalis complementi p, hanc quidem, tribus terminis ultimis adhibitis ſemper explorare convenit ubi in æquatione $A + Bp + Cp^2$ &c. $= 0$, quadratum coefficientis penultimi termini B^2 non ſit decies majus quam factus ex ultimo termino A in coefficientem termini antepenultimi C.

Nam quoniam valor p, adhibitis duobus terminis ultimis tantum æquationis reſultantis, ſit $\frac{-A}{B}$ fere; adhibitis autem tribus, fit $\frac{-AB}{B^2-AC}$ accuratius, $= \frac{-A}{B} - \frac{A^2C}{B^3}$ circiter: (ut patebit dividendo — AB per B^2 — AC) differentiam horum valorum fere exponet $\frac{-A^2C}{B^3} = \frac{AC}{B^2} \times \frac{-A}{B}$. Unde poſito $\frac{AC}{B^2} = \frac{1}{10}$, differentia valorum evadet $\frac{1}{10} \times \frac{-A}{B}$ fere, hoc eſt, pars decima fere totius valoris prioris; poſito $\frac{AC}{B^2} = \frac{1}{9}$, evadet major parte decimâ totius; poſito $\frac{AC}{B^2} = \frac{1}{8}$, evadet adhuc major parte decimâ totius: et ſic deinceps, creſcente valore $\frac{AC}{B^2}$, ſeu quod perinde eſt, decreſcente ratione B^2 ad AC, eo magis recedet prior valor $\frac{-A}{B}$ a poſteriore $\frac{-AC}{B^2-AC}$, ſive, *a fortiore*, a valore accuratiore $\frac{-\frac{1}{2}B + \sqrt{\frac{1}{4}B^2 - AC}}{C}$. Hinc, ſi B^2 non ſit decies majus quam AC, differentia valorum tanta eſſe poteſt, quanta errorem in ipſam primam figuram decimalem prioris valoris $\frac{-A}{B}$ inducat. Quod ſi B^2 ſit decies aut plus quam decies majus quam AC, differentia iſta figuras decimales in $\frac{-A}{B}$ poſt primum locum tantùm afficiet. Et ſimiliter, ſi B^2 ſit centies majus quam AC, figuras decimales poſt ſecundum locum tantum; ſi millies, poſt tertium locum tantum afficiet. Atque ita in cæteris q, r, s, &c.

Obſervandum eſt, quod ope theorematis *irrationalis* qualibet vice triplicatur numerus locorum decimalium radici per præcedentem operationem adjiciendorum, per § 250. Hoc pacto igitur figuras decimales duplo plures quam per methodum præcedentem, qualibet vice radici lucraberis.

255. Hinc ſi radicem ad uſque quintum aut ſextum decimalium locum tantum quæras, in praxi plerumque facilius erit utramque methodum adhibere. Eruendo nimirum valorem primi complementi p ex tribus ultimis terminis æquationis reſultantis; deinde *Halleium* ſecuti, opus de integro repetere, ſubſtituendo in æquationem datam, valorem radicis jam inventum $+q$ pro x; (præſertim ubi aliquis terminus datæ æquationis deeſt, hoc pacto enim minuitur ſubſtituendi opus) unde ex binis terminis æquationis ultimò reſultantis habebitur q. Sic ex conſenſu theorematis *rationalis* & *irrationalis* prodiit $p = ,094$: deinde ſubſtituendo in æquationem datam $2,094 + q$ pro x, prodit nova $- 0,006153416 + 11,154508q +$ &c. $= 0$, adeoque $q = + \frac{0,006153416}{11,154508} = + 0,000551$. Adeoque $x = 2,094 + q = 2,094551$.

256. Quod ſi radix ad plurima decimalium loca deſideratur quam proxime, laborem plerunque minues, præſertim in æquationibus ſuperiorum dimenſionum, ſi ſingula complementa p, q, r, &c. extrahendo minorem radicum ex tribus ultimis terminis æquationis noviſſimè reſultantis exquiras.

Ex. 2. Proponatur æquatio $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$, cujus radicem in ſolutionem difficiſſimi problematis Arithmetici, *Vietæ* methodo, accuratiſſime aſſecutus eſt *Wallis*, *Algeb.* Cap. 62; quamque itidem methodi ſuæ exemplum affert *Ralphſon*, p. 25, 26. Ejuſmodi autem formæ eſt hæc æquatio, ut plures habeat radices affirmativas, & quod reſolvendi difficultatem adaugeat, prægrandes ſunt coefficientes ejus reſpectu termini ultimi.

Minuendi gratiâ, dividantur hujus radices per denarium, & prodibit $y^4 - 8y^3 + 20y^2 - 15y + 0,5 = 0$. Inveniatur limes hujus $k = 1$, & ſubſtituendo $1 + p$ pro y, prodibit nova $- 1,5 + 5p + 2p^2 -$ &c. $= 0$, adeoque per extractionem radicis minoris, erit $p = \frac{-5 + \sqrt{37}}{4} = 0,27$. & $y = 1 + p = 1,27$: adeoque $x = 10y = 12,7$. Subſtituatur iterum $12,7 + q$ pro x, in æquationem datam, & prodibit $- 298,6559 + 5296,132q - 82,26q^2 -$ &c. $= 0$, adeoque $q = \frac{2648,066 - \sqrt{6987686,106022}}{82,26} = 0,05644080$ &c. Per theorema *rationale* quoque prodibit $q =$

$$= \frac{298,6559}{5296,132 - \left(\frac{82,26 \times 298,6559}{5296,132}\right)} = \frac{298,6559}{5296,132 - 4,63875} =$$

0,05644075 &c. Unde, his valoribus collatis, certi eſſe poſſumus q fere deſignari ad uſque quinque decimalium loca ſaltem; adeoque erit $x = 12,7 + q = 12,75644$ proxime. Et ſi plures deſiderentur radicis figuræ, de integro repetatur calculus, ſubſtituendo $12,75644 + r$ pro x in æquationem datam, & aſſumptis tribus terminis æquationis reſultantis, prodibit $r = 0,00000179448074$: adeoque radix accuratiſſima fit 12,75644179448074. redintegratione calculi nimirum triplicante numerum figurarum decimalium prius inventarum.

257. Labor hujus methodi præcipuus in operatione ſubſtituendi quantitates unas pro aliis reperietur: id quod varie perficias, ſequentem modum maxime expeditum cenſet *Newtonus*, præſertim ubi numeri coefficientes conſtant ex pluribus figuris.

Repetatur æquatio præcedens $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$. Reſolvi hæc poteſt in factores hujus formæ,

$$\overline{\overline{\overline{\overline{\overline{(x - 80}\times x} + 1998}\times x} - 14937}\times x + 5000 = 0.$$

Ubi prodit primus $x - 80$; ſecundus, x; tertius, factus ex præcedentibus una cum 1998; quartus, x; quintus, factus ex præcedentibus una cum $-$ 14937; ſextus, x; et facto ſub omnibus adjiciendus 5000. Subſtituendo autem in ſingulis factoribus $12{,}7 + q$ pro x, erit

$$\begin{array}{lll}
\text{I.} & - & 67{,}3 + q \\
 & \times & 12{,}7 + q \\ \hline
\text{II.} & - & 854{,}71 - 54{,}6q + q^2 \\
 & + & 1998 \\ \hline
\text{III.} & + & 1143{,}29 - 54{,}6q + q^2 \\
 & \times & 12{,}7 + q \\ \hline
\text{IV.} & + & 14519{,}783 + 449{,}87q - 41{,}9q^2 + \&c. \\
 & - & 14937 \\ \hline
\text{V.} & - & 417{,}217 + 449{,}87q - 41{,}9q^2 \\
 & \times & 12{,}7 + q \\ \hline
\text{VI.} & - & 5298{,}6559 + 5296{,}132q - 82{,}26q^2 - \&c. \\
 & + & 5000 \\ \hline
\text{VII.} & - & 298{,}6559 + 5296{,}132q - 82{,}26q^2 - \&c. = 0.
\end{array}$$

258. Hæc methodus radicibus æquationum tam purarum quam affectarum educendis inſervit, & proinde in extrahendis radicibus numerorum altioribus utilis erit. Vide *Halley*, loco. cit. Porro, inventâ quotiente k, p, &c. æquatio ſemper relinquitur cujus radix unà cum acquiſitâ quotiente adæquat radicem æquationis primo propoſitæ. Unde examinatio operis hic æque poterit inſtitui ac in reliquâ Arithmeticâ, auferendo nempe quotientem a radice primæ æquationis (ſicut analyſtis notum eſt) ut æquatio ultima, vel termini ejus duo treſve ultimi, inde producantur. Sic invento $p = 0,1$, erit $k + p = 2,1$, adeoque ſubſtituendo $2,1 + q$ pro y, in æquationem primam $y^3 - 2y - 5 = 0$, prodibit $0,061 + 11,23q + 6,3q^2 + q^3 = 0$; ipſa nempe æquatio quæ ex ſubſtitutione $0,1 + q$ pro p in æquationem primo reſultantem prodiit: atque ita iſtius operationis veritas comprobatur.

259. Ex dictis facile judicabit lector, an valent multi nominis viri *La Grange* objectiones, *Diſſert. Acad. Berlin*, an. 1768. nempe, quod methodo *Newtonianâ* ignoratur quantum ad veritatem accedit tum valor (k) primo aſſumptus, tum valor cujuſque complementi inſequentis, (p), (q), &c. Methodus certe, quam ut accuratiorem commendat, *Newtonianæ* facilitate longe cedit, quippe quæ ex inventione quantitatis $\sqrt{1}$, § 234. non niſi per tentamina valde laborioſa in æquationibus plurimarum dimenſionum eruendæ, auſpicatur; per inventionem limitum æquationum ex binomiis compluribus ſucceſſive ſubſtitutis reſultantium, procedit; et per combinationem plurium fractionum continuarum ex hiſce operationibus elicitarum, concludit.

260. Ad evitandum tertiæ ſubſtitutionis laborem in exemplo ſecundo, correctiones duas ſatis artificioſas adhibet *Halley*, quarum priore, notas veras in aſſumptâ (k) quintu-

plices reddit; poſteriore autem, etiam duplum numerum cifrarum adjungit: ut omnino aſſumptas ſeptuplicet. Cæterum iteratâ hâc operatione parùm promovetur calculus. Simili methodo inſiſtit *Bernoulli*, *Lect. Calculi integ.* Lect. 53.

Poſſimus quoque *Le Marquis de Courtivron* atque *Eulerum* ſecuti, valorem p ex pluribus terminis æquationis reſultantis elicere, ſubſtituendo nimirum in hanc æquationem valores p^2, p^3, &c. ex quaque præcedente operatione inventos, ſcil. ſubſtituendo $\frac{A^2}{B^2}$ pro p^2; deinde, ſubſtituendo $\left(-\frac{A}{B}-\frac{C}{B}\times\frac{A^2}{B^2}\right)^3$ pro p^3; & ſic deinceps. Unde elicitur formula inſequens, $p =$

$$-\frac{A}{B}-\frac{C}{B}\left(\frac{A}{B}\right)^2-\left(\frac{2C^2-BD}{B^2}\right)\left(\frac{A}{B}\right)^3-\left(\frac{5C^3-5BCD+B^2E}{B^3}\right)\left(\frac{A}{B}\right)^4-$$

&c. Sic repetito exemplo 1mo. prodit $\frac{A}{B} = -\frac{1}{10} = -0{,}1$. $\frac{A^2}{B^2} = 0{,}01$. $\frac{A^3}{B^3} = -0{,}001$. $\frac{A^4}{B^4} = 0{,}0001$. $\frac{C}{B} = 0{,}6$. $\frac{2C^2-BD}{B^2} = 0{,}62$. $\frac{5C^3-5BCD}{B^3} = 0{,}78$. Unde prodit $p = +0{,}1 - 0{,}006 + 0{,}00062 - 0{,}000078 = 0{,}94542$. Vide *Mem. De l'Acad. Paris*, 1744, p. 405. Præſtat autem methodus *Newtoniana*, utpote quæ iteratâ operatione ad verum radicis valorem citius convergit.

Aliam approximandi methodum tradit *Taylor*, ex ipſius *Incrementorum* theoriâ petitam, *Phil. Tranſ.* an. 1717. Duas quoque, & egregias quidem, excogitavit *Simpſon*, quarum prior per ſeriem infinitam citiſſime convergentem procedit, *Mathematical Diſſertations*, p. 102; poſterior verò ex calculo fluxionali pendet, *Eſſays on various ſubjects*, p. 81.

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